OCR
RECOGNISING ACHIEVEMENT

**ADVANCED GCE**

# MATHEMATICS (MEI) 4777

Numerical Computation

Candidates answer on the answer booklet.

**OCR supplied materials:**
- 8 page answer booklet (sent with general stationery)
- MEI Examination Formulae and Tables (MF2)
- Graph paper

**Other materials required:**
- Scientific or graphical calculator
- Computer with appropriate software and printing facilities

**Friday 24 June 2011**
**Afternoon**

**Duration:** 2 hours 30 minutes

## INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet. Please write clearly and in capital letters.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer any **three** questions.
- Additional sheets, including computer print-outs, should be fastened securely to the Answer Booklet.
- Do **not** write in the bar codes.

## COMPUTING RESOURCES

- Candidates will require access to a computer with a spreadsheet program and suitable printing facilities throughout the examination.

## INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- In each of the questions you are required to write spreadsheet routines to carry out various numerical analysis processes.
- You will not receive credit for using any numerical analysis functions which are provided within the spreadsheet. For example, many spreadsheets provide a solver routine; you will not receive credit for using this routine when asked to write your own procedure for solving an equation.
  You may use the following built-in mathematical functions: square root, sin, cos, tan, arcsin, arccos, arctan, ln, exp.
- For each question you attempt, you should submit print-outs showing the spreadsheet routine you have written and the output it generates. It will be necessary to print out the formulae in the cells as well as the values in the cells.
  You are not expected to print out and submit everything your routine produces, but you are required to submit sufficient evidence to convince the examiner that a correct procedure has been used.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

**1** In this question, the notation $\exp(t)$ is used to denote $e^t$.

**(i)** Show, graphically or otherwise, that the equation $x = \exp(-x^2)$ has exactly one root, $\alpha$.

Use a spreadsheet to show that the iteration $x_{r+1} = \exp(-x_r^2)$, with a suitable starting value, converges slowly to $\alpha$. Confirm your findings by considering the derivative of $\exp(-x^2)$. **[9]**

**(ii)** Obtain the relaxed iteration $x_{r+1} = (1 - \lambda)\, x_r + \lambda \exp(-x_r^2)$.

On a spreadsheet investigate the speed of convergence of the relaxed iteration for various values of $\lambda$. Hence find, correct to 1 decimal place, the value of $\lambda$ that gives fastest convergence. **[6]**

**(iii)** Show that, theoretically, the best value of $\lambda$ is given by

$$\lambda = \frac{1}{1 + 2\alpha \exp(-\alpha^2)}.$$

Evaluate this expression. **[3]**

**(iv)** On a spreadsheet, carry out the iteration $x_{r+1} = (1 - \lambda_r)\, x_r + \lambda_r \exp(-x_r^2)$ where

$$\lambda_r = \frac{1}{1 + 2x_r \exp(-x_r^2)}.$$

Show that the convergence of this iteration is faster than first order. **[6]**

**2** **(i)** Obtain from first principles the Gaussian two-point rule for numerical integration:

$$\int_{-h}^{h} \mathrm{f}(x)\,\mathrm{d}x \approx h\left(\mathrm{f}\left(-\frac{h}{\sqrt{3}}\right) + \mathrm{f}\left(\frac{h}{\sqrt{3}}\right)\right).$$

Find the error when this rule is used to evaluate $\int_{-h}^{h} x^3\,\mathrm{d}x$ and $\int_{-h}^{h} x^4\,\mathrm{d}x$. Hence state the orders of the local and global errors in the rule. **[10]**

**(ii)** Use the Gaussian two-point rule to find, correct to 6 decimal places, the value of the integral

$$I = \int_0^1 \sqrt{2 + \sin x}\,\mathrm{d}x.$$

You should begin with $h = 0.5$ and then take $h = 0.25, 0.125, \ldots$ as necessary.
Show, by considering ratios of differences, that the global error is as stated in part **(i)**. **[9]**

**(iii)** Modify your routine so that it calculates values of the integral

$$J = \int_0^1 (2 + \sin x)^k\,\mathrm{d}x$$

for any specified $k$. Find, correct to 2 decimal places, the value of $k$ for which $J = 3$. **[5]**

**3** The differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$ with initial conditions $x = x_0$, $y = y_0$, is to be solved by using the Runge-Kutta methods given below.

| Method A | Method B |
|---|---|
| $k_1 = h\,\mathrm{f}(x_r, y_r)$ | $k_1 = h\,\mathrm{f}(x_r, y_r)$ |
| $k_2 = h\,\mathrm{f}(x_r + \frac{1}{2}h, y_r + \frac{1}{2}k_1)$ | $k_2 = h\,\mathrm{f}(x_r + h, y_r + k_1)$ |
| $y_{r+1} = y_r + k_2$ | $y_{r+1} = y_r + \frac{1}{2}(k_1 + k_2)$ |
| $x_{r+1} = x_r + h$ | $x_{r+1} = x_r + h$ |

**(i)** Set up a spreadsheet to obtain a numerical solution by each method to the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{1 + x + y}, \text{ with } y = 0 \text{ when } x = 0.$$

For $h = 0.2, 0.1, 0.05, 0.025$, find the estimates given by each method for $y$ when $x = 2$. By considering ratios of differences show that each method is second order. Show that the errors in one method are substantially less than the errors in the other. **[20]**

**(ii)** Obtain a graph of the solution curve.

Determine, correct to 2 decimal places, the value of $x$ on the solution curve for which $y = 2x$. **[4]**

**[Question 4 is printed overleaf.]**

**4** The variables $x$ and $y$ are thought to be related by an equation of the form

$$y = ax + bx^2 + cx^3, \qquad (*)$$

for some constants $a$, $b$ and $c$.

The following experimental data are available. The $x$ values are exact but the $y$ values contain experimental error.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ | –3 | –2 | –1 | 0 | 1 | 2 | 3 |
| $y$ | –35.25 | –8.01 | 2.51 | –0.09 | –4.07 | –5.06 | 0.65 |

**(i)** Use a spreadsheet to obtain a sketch of the data points and use it to explain why (*) looks like a reasonable fit. **[4]**

**(ii)** Show that one of the normal equations for finding the least squares estimates of $a$, $b$ and $c$ is

$$\sum xy = a\sum x^2 + b\sum x^3 + c\sum x^4.$$

Write down the other two normal equations. **[5]**

**(iii)** Find

- the least squares estimates of $a$, $b$ and $c$,
- the fitted values of $y$,
- the sum of the residuals,
- the sum of the squares of the residuals. **[12]**

**(iv)** Obtain a sketch of the fitted curve and the data points. Comment briefly on the fit of the curve to the data. **[3]**

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