OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE **4756/01**

MATHEMATICS (MEI)

Further Methods for Advanced Mathematics (FP2)

THURSDAY 15 MAY 2008 Morning

Time: 1 hour 30 minutes

Additional materials (enclosed): None

Additional materials (required):
Answer Booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer **all** the questions in Section A and **one** question from Section B.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is **72**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages.

Section A (54 marks)

Answer all the questions

1 **(a)** A curve has cartesian equation $(x^2 + y^2)^2 = 3xy^2$.

(i) Show that the polar equation of the curve is $r = 3\cos\theta\sin^2\theta$. [3]

(ii) Hence sketch the curve. [3]

(b) Find the exact value of $\displaystyle\int_0^1 \frac{1}{\sqrt{4-3x^2}}\,dx$. [5]

(c) **(i)** Write down the series for $\ln(1+x)$ and the series for $\ln(1-x)$, both as far as the term in x^5. [2]

(ii) Hence find the first three non-zero terms in the series for $\ln\left(\frac{1+x}{1-x}\right)$. [2]

(iii) Use the series in part **(ii)** to show that $\displaystyle\sum_{r=0}^{\infty} \frac{1}{(2r+1)4^r} = \ln 3$. [3]

2 You are given the complex numbers $z = \sqrt{32}(1 + j)$ and $w = 8\left(\cos\frac{7}{12}\pi + j\sin\frac{7}{12}\pi\right)$.

(i) Find the modulus and argument of each of the complex numbers z, z^*, zw and $\frac{z}{w}$. [7]

(ii) Express $\frac{z}{w}$ in the form $a + bj$, giving the exact values of a and b. [2]

(iii) Find the cube roots of z, in the form $re^{j\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. [4]

(iv) Show that the cube roots of z can be written as

$$k_1 w^*, \quad k_2 z^* \quad \text{and} \quad k_3 jw,$$

where k_1, k_2 and k_3 are real numbers. State the values of k_1, k_2 and k_3. [5]

3 **(i)** Given the matrix $\mathbf{Q} = \begin{pmatrix} 2 & -1 & k \\ 1 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix}$ (where $k \neq 3$), find $\mathbf{Q}^{-1}$ in terms of k.

Show that, when $k = 4$, $\mathbf{Q}^{-1} = \begin{pmatrix} -1 & 6 & -1 \\ 1 & -8 & 2 \\ 1 & -5 & 1 \end{pmatrix}$. [6]

The matrix $\mathbf{M}$ has eigenvectors $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$, with corresponding eigenvalues 1, −1 and 3 respectively.

(ii) Write down a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{MP} = \mathbf{D}$, and hence find the matrix $\mathbf{M}$. [7]

(iii) Write down the characteristic equation for $\mathbf{M}$, and use the Cayley-Hamilton theorem to find integers a, b and c such that $\mathbf{M}^4 = a\mathbf{M}^2 + b\mathbf{M} + c\mathbf{I}$. [5]

Section B (18 marks)

Answer one question

Option 1: Hyperbolic functions

4 **(i)** Starting from the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials, prove that

$$\cosh^2 x - \sinh^2 x = 1.$$ [3]

(ii) Solve the equation $4\cosh^2 x + 9\sinh x = 13$, giving the answers in exact logarithmic form. [6]

(iii) Show that there is only one stationary point on the curve

$$y = 4\cosh^2 x + 9\sinh x,$$

and find the y-coordinate of the stationary point. [4]

(iv) Show that $\int_0^{\ln 2} (4\cosh^2 x + 9\sinh x)\,dx = 2\ln 2 + \frac{33}{8}$. [5]

[Question 5 is printed overleaf.]

Option 2: Investigation of curves

This question requires the use of a graphical calculator.

5 A curve has parametric equations $x = \lambda \cos\theta - \frac{1}{\lambda}\sin\theta$, $y = \cos\theta + \sin\theta$, where λ is a positive constant.

