OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE **4777/01**

MATHEMATICS (MEI)

Numerical Computation

WEDNESDAY 18 JUNE 2008 Morning

Time: 2 hour 30 minutes

Additional materials: Answer Booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

CUP/T39242

INSTRUCTIONS TO CANDIDATES

- Write your name in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer any **three** questions.
- Additional sheets, including computer print-outs, should be fastened securely to the Answer Booklet.

COMPUTING RESOURCES

- Candidates will require access to a computer with a spreadsheet program and suitable printing facilities throughout the examination.

INFORMATION FOR CANDIDATES

- The number of marks for each question is given in brackets [ ] at the end of each question or part question.
- In each of the questions you are required to write spreadsheet routines to carry out various numerical analysis processes. You should note the following points.
- You will not receive credit for using any numerical analysis functions which are provided within the spreadsheet. For example, many spreadsheets provide a solver routine; you will not receive credit for using this routine when asked to write your own procedure for solving an equation.

  You may use the following built-in mathematical functions: square root, sin, cos, tan, arcsin, arccos, arctan, ln, exp.
- For each question you attempt, you should submit print-outs showing the spreadsheet routine you have written and the output it generates. It will be necessary to print out the *formulae* in the cells as well as the *values* in the cells.

  You are not expected to print out and submit everything your routine produces, but you are required to submit sufficient evidence to convince the examiner that a correct procedure has been used.
- The total number of marks for this paper is **72**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages.

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1 **(i)** Explain carefully what it means to say that an iteration has first order convergence.
Show that, if y_0, y_1, y_2 are three terms in a first order iteration converging to α, then α may be estimated as $\dfrac{{y_1}^2 - y_0 y_2}{2y_1 - y_0 - y_2}$. [6]

A curve has equation $x^y + y^x = 2$, where $x > 0$ and $y > 0$. Note that the point (1, 1) lies on the curve and that the curve is symmetrical about the line $y = x$. You are given that, for any value of y, there is only one value of x.

(ii) Show that, for $x = 1.1$, the equation may be re-arranged as $y = (2 - 1.1^y)^{\frac{1}{1.1}}$. Set up a spreadsheet to perform the iteration based on this rearrangement. Starting with $y_0 = 1$, obtain y_1 and y_2. Use the result in part **(i)** to obtain a more accurate value of y when $x = 1.1$.

Repeat this process beginning with the more accurate value of y. Comment on the likely accuracy of your new estimate. [6]

(iii) Repeat the process in part **(ii)** to obtain estimates of y for $x = 1.2, 1.3, \dots 2.0$.
Comment on the likely accuracy of your result for $x = 2$.

Use the spreadsheet to obtain a sketch of the curve for $0 < x \leqslant 2$, $0 < y \leqslant 2$. [12]

2 The trapezium rule, using n strips of width h, is used to find an estimate T_n of the integral

$$I = \int_a^b \mathrm{f}(x)\,\mathrm{d}x,$$

where $b - a = nh$. You may assume that the global error in T_n is of the form

$$A_2 h^2 + A_4 h^4 + A_6 h^6 + \dots ,$$

where the coefficients $A_2, A_4, A_6, \dots$ are independent of n and h.

(i) Show that $T_n{}^* = \dfrac{4T_{2n} - T_n}{3}$ is an estimate of I with global error of order h^4.

Write down an expression, $T_n{}^{**}$, in terms of $T_{2n}{}^*$ and $T_n{}^*$, that represents an estimate of I with global error of order h^6. [6]

(ii) Use Romberg's method on a spreadsheet to find the value of

$$I = \int_0^2 \frac{x^2}{1 + \mathrm{e}^{-x}}\,\mathrm{d}x$$

correct to 6 decimal places. [9]

(iii) Modify your spreadsheet to find the value of

$$J = \int_0^k \frac{x^2}{1 + \mathrm{e}^{-x}}\,\mathrm{d}x$$

for $k = 0, 0.25, \dots, 2$. Hence obtain a sketch of J against k. [6]

(iv) Use your spreadsheet to determine, correct to 2 decimal places, the value of k for which $J = 1$. [3]

3 The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \sqrt{x + y}, \text{ with } y = 0 \text{ when } x = 0,$$

is to be solved numerically.

(i) Use the Runge-Kutta order 4 method with $h = 0.2$ to obtain a sketch of the solution curve for $0 < x < 3$. Give a rough estimate of the coordinates of the turning point (p, q) on the solution curve. Also give a rough estimate of α, the value of x for which the curve crosses the horizontal axis. [11]

(ii) By reducing h appropriately, obtain the values of p, q and α correct to 2 decimal places. [5]

(iii) The differential equation is now generalised to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = s - \sqrt{x + y}, \text{ with } y = 0 \text{ when } x = 0.$$

Modify your spreadsheet to find, correct to 2 decimal places, the value of s for which $\alpha = 1$. [8]

4 A curve of the form

$$y = a + bx + cx^2 \qquad (1)$$

is to be fitted, using least squares, to a set of data points (x_i, y_i), $i = 1, 2, \ldots, n$.

