RECOGNISING ACHIEVEMENT

ADVANCED GCE **4764/01**

MATHEMATICS (MEI)

Mechanics 4

WEDNESDAY 18 JUNE 2008 Morning

Time: 1 hour 30 minutes

Additional materials (enclosed): None

Additional materials (required):
Answer Booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is **72**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages.

Section A (24 marks)

1 A rocket in deep space starts from rest and moves in a straight line. The initial mass of the rocket is m_0 and the propulsion system ejects matter at a constant mass rate k with constant speed u relative to the rocket. At time t the speed of the rocket is v.

(i) Show that while mass is being ejected from the rocket, $(m_0 - kt)\dfrac{dv}{dt} = uk$. [5]

(ii) Hence find an expression for v in terms of t. [4]

(iii) Find the speed of the rocket when its mass is $\frac{1}{3}m_0$. [3]

2 A car of mass m kg starts from rest at a point O and moves along a straight horizontal road. The resultant force in the direction of motion has power P watts, given by $P = m(k^2 - v^2)$, where $v\,\text{m s}^{-1}$ is the velocity of the car and k is a positive constant. The displacement from O in the direction of motion is x m.

(i) Show that $\left(\dfrac{k^2}{k^2 - v^2} - 1\right)\dfrac{dv}{dx} = 1$, and hence find x in terms of v and k. [9]

(ii) How far does the car travel before reaching 90% of its terminal velocity? [3]

Section B (48 marks)

3 A circular disc of radius a m has mass per unit area $\rho\,\text{kg m}^{-2}$ given by $\rho = k(a + r)$, where r m is the distance from the centre and k is a positive constant. The disc can rotate freely about an axis perpendicular to it and through its centre.

(i) Show that the mass, M kg, of the disc is given by $M = \frac{5}{3}k\pi a^3$, and show that the moment of inertia, $I\,\text{kg m}^2$, about this axis is given by $I = \frac{27}{50}Ma^2$. [9]

For the rest of this question, take $M = 64$ and $a = 0.625$.

The disc is at rest when it is given a tangential impulsive blow of 50 N s at a point on its circumference.

(ii) Find the angular speed of the disc. [4]

The disc is then accelerated by a constant couple reaching an angular speed of $30\,\text{rad s}^{-1}$ in 20 seconds.

(iii) Calculate the magnitude of this couple. [3]

When the angular speed is $30\,\text{rad s}^{-1}$, the couple is removed and brakes are applied to bring the disc to rest. The effect of the brakes is modelled by a resistive couple of $3\dot{\theta}$ N m, where $\dot{\theta}$ is the angular speed of the disc in rad s^{-1}.

(iv) Formulate a differential equation for $\dot{\theta}$ and hence find $\dot{\theta}$ in terms of t, the time in seconds from when the brakes are first applied. [7]

(v) By reference to your expression for $\dot{\theta}$, give a brief criticism of this model for the effect of the brakes. [1]

4 A uniform smooth pulley can rotate freely about its axis, which is fixed and horizontal. A light elastic string AB is attached to the pulley at the end B. The end A is attached to a fixed point such that the string is vertical and is initially at its natural length with B at the same horizontal level as the axis. In this position a particle P is attached to the highest point of the pulley. This initial position is shown in Fig. 4.1.

The radius of the pulley is a, the mass of P is m and the stiffness of the string AB is $\frac{mg}{10a}$.

Fig. 4.1 **Fig. 4.2**

(i) Fig. 4.2 shows the system with the pulley rotated through an angle θ and the string stretched. Write down the extension of the string and hence find the potential energy, V, of the system in this position. Show that $\frac{dV}{d\theta} = mga\left(\frac{1}{10}\theta - \sin\theta\right)$. [6]

(ii) Hence deduce that the system has a position of unstable equilibrium at $\theta = 0$. [6]

(iii) Explain how your expression for V relies on smooth contact between the string and the pulley. [2]

Fig. 4.3 shows the graph of the function $f(\theta) = \frac{1}{10}\theta - \sin\theta$.