(i) Use your calculator to obtain a sketch of the curve in each of the cases

$$\lambda = 0.5, \quad \lambda = 3 \quad \text{and} \quad \lambda = 5.$$ [3]

(ii) Given that the curve is a conic, name the type of conic. [1]

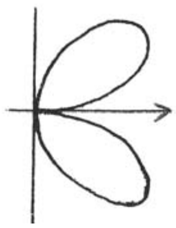
(iii) Show that y has a maximum value of $\sqrt{2}$ when $\theta = \frac{1}{4}\pi$. [2]

(iv) Show that $x^2 + y^2 = (1 + \lambda^2) + \left(\frac{1}{\lambda^2} - \lambda^2\right)\sin^2\theta$, and deduce that the distance from the origin of any point on the curve is between $\sqrt{1 + \frac{1}{\lambda^2}}$ and $\sqrt{1 + \lambda^2}$. [6]

(v) For the case $\lambda = 1$, show that the curve is a circle, and find its radius. [2]

(vi) For the case $\lambda = 2$, draw a sketch of the curve, and label the points A, B, C, D, E, F, G, H on the curve corresponding to $\theta = 0, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{3}{4}\pi, \pi, \frac{5}{4}\pi, \frac{3}{2}\pi, \frac{7}{4}\pi$ respectively. You should make clear what is special about each of these points. [4]

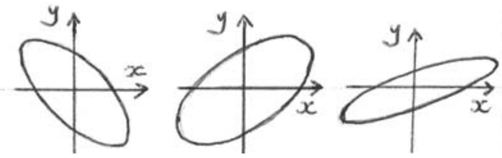
4756 (FP2) Further Methods for Advanced Mathematics

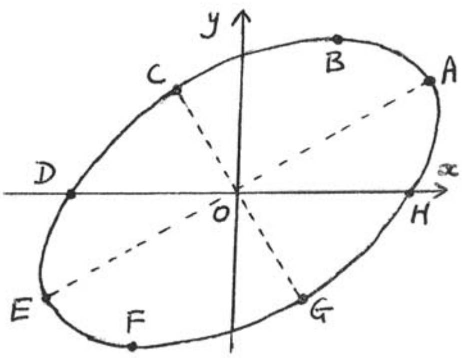
1(a)(i)	$x = r\cos\theta,\ \ y = r\sin\theta$ $(r^2\cos^2\theta + r^2\sin^2\theta)^2 = 3(r\cos\theta)(r\sin\theta)^2$ $r^4 = 3r^3\cos\theta\sin^2\theta$ $r = 3\cos\theta\sin^2\theta$	M1 A1 A1 ag **3**	(M0 for $x = \cos\theta,\ \ y = \sin\theta$)
(ii)		B1 B1 B1 **3**	Loop in 1st quadrant Loop in 4th quadrant Fully correct curve *Curve may be drawn using continuous or broken lines in any combination*
(b)	$\int_0^1 \frac{1}{\sqrt{4-3x^2}}\,dx = \left[\frac{1}{\sqrt{3}}\arcsin\frac{\sqrt{3}\,x}{2}\right]_0^1$ $= \frac{1}{\sqrt{3}}\arcsin\frac{\sqrt{3}}{2}$ $= \frac{\pi}{3\sqrt{3}}$	M1 A1A1 M1 A1 **5**	For arcsin For $\frac{1}{\sqrt{3}}$ and $\frac{\sqrt{3}\,x}{2}$ Exact numerical value *Dependent on first M1* (M1A0 for $60/\sqrt{3}$)
	OR M1 Put $\sqrt{3}\,x = 2\sin\theta$ A1 $\int_0^1 \frac{1}{\sqrt{4-3x^2}}\,dx = \int_0^{\frac{\pi}{3}} \frac{1}{\sqrt{3}}\,d\theta$ A1 $= \frac{\pi}{3\sqrt{3}}$ M1A1		Any sine substitution For $\int \frac{1}{\sqrt{3}}\,d\theta$ *M1 dependent on first M1*
(c)(i)	$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \ldots$ $\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 - \ldots$	B1 B1 **2**	*Accept unsimplified forms*
(ii)	$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$ $= 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \ldots$	M1 A1 **2**	Obtained from two correct series *Terms need not be added* If M0, then B1 for $2x + \frac{2}{3}x^3 + \frac{2}{5}x^5$

