OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE **4758/01**

MATHEMATICS (MEI)

Differential Equations

THURSDAY 24 JANUARY 2008 Morning

Time: 1 hour 30 minutes

Additional materials (enclosed): None

Additional materials (required):
Answer Booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is **72**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages.

1 The differential equation $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + y = f(t)$ is to be solved for $t \geqslant 0$ subject to the conditions that $\dfrac{dy}{dt} = 0$ and $y = 0$ when $t = 0$.

Firstly consider the case $f(t) = 2$.

(i) Find the solution for y in terms of t. [10]

Now consider the case $f(t) = e^{-t}$.

(ii) Explain briefly why a particular integral cannot be of the form ae^{-t} or ate^{-t}. Find a particular integral and hence solve the differential equation, subject to the given conditions. [8]

(iii) For $t > 0$, show that $y > 0$ and find the maximum value of y. Hence sketch the solution for $t \geqslant 0$. [You may assume that $t^k e^{-t} \to 0$ as $t \to \infty$ for any k.] [6]

2 A raindrop falls from rest through mist. Its velocity, $v\,\mathrm{m\,s^{-1}}$ vertically downwards, at time t seconds after it starts to fall is modelled by the differential equation

$$(1+t)\frac{dv}{dt} + 3v = (1+t)g - 3.$$

(i) Solve the differential equation to show that $v = \frac{1}{4}g(1+t) - 1 + (1 - \frac{1}{4}g)(1+t)^{-3}$. [10]

The model is refined and the term -3 is replaced by the term $-2v$, giving the differential equation

$$(1+t)\frac{dv}{dt} + 3v = (1+t)g - 2v.$$

(ii) Find the solution subject to the same initial conditions as before. [9]

(iii) For each model, describe what happens to the acceleration of the raindrop as $t \to \infty$. [5]

3 The population, P, of a species at time t years is to be modelled by a differential equation. The initial population is 2000.

At first the model $\frac{\mathrm{d}P}{\mathrm{d}t} = 0.5P$ is used.

(i) Find P in terms of t. [3]

To take account of observed fluctuations, the model is refined to give $\frac{\mathrm{d}P}{\mathrm{d}t} = 0.5P + 170\sin 2t$.

(ii) State the complementary function for this differential equation. Find a particular integral and hence state the general solution. [8]

(iii) Find the solution subject to the given initial condition. [2]

The model is further refined to give $\frac{\mathrm{d}P}{\mathrm{d}t} = 0.5P + P^{\frac{2}{3}}\sin 2t$. This is to be solved using Euler's method. The algorithm is given by $t_{r+1} = t_r + h,\ P_{r+1} = P_r + h\dot{P}_r$.

(iv) Using a step length of 0.1 and the given initial conditions, perform two iterations of the algorithm to estimate the population when $t = 0.2$. [4]

The population is observed to tend to a non-zero finite limit as $t \to \infty$, so a further model is proposed, given by

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 0.5P\left(1 - \frac{P}{12\,000}\right)^{\frac{1}{2}}.$$

(v) Without solving the differential equation,

(A) find the limiting value of P as $t \to \infty$, [3]

(B) find the value of P for which the rate of population growth is greatest. [4]

4 The simultaneous differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -3x + y + 9,$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -5x + y + 15,$$

are to be solved for $t \geqslant 0$.

(i) Show that $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} + 2x = 6$. [5]

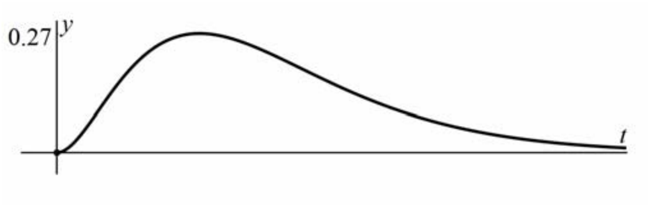
(ii) Find the general solution for x. [7]

(iii) Hence find the corresponding general solution for y. [3]

(iv) Find the solutions subject to the conditions that $x = y = 0$ when $t = 0$. [4]