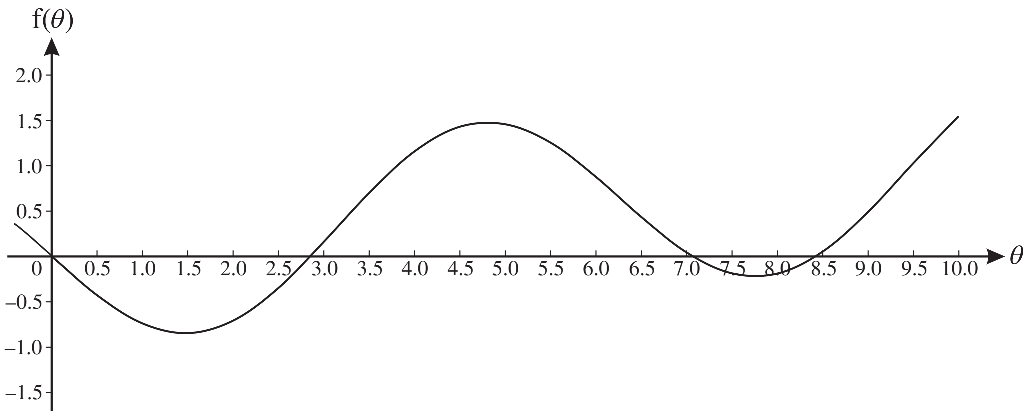

Fig. 4.3

(iv) Use the graph to give rough estimates of three further values of θ (other than $\theta = 0$) which give positions of equilibrium. In each case, state with reasons whether the equilibrium is stable or unstable. [6]

(v) Show on a sketch the physical situation corresponding to the least value of θ you identified in part **(iv)**. On your sketch, mark clearly the positions of P and B. [2]

(vi) The equation $f(\theta) = 0$ has another root at $\theta \approx -2.9$. Explain, with justification, whether this necessarily gives a position of equilibrium. [2]

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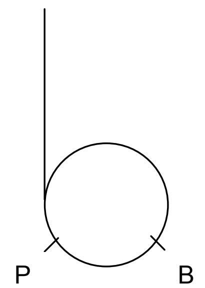
1(i)	If δm is change in mass over time δt PCLM $mv = (m+\delta m)(v+\delta v) + \lvert\delta m\rvert(v-u)$ [N.B. $\delta m < 0$]	M1	Change in momentum over time δt	
	$(m+\delta m)\frac{\delta v}{\delta t} + u\frac{\delta m}{\delta t} = 0 \Rightarrow m\frac{\mathrm{d}v}{\mathrm{d}t} = -u\frac{\mathrm{d}m}{\mathrm{d}t}$	M1 A1	Rearrange to produce DE Accept sign error	
	$\frac{\mathrm{d}m}{\mathrm{d}t} = -k \Rightarrow m = m_0 - kt$	M1	Find *m* in terms of *t*	
	$\Rightarrow (m_0 - kt)\frac{\mathrm{d}v}{\mathrm{d}t} = uk$	E1	Convincingly shown	
				5
(ii)	$v = \int \frac{uk}{m_0 - kt}\mathrm{d}t$	M1	Separate and integrate	
	$= -u\ln(m_0 - kt) + c$	A1	cao (allow no constant)	
	$t = 0, v = 0 \Rightarrow c = u\ln m_0$	M1	Use initial condition	
	$v = u\ln\left(\frac{m_0}{m_0 - kt}\right)$	A1	All correct	
				4
(iii)	$m = \frac{1}{3}m_0 \Rightarrow m_0 - kt = \frac{1}{3}m_0$	M1 A1	Find expression for mass or time Or $t = 2m_0/3k$	
	$\Rightarrow v = u\ln 3$	A1		
				3

2(i)	$P = Fv$	M1	Used, not just quoted	
	$= mv\frac{\mathrm{d}v}{\mathrm{d}x}v$	M1	Use N2L and expression for acceleration	
	$\Rightarrow mv^2\frac{\mathrm{d}v}{\mathrm{d}x} = m\left(k^2 - v^2\right)$	A1	Correct DE	
	$\Rightarrow \frac{v^2}{k^2 - v^2}\frac{\mathrm{d}v}{\mathrm{d}x} = 1$	M1	Rearrange	
	$\Rightarrow \left(\frac{k^2}{k^2 - v^2} - 1\right)\frac{\mathrm{d}v}{\mathrm{d}x} = 1$	E1	Convincingly shown	
	$\int\left(\frac{k^2}{k^2 - v^2} - 1\right)\mathrm{d}v = \int \mathrm{d}x$	M1	Separate and integrate	
	$\frac{1}{2}k\ln\left(\frac{k+v}{k-v}\right) - v = x + c$	A1	LHS	
	$x = 0, v = 0 \Rightarrow c = 0$	M1	Use condition	
	$x = \frac{1}{2}k\ln\left(\frac{k+v}{k-v}\right) - v$	A1	cao	
				9
(ii)	Terminal velocity when acceleration zero $\Rightarrow v = k$	M1 A1		
	$v = 0.9k \Rightarrow x = \frac{1}{2}k\ln\left(\frac{1.9}{0.1}\right) - 0.9k = \left(\frac{1}{2}\ln 19 - 0.9\right)k \approx$ 0.572*k*	F1	Follow their solution to (i)	
				3