(iii)	$\sum_{r=0}^{\infty}\frac{1}{(2r+1)4^r}=1+\frac{1}{3\times4}+\frac{1}{5\times4^2}+\ldots$ $=2\times\frac{1}{2}+\frac{2}{3}\times(\frac{1}{2})^3+\frac{2}{5}\times(\frac{1}{2})^5+\ldots$ $=\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right)=\ln 3$	B1 B1 B1 ag **3**	*Terms need not be added* For $x=\frac{1}{2}$ seen or implied Satisfactory completion
2 (i)	$\lvert z\rvert=8,\quad \arg z=\frac{1}{4}\pi$ $\lvert z^*\rvert=8,\quad \arg z^*=-\frac{1}{4}\pi$ $\lvert zw\rvert=8\times8=64$ $\arg(zw)=\frac{1}{4}\pi+\frac{7}{12}\pi=\frac{5}{6}\pi$ $\left\lvert\frac{z}{w}\right\rvert=\frac{8}{8}=1$ $\arg(\frac{z}{w})=\frac{1}{4}\pi-\frac{7}{12}\pi=-\frac{1}{3}\pi$	B1B1 B1 ft B1 ft B1 ft B1 ft B1 ft **7**	*Must be given separately* *Remainder may be given in exponential or $r\,\mathrm{cjs}\,\theta$ form* (B0 for $\frac{7}{4}\pi$) (B0 if left as $8/8$)
(ii)	$\frac{z}{w}=\cos(-\frac{1}{3}\pi)+\mathrm{j}\sin(-\frac{1}{3}\pi)$ $=\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{j}$ $a=\frac{1}{2},\quad b=-\frac{1}{2}\sqrt{3}$	M1 A1 **2**	If M0, then B1B1 for $\frac{1}{2}$ and $-\frac{\sqrt{3}}{2}$
(iii)	$r=\sqrt[3]{8}=2$ $\theta=\frac{1}{12}\pi$ $\theta=\frac{\pi}{12}+\frac{2k\pi}{3}$ $\theta=-\frac{7}{12}\pi,\quad\frac{3}{4}\pi$	B1 ft B1 M1 A1 **4**	Accept $\sqrt[3]{8}$ Implied by one further correct (ft) value *Ignore values outside the required range*
(iv)	$w^*=8\mathrm{e}^{-\frac{7}{12}\pi\mathrm{j}}$, so $2\mathrm{e}^{-\frac{7}{12}\pi\mathrm{j}}=\frac{1}{4}w^*$ $k_1=\frac{1}{4}$ $z^*=8\mathrm{e}^{-\frac{1}{4}\pi\mathrm{j}}=-8\mathrm{e}^{\frac{3}{4}\pi\mathrm{j}}$ So $2\mathrm{e}^{\frac{3}{4}\pi\mathrm{j}}=-\frac{1}{4}z^*$ $k_2=-\frac{1}{4}$ $\mathrm{j}w=8\mathrm{e}^{(\frac{1}{2}\pi+\frac{7}{12}\pi)\mathrm{j}}=8\mathrm{e}^{\frac{13}{12}\pi\mathrm{j}}$ $=-8\mathrm{e}^{\frac{1}{12}\pi\mathrm{j}}$, so $2\mathrm{e}^{\frac{1}{12}\pi\mathrm{j}}=-\frac{1}{4}\mathrm{j}w$ $k_3=-\frac{1}{4}$	B1 ft M1 A1 ft M1 A1 ft **5**	Matching *w** to a cube root with argument $-\frac{7}{12}\pi$ and $k_1=\frac{1}{4}$ or ft ft is $\frac{r}{8}$ Matching *z** to a cube root with argument $\frac{3}{4}\pi$ *May be implied* ft is $-\frac{r}{\lvert z^*\rvert}$ Matching j*w* to a cube root with argument $\frac{1}{12}\pi$ *May be implied* OR M1 for $\arg(\mathrm{j}w)=\frac{1}{2}\pi+\arg w$ *(implied by $\frac{13}{12}\pi$ or $-\frac{11}{12}\pi$)* ft is $-\frac{r}{8}$