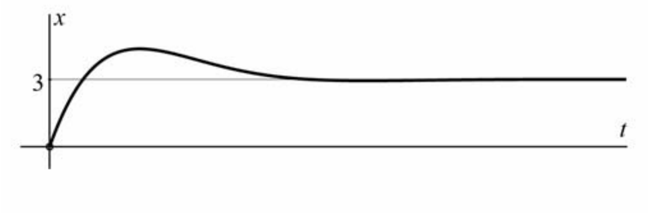
(v) Sketch, on separate axes, graphs of the solutions for $t \geqslant 0$. [5]

4758 Differential Equations

1(i)	$\alpha^2 + 2\alpha + 1 = 0$	M1	Auxiliary equation	
	$\alpha = -1$ (repeated)	A1		
	CF $y = (A + Bt)e^{-t}$	F1	CF for their roots	
	PI $y = a$	B1	Constant PI	
	in DE $\Rightarrow y = 2$	B1	PI correct	
	$y = 2 + (A + Bt)e^{-t}$	F1	Their PI + CF (with two arbitrary constants)	
	$t = 0, y = 0 \Rightarrow 0 = 2 + A \Rightarrow A = -2$	M1	Condition on y	
	$\dot{y} = (B - A - Bt)e^{-t}$	M1	Differentiate (product rule)	
	$t = 0, \dot{y} = 0 \Rightarrow 0 = B - A \Rightarrow B = -2$	M1	Condition on $\dot{y}$	
	$y = 2 - (2 + 2t)e^{-t}$	A1		
				10
(ii)	Both terms in CF hence will give zero if substituted in LHS	E1		
	PI $y = bt^2 e^{-t}$	B1		
	$\dot{y} = (2bt - bt^2)e^{-t}, \ddot{y} = (2b - 4bt + bt^2)e^{-t}$			
	in DE $\Rightarrow (2b - 4bt + bt^2 + 2(2bt - bt^2) + bt^2)e^{-t} = e^{-t}$	M1	Differentiate twice and substitute	
	$\Rightarrow b = \frac{1}{2}$	A1	PI correct	
	$y = (C + Dt + \frac{1}{2}t^2)e^{-t}$	F1	Their PI + CF (with two arbitrary constants)	
	$t = 0, y = 0 \Rightarrow 0 = C$	M1	Condition on y	
	$\dot{y} = (D + t - C - Dt - \frac{1}{2}t^2)e^{-t}$			
	$t = 0, \dot{y} = 0 \Rightarrow 0 = D - C \Rightarrow D = 0$	M1	Condition on $\dot{y}$	
	$y = \frac{1}{2}t^2 e^{-t}$	A1		
				8
(iii)	$t > 0 \Rightarrow \frac{1}{2}t^2 > 0$ and $e^{-t} > 0 \Rightarrow y > 0$	E1		
	$\dot{y} = (t - \frac{1}{2}t^2)e^{-t}$ so $\dot{y} = 0 \Leftrightarrow t - \frac{1}{2}t^2 = 0 \Leftrightarrow t = 0$ or 2	M1	Solve $\dot{y} = 0$	
	Maximum at $t = 2, y = 2e^{-2}$	A1	Maximum value of y	
		B1 B1 B1	Starts at origin Maximum at their value of y $y > 0$	
				6