3(i)	$M=\int_0^a k(a+r)2\pi r\,\mathrm{d}r$	M1 M1	Use circular elements (for *M* or *I*) Integral for mass	
	$=2k\pi\left[\frac{1}{2}ar^2+\frac{1}{3}r^3\right]_0^a$	M1 A1	Integrate (for *M* or *I*) For […]	
	$=\frac{5}{3}k\pi a^3$	E1		
	$I=\int_0^a k(a+r)2\pi r\cdot r^2\,\mathrm{d}r$	M1	Integral for *I*	
	$=2k\pi\left[\frac{1}{4}ar^4+\frac{1}{5}r^5\right]_0^a$	A1	For […]	
	$=\frac{9}{10}k\pi a^5$	A1	cao	
	$=\frac{27}{50}Ma^2$	E1	Complete argument (including mass)	9
(ii)	$I=13.5$	B1	Seen or used (here or later)	
	$0.625\times 50=I\omega$	M1 M1	Use angular momentum Use moment of impulse	
	$\Rightarrow \omega\approx 2.31$	A1	cao	4
(iii)	$\ddot{\theta}=\frac{30-2.31}{20}\approx 1.38$	M1	Find angular acceleration	
	Couple $=I\ddot{\theta}$	M1	Use equation of motion	
	≈ 18.7	F1	Follow their ω and *I*	3
(iv)	$I\ddot{\theta}=-3\dot{\theta}$	B1	Allow sign error and follow their *I* (but not *M*)	
	$I\frac{\mathrm{d}\dot{\theta}}{\mathrm{d}t}=-3\dot{\theta}$	M1	Set up DE for $\dot{\theta}$ (first order)	
	$\int\frac{\mathrm{d}\dot{\theta}}{\dot{\theta}}=\int-\frac{3}{I}\mathrm{d}t$	M1	Separate and integrate	
	$\ln\lvert\dot{\theta}\rvert=-\frac{t}{4.5}+c$	B1	$\ln(\text{multiple of }\dot{\theta})$ seen	
	$\dot{\theta}=A\mathrm{e}^{-t/4.5}$	M1	Rearrange, dealing properly with constant	
	$t=0,\dot{\theta}=30\Rightarrow A=30$	M1	Use condition on $\dot{\theta}$	
	$\dot{\theta}=30\,\mathrm{e}^{-t/4.5}$	A1		7
(v)	Model predicts $\dot{\theta}$ never zero in finite time.	B1		1

4(i)	$V=\frac{1}{2}\left(\frac{mg}{10a}\right)(a\theta)^2+mga\cos\theta$ (relative to centre of pulley)	M1	EPE term	
		B1	Extension $=a\theta$	
		M1	GPE relative to any zero level	
		A1	(± constant)	
	$\frac{dV}{d\theta}=\frac{1}{2}\left(\frac{mg}{10a}\right)\cdot 2a^2\theta-mga\sin\theta$	M1	Differentiate	
	$\frac{dV}{d\theta}=mga\left(\frac{1}{10}\theta-\sin\theta\right)$	E1		6
(ii)	$\theta=0\Rightarrow\frac{dV}{d\theta}=mga\left(\frac{1}{10}(0)-\sin 0\right)=0$	M1	Consider value of $\frac{dV}{d\theta}$	
	hence equilibrium	E1		
	$\frac{d^2V}{d\theta^2}=mga\left(\frac{1}{10}-\cos\theta\right)$	M1	Differentiate again	
		A1		
	$V''(0)=-0.9mga<0$	M1	Consider sign of V''	
	hence unstable	E1	V'' must be correct	6
(iii)	If the pulley is smooth, then the tension in the string is constant.	B1		
	Hence the EPE term is valid.	B1		2
(iv)	Equilibrium positions at $\theta=2.8$,	B1	One correct	
	$\theta=7.1$ and $\theta=8.4$	B1	All three correct, no extras Accept answers in [2.7,3.0), [7,7.2], [8.3,8.5]	
	From graph, $V''(2.8)=mga\mathrm{f}'(2.8)>0$	M1	Consider sign of V'' or f'	
	hence stable at $\theta=2.8$	A1	Accept no reference to V'' for one conclusion but other two must relate to sign of V'', not just f'.	
	$V''(7.1)=mga\mathrm{f}'(7.1)<0\Rightarrow$ unstable at $\theta=7.1$	A1		
	$V''(8.4)=mga\mathrm{f}'(8.4)>0\Rightarrow$ stable at $\theta=8.4$	A1		6
(v)		B1	P in approximately correct place	
		B1	B in approximately correct place	2
(vi)	If $\theta<0$ then expression for EPE not valid	M1		
	hence not necessarily an equilibrium position.	A1		2