3 (i)	$\mathbf{Q}^{-1} = \dfrac{1}{k-3}\begin{pmatrix} -1 & k+2 & -1 \\ 1 & 4-3k & k-2 \\ 1 & -5 & 1 \end{pmatrix}$ When $k=4$, $\mathbf{Q}^{-1} = \begin{pmatrix} -1 & 6 & -1 \\ 1 & -8 & 2 \\ 1 & -5 & 1 \end{pmatrix}$	M1 A1 M1 A1 M1 A1 **6**	Evaluation of determinant *(must involve k)* For $(k-3)$ Finding at least four cofactors *(including one involving k)* Six signed cofactors correct *(including one involving k)* Transposing and dividing by det *Dependent on previous M1M1* $\mathbf{Q}^{-1}$ correct (in terms of *k*) and result for $k=4$ stated After 0, SC1 for $\mathbf{Q}^{-1}$ when $k=4$ obtained correctly with some working
(ii)	$\mathbf{P} = \begin{pmatrix} 2 & -1 & 4 \\ 1 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ $\mathbf{M} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ $= \begin{pmatrix} 2 & -1 & 4 \\ 1 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} -1 & 6 & -1 \\ 1 & -8 & 2 \\ 1 & -5 & 1 \end{pmatrix}$ $= \begin{pmatrix} 2 & 1 & 12 \\ 1 & 0 & 3 \\ 3 & -1 & 6 \end{pmatrix}\begin{pmatrix} -1 & 6 & -1 \\ 1 & -8 & 2 \\ 1 & -5 & 1 \end{pmatrix}$ $= \begin{pmatrix} 11 & -56 & 12 \\ 2 & -9 & 2 \\ 2 & -4 & 1 \end{pmatrix}$	B1B1 B2 M1 A2 **7**	For B2, order must be consistent Give B1 for $\mathbf{M} = \mathbf{P}^{-1}\mathbf{D}\mathbf{P}$ or $\begin{pmatrix} 2 & -1 & 4 \\ 1 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 6 & -1 \\ -1 & 8 & -2 \\ 3 & -15 & 3 \end{pmatrix}$ Good attempt at multiplying two matrices (no more than 3 errors), leaving third matrix in correct position Give A1 for five elements correct Correct **M** implies B2M1A2 5-8 elements correct implies B2M1A1
(iii)	Characteristic equation is $(\lambda-1)(\lambda+1)(\lambda-3)=0$ $\lambda^3 - 3\lambda^2 - \lambda + 3 = 0$ $\mathbf{M}^3 = 3\mathbf{M}^2 + \mathbf{M} - 3\mathbf{I}$ $\mathbf{M}^4 = 3\mathbf{M}^3 + \mathbf{M}^2 - 3\mathbf{M}$ $= 3(3\mathbf{M}^2 + \mathbf{M} - 3\mathbf{I}) + \mathbf{M}^2 - 3\mathbf{M}$ $= 10\mathbf{M}^2 - 9\mathbf{I}$ $a=10,\ b=0,\ c=-9$	B1 M1 A1 M1 A1 **5**	In any correct form *(Condone omission of =0)* **M** satisfies the characteristic equation Correct expanded form *(Condone omission of* **I** *)*

4 (i)	$\cosh^2 x = [\frac{1}{2}(e^x + e^{-x})]^2 = \frac{1}{4}(e^{2x} + 2 + e^{-2x})$ $\sinh^2 x = [\frac{1}{2}(e^x - e^{-x})]^2 = \frac{1}{4}(e^{2x} - 2 + e^{-2x})$ $\cosh^2 x - \sinh^2 x = \frac{1}{4}(2+2) = 1$	B1 B1 B1 ag **3**	For completion
	OR $\cosh x + \sinh x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x}) = e^x$ B1 $\cosh x - \sinh x = \frac{1}{2}(e^x + e^{-x}) - \frac{1}{2}(e^x - e^{-x}) = e^{-x}$ B1 $\cosh^2 x - \sinh^2 x = e^x \times e^{-x} = 1$ B1		Completion
(ii)	$4(1+\sinh^2 x) + 9\sinh x = 13$ $4\sinh^2 x + 9\sinh x - 9 = 0$ $\sinh x = \frac{3}{4},\ -3$ $x = \ln 2,\ \ln(-3+\sqrt{10})$	M1 M1 A1A1 A1A1 ft **6**	(M0 for $1 - \sinh^2 x$) Obtaining a value for $\sinh x$ Exact logarithmic form *Dep on M1M1* Max A1 if any extra values given
	OR $2e^{4x} + 9e^{3x} - 22e^{2x} - 9e^x + 2 = 0$ $(2e^{2x} - 3e^x - 2)(e^{2x} + 6e^x - 1) = 0$ M1 M1 $e^x = 2,\ -3+\sqrt{10}$ A1A1 $x = \ln 2,\ \ln(-3+\sqrt{10})$ A1A1 ft		Quadratic and / or linear factors Obtaining a value for e^x Ignore extra values *Dependent on M1M1* Max A1 if any extra values given *Just* $x = \ln 2$ *earns M0M1A1A0A0A0*
(iii)	$\frac{dy}{dx} = 8\cosh x \sinh x + 9\cosh x$ $= \cosh x(8\sinh x + 9)$ $= 0$ only when $\sinh x = -\frac{9}{8}$ $\cosh^2 x = 1 + (-\frac{9}{8})^2 = \frac{145}{64}$ $y = 4\times\frac{145}{64} + 9\times(-\frac{9}{8}) = -\frac{17}{16}$	B1 B1 M1 A1 **4**	Any correct form *or* $y = (2\sinh x + \frac{9}{4})^2 + \ldots\ (-\frac{17}{16})$ Correctly showing there is only one solution Exact evaluation of y or $\cosh^2 x$ or $\cosh 2x$ Give B2 (replacing M1A1) for -1.06 or better
(iv)	$\int_0^{\ln 2} (2 + 2\cosh 2x + 9\sinh x)\,dx$ $= [\ 2x + \sinh 2x + 9\cosh x\ ]_0^{\ln 2}$ $= \left\{ 2\ln 2 + \frac{1}{2}\left(4 - \frac{1}{4}\right) + \frac{9}{2}\left(2 + \frac{1}{2}\right) \right\} - 9$ $= 2\ln 2 + \frac{33}{8}$	M1 A2 M1 A1 ag **5**	Expressing in integrable form Give A1 for two terms correct $\sinh(2\ln 2) = \frac{1}{2}(4 - \frac{1}{4})$ *Must see both terms for M1* *Must also see* $\cosh(\ln 2) = \frac{1}{2}(2 + \frac{1}{2})$ *for A1*