(i) Show, using partial differentiation, that one of the normal equations is

$$\Sigma y = na + b\,\Sigma x + c\,\Sigma x^2.$$

Write down the other two normal equations. [5]

(ii) Use a spreadsheet to obtain a scatter diagram for the following data.

x_i	0	0.5	1	1.5	2	2.5	3
y_i	1.02	2.08	2.73	3.14	2.87	2.22	1.43

What feature of the data suggests that a curve of the form (1) might be a suitable fit? [3]

(iii) Use a spreadsheet to

(*A*) formulate the normal equations,

(*B*) solve for a, b, c, using Gaussian elimination,

(*C*) find, and comment on, the sum of the residuals,

(*D*) find the residual sum of squares. [16]

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4777 Numerical Computation

1 (i) Eg: e_{r+1} is approximately ke_r **[E2]**

Uses $y_0 = \alpha + e_0$, $y_1 = \alpha + ke_0$, $y_2 = \alpha + k^2e_0$ or equivalent **[M1A1]**

Convincing algebra to eliminate k hence given result **[A1A1]**

[subtotal 6]

(ii) Convincing re-arrangment **[A1]**

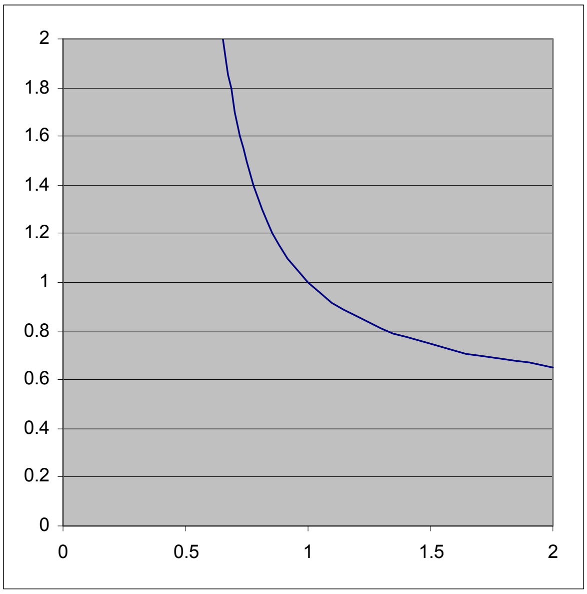
y_0	y_1	y_2	extrap (new y_o)	new y_1	new y_2	extrap		
							once	**[M1A1]**
1	0.908662	0.917409	0.916644	0.916648	0.916647	0.916647	*twice*	**[M1A1]**
					4 or 5 sf looks secure			**[A1]**

[subtotal 6]

(iii)

x	y_0	y_1	y_2	extrap (new y_o)	new y_1	new y_2	extrap	
1.1	1	0.908662	0.917409	0.916644	0.916648	0.916647	0.916647	
1.2	0.916647	0.845937	0.858962	0.856936	0.85695	0.856947	0.856948	**set up**
1.3	0.856948	0.799744	0.815042	0.811814	0.81184	0.811833	0.811835	**SS**
1.4	0.811835	0.763904	0.780556	0.776263	0.776302	0.776288	0.776292	**[M2A2]**
1.5	0.776292	0.734953	0.752555	0.747298	0.747351	0.747329	0.747335	
1.6	0.747335	0.7108	0.729213	0.723043	0.72311	0.723076	0.723087	**values**
1.7	0.723087	0.690112	0.70934	0.702258	0.70234	0.702292	0.70231	**[A3]**
1.8	0.70231	0.671996	0.692131	0.684095	0.684194	0.684128	0.684155	
1.9	0.684155	0.655831	0.677026	0.667954	0.668075	0.667985	0.668023	
2	0.668023	0.641175	0.663627	0.653402	0.65355	0.653427	0.653483	
					3 or 4 sf looks secure			**[A1]**

x	y
0.653483	2
0.668023	1.9
0.684155	1.8
0.70231	1.7
0.723087	1.6
0.747335	1.5
0.776292	1.4
0.811835	1.3
0.856948	1.2
0.916647	1.1
1	1
1.1	0.916647
1.2	0.856948
1.3	0.811835
1.4	0.776292
1.5	0.747335
1.6	0.723087
1.7	0.70231
1.8	0.684155
1.9	0.668023
2	0.653483



organise data [M1A1]

graph G2

Sub Total 12

TOTAL 24

2 (i) $T_n - I = A_2h^2 + A_4h^4 + A_6h^6 + \ldots$

$T_{2n} - I = A_2(h/2)^2 + A_4(h/2)^4 + A_6(h/2)^6 + \ldots$	**[M1A1]**
$4(T_{2n} - I) - (T_n - I) = b_4h^4 + b_6h^6 + \ldots$	**[M1]**
$4T_{2n} - T_n - 3\,I = b_4h^4 + b_6h^6 + \ldots$	**[A1]**
$(4T_{2n} - T_n)/3 - I = B_4h^4 + B_6h^6 + \ldots$	**[A1]**
($T_n{}^* = (4T_{2n} - T_n)/3$ has error of order h^4 as given)	
$T_n{}^{**} = (16T_{2n}{}^* - T_n{}^*)/15$ has error of order h^6	**[B1]**
	[subtotal 6]

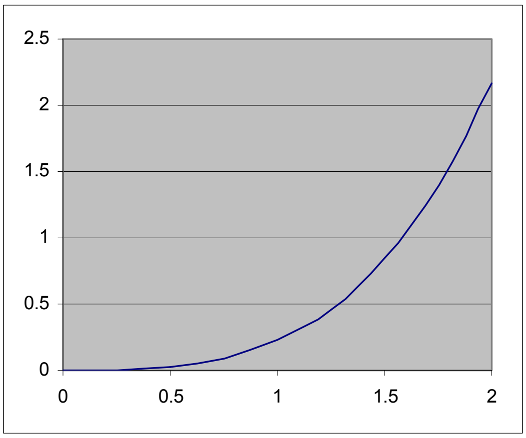
(ii)

x	f(x)	T	T*	T**	(T***)		
0	0						
2	3.523188	3.523188					
1	0.731059	2.492653	2.149141				
0.5	0.155615					f:	**[A1]**
1.5	1.839543	2.243905	2.160989	2.161779			
0.25	0.035136					*T:*	**[M1A2]**
0.75	0.382038						
1.25	1.214531					*T*:*	**[M1A1]**
1.75	2.609105	2.182155	2.161572	2.161611	2.161608		
0.125	0.0083					*T**:*	**[M1A1]**
0.375	0.083344						
0.625	0.254435					*answer:*	**[A1]**
0.875	0.540367						
1.125	0.955439						
1.375	1.509072						
1.625	2.206199						
1.875	3.048173	2.166744	2.161606	2.161609	2.161609		

[subtotal 9]

(iii)

k	I
0	0
0.25	0.002847
0.5	0.024686
0.75	0.089495
1	0.225935
1.25	0.466242
1.5	0.845007
1.75	1.398068
2	2.161609



modify SS **[M2]**

values of I **[A2]**

graph **[G2]**

[subtotal 6]

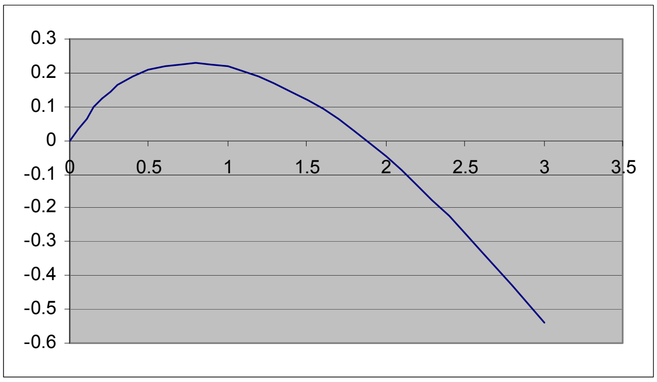
(iv)

k	I			
1.57	0.980739	accept 1.57	*evidence of t&e:*	**[M2]**
1.58	1.001291	or 1.58	*result:*	**[A1]**
1.579	0.999223	(or in between)		

[subtotal 3]
[TOTAL 24]

3 (i)

h	x	y	k 1	k 2	k 3	k 4	
0.2	0	0	0.2	0.110557	0.121189	0.086653	
0.2	0.2	0.125024	0.085978	0.063177	0.064854	0.046393	
0.2	0.4	0.189763	0.046408	0.031125	0.032033	0.018694	
0.2	0.6	0.221666	0.018708	0.007021	0.007628	-0.00291	**setup**
0.2	0.8	0.229182	-0.0029	-0.01239	-0.01194	-0.02066	**[M3]**
0.2	1	0.217146	-0.02065	-0.02863	-0.02828	-0.0357	
0.2	1.2	0.188783	-0.03569	-0.04256	-0.04228	-0.04872	**values**
0.2	1.4	0.146433	-0.04871	-0.05472	-0.05449	-0.06015	**[A3]**
0.2	1.6	0.091887	-0.06015	-0.06547	-0.06527	-0.07031	
0.2	1.8	0.026567	-0.0703	-0.07506	-0.07488	-0.07941	
0.2	2	-0.04836	-0.0794	-0.08369	-0.08353	-0.08762	
0.2	2.2	-0.13194	-0.08761	-0.0915	-0.09136	-0.09507	
0.2	2.4	-0.22334	-0.09507	-0.0986	-0.09849	-0.10187	
0.2	2.6	-0.32186	-0.10187	-0.1051	-0.105	-0.1081	
0.2	2.8	-0.42689	-0.1081	-0.11107	-0.11097	-0.11382	
0.2	3	-0.53789	-0.11382	-0.11656	-0.11647	-0.1191	