	OR $\int_0^{\ln 2} (e^{2x}+2+e^{-2x}+\frac{9}{2}(e^x-e^{-x}))\,dx$ M1 $=\left[\frac{1}{2}e^{2x}+2x-\frac{1}{2}e^{-2x}+\frac{9}{2}e^x+\frac{9}{2}e^{-x}\right]_0^{\ln 2}$ A2 $=\left(2+2\ln 2-\frac{1}{8}+9+\frac{9}{4}\right)-\left(\frac{1}{2}-\frac{1}{2}+\frac{9}{2}+\frac{9}{2}\right)$ M1 $=2\ln 2+\frac{33}{8}$ A1 ag		Expanded exponential form (M0 if the 2 is omitted) Give A1 for three terms correct $e^{2\ln 2}=4$ and $e^{-2\ln 2}=\frac{1}{4}$ *both seen* *Must also see* $e^{\ln 2}=2$ and $e^{-\ln 2}=\frac{1}{2}$ *for A1*
5 (i)	$\lambda=0.5$ $\lambda=3$ $\lambda=5$ 	B1B1B1 **3**	
(ii)	Ellipse	B1 **1**	
(iii)	$y=\sqrt{2}\cos(\theta-\frac{1}{4}\pi)$ Maximum $y=\sqrt{2}$ when $\theta=\frac{1}{4}\pi$	M1 A1 ag **2**	or $\sqrt{2}\sin(\theta+\frac{1}{4}\pi)$
	OR $\frac{dy}{d\theta}=-\sin\theta+\cos\theta=0$ when $\theta=\frac{1}{4}\pi$ M1 $y=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\sqrt{2}$ A1		
(iv)	$x^2+y^2=\lambda^2\cos^2\theta-2\cos\theta\sin\theta+\frac{1}{\lambda^2}\sin^2\theta$ $+\cos^2\theta+2\cos\theta\sin\theta+\sin^2\theta$ $=(\lambda^2+1)(1-\sin^2\theta)+(\frac{1}{\lambda^2}+1)\sin^2\theta$ $=1+\lambda^2+(\frac{1}{\lambda^2}-\lambda^2)\sin^2\theta$ When $\sin^2\theta=0$, $x^2+y^2=1+\lambda^2$ When $\sin^2\theta=1$, $x^2+y^2=1+\frac{1}{\lambda^2}$ Since $0\le\sin^2\theta\le 1$, distance from O, $\sqrt{x^2+y^2}$, is between $\sqrt{1+\frac{1}{\lambda^2}}$ and $\sqrt{1+\lambda^2}$	M1 M1 A1 ag M1 M1 A1 ag **6**	Using $\cos^2\theta=1-\sin^2\theta$
(v)	When $\lambda=1$, $x^2+y^2=2$ Curve is a circle (centre O) with radius $\sqrt{2}$	M1 A1 **2**	

(vi)		B4	**4**	A, E at maximum distance from O C, G at minimum distance from O B, F are stationary points D, H are on the *x*-axis Give ½ mark for each point, then round down Special properties must be clear from diagram, or stated *Max 3 if curve is not the correct shape*