[G2]

Maximum about (0.8, 0.23) root about 1.8 **[A1A1A1]**

[subtotal11]

(ii) Eg: **[M2]**

h = 0.01 gives (p, q) as (0.77, 0.22743) hence (0.77, 0.23) **[A1A1]**

h = 0.01 gives root as between 1.87 and 1.88 accept either **[A1]**

[subtotal5]

(iii) Eg:

s	h	x	y	k 1	k 2	k 3	k 4	
1	0.01	0	0	0.01	0.009	0.009025	0.008621	
1	0.01	0.01	0.009112	0.008618	0.008314	0.008319	0.008065	
1	0.01	0.02	0.017437	0.008065	0.007844	0.007847	0.007649	**Mods**
1	0.01	0.03	0.025286	0.007649	0.007468	0.00747	0.007303	**[M3]**
1	0.01	0.04	0.032757	0.007303	0.007147	0.007148	0.007002	**t & e**
								[M3]

s = 0.715, h = 0.01 gives root closest to x = 1 accept 0.71 to 0.72 **[A2]**

[subtotal8]

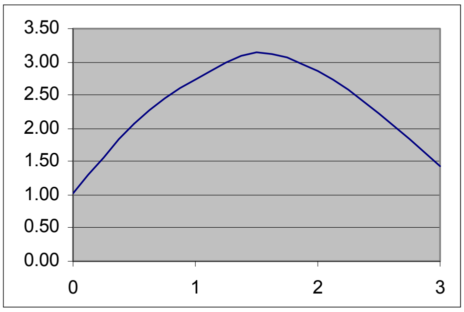
[TOTAL 24]

4 (i) $Q = \Sigma (y - a - bx - cx^2)^2$ **[M1]**

dQ/da = 0 gives	$\Sigma y =$	$na + b \Sigma x + c \Sigma x^2$	as given	**[M1A1]**
other equations:	$\Sigma xy =$	$a \Sigma x + b \Sigma x^2 + c \Sigma x^3$		**[B1]**
	$\Sigma x^2y =$	$a \Sigma x^2 + b \Sigma x^3 + c \Sigma x^4$		**[B1]**
				[subtotal 5]

(ii)

X	Y
0	1.02
0.5	2.08
1	2.73
1.5	3.14
2	2.87
2.5	2.22
3	1.43



[G2]

roughly parabolic (quadratic) in shape **[E1]**

[subtotal 3]

(iii)

x	y	xy	x^2y	x^2	x^3	x^4	
0	1.02	0	0	0	0	0	
0.5	2.08	1.04	0.52	0.25	0.125	0.0625	
1	2.73	2.73	2.73	1	1	1	
1.5	3.14	4.71	7.065	2.25	3.375	5.0625	
2	2.87	5.74	11.48	4	8	16	
2.5	2.22	5.55	13.875	6.25	15.625	39.0625	
3	1.43	4.29	12.87	9	27	81	**[M2]**
10.5	**15.49**	**24.06**	**48.54**	**22.75**	**55.125**	**142.1875**	**[A2]**

normal equations:

7	10.5	22.75	15.49			**form equations**
10.5	22.75	55.125	24.06			**[M1A1]**
$\boxed{22.75}$	55.125	142.1875	48.54	a=	1.017619	**solution**
	$\boxed{-6.46154}$	-21	0.554615	b=	2.562143	**[M2A2]**
	-2.69231	-10.5	1.656923			
		$\boxed{-1.75}$	1.425833	c=	-0.81476	

x	y	y fitted	residual	res^2	
0	1.02	1.017619	0.002381	5.67E-06	
0.5	2.08	2.095	-0.015	0.000225	
1	2.73	2.765	-0.035	0.001225	**y fitted**
1.5	3.14	3.027619	0.112381	0.012629	**[M1A1]**
2	2.87	2.882857	-0.01286	0.000165	
2.5	2.22	2.330714	-0.11071	0.012258	**residuals**
3	1.43	1.37119	0.05881	0.003459	**[M1A1]**
			-3.6E-15	**0.029967**	

residual sum is zero (except for rounding errors) as it should be **[E1]**

residual sum of squares is 0.029967 **[A1]**

[subtotal 16]

[TOTAL 24]