2(i)	$\frac{dv}{dt}+\frac{3}{1+t}v=g-\frac{3}{1+t}$	M1	Rearrange	
	$I=\exp\left(\int\frac{3}{1+t}\,dt\right)=e^{3\ln(1+t)}=(1+t)^3$	M1 A1 A1	Attempt integrating factor Correct Simplified	
	$(1+t)^3\frac{dv}{dt}+3(1+t)^2v=g(1+t)^3-3(1+t)^2$	F1	Multiply DE by their *I*	
	$\frac{d}{dt}\left((1+t)^3v\right)=g(1+t)^3-3(1+t)^2$			
	$(1+t)^3v=\int\left(g(1+t)^3-3(1+t)^2\right)dx$	M1	Integrate	
	$=\frac{1}{4}g(1+t)^4-(1+t)^3+A$	A1	RHS	
	$v=\frac{1}{4}g(1+t)-1+A(1+t)^{-3}$	F1	Divide by their *I* (must also divide constant)	
	$t=0,v=0\Rightarrow 0=\frac{1}{4}g-1+A$	M1	Use condition	
	$v=\frac{1}{4}g(1+t)-1+\left(1-\frac{1}{4}g\right)(1+t)^{-3}$	E1	Convincingly shown	
				10
(ii)	$(1+t)\frac{dv}{dt}+5v=(1+t)g$	M1	Rearrange	
	$\frac{dv}{dt}+\frac{5}{1+t}v=g$			
	$I=\exp\left(\int\frac{5}{1+t}\,dt\right)=e^{5\ln(1+t)}=(1+t)^5$	M1 A1	Attempt integrating factor Simplified	
	$(1+t)^5\frac{dv}{dt}+5(1+t)^4v=g(1+t)^5$	F1	Multiply DE by their *I*	
	$\frac{d}{dt}\left((1+t)^5v\right)=g(1+t)^5$			
	$(1+t)^5v=\int g(1+t)^5\,dx$	M1	Integrate	
	$=\frac{1}{6}g(1+t)^6+B$	A1	RHS	
	$v=\frac{1}{6}g(1+t)+B(1+t)^{-5}$	F1	Divide by their *I* (must also divide constant)	
	$t=0,v=0\Rightarrow 0=\frac{1}{6}g+B$	M1	Use condition	
	$v=\frac{1}{6}g\left(1+t-(1+t)^{-5}\right)$	F1	Follow a non-trivial GS	
				9
(iii)	First model: $\frac{dv}{dt}=\frac{1}{4}g-3\left(1-\frac{1}{4}g\right)(1+t)^{-4}$	M1	Find acceleration	
	As $t\to\infty,(1+t)^{-4}\to 0$	B1	Identify term(s) → 0 in their solution for either model	
	Hence acceleration tends to $\frac{1}{4}g$	A1		
	Second model $\frac{dv}{dt}=\frac{1}{6}g\left(1+5(1+t)^{-6}\right)$	M1	Find acceleration	
	Hence acceleration tends to $\frac{1}{6}g$	A1		
				5

3(i)	$P = A\mathrm{e}^{0.5t}$	M1	Any valid method	
	$t = 0, P = 2000 \Rightarrow A = 2000$	M1	Use condition	
	$P = 2000\mathrm{e}^{0.5t}$	A1		
				3
(ii)	CF $P = A\mathrm{e}^{0.5t}$	F1	Correct or follows (i)	
	PI $P = a\cos 2t + b\sin 2t$	B1		
	$\dot{P} = -2a\sin 2t + 2b\cos 2t$	M1	Differentiate	
	$-2a\sin 2t + 2b\cos 2t = 0.5(a\cos 2t + b\sin 2t) + 170\sin 2t$	M1	Substitute	
	$-2a = 0.5b + 170$	M1	Compare coefficients	
	$2b = 0.5a$	M1	Solve	
	solving $\Rightarrow a = -80, b = -20$	A1		
	GS $P = A\mathrm{e}^{0.5t} - 80\cos 2t - 20\sin 2t$	F1	Their PI + CF (with one arbitrary constant)	
				8
(iii)	$t = 0, P = 2000 \Rightarrow A = 2080$	M1	Use condition	
	$P = 2080\mathrm{e}^{0.5t} - 80\cos 2t - 20\sin 2t$	F1	Follow a non-trivial GS	
				2
(iv)	*t* *P* $\dot{P}$	M1	Use of algorithm	
	0 2000 1000	A1	2100	
	0.1 2100 1082.58	A1	1082.5…	
	0.2 2208	A1	2208	
				4
(v)	*(A)* Limiting value $\Rightarrow \dot{P} = 0$	M1	Set $\dot{P} = 0$	
	$\Rightarrow P\left(1 - \frac{P}{12\,000}\right)^{\frac{1}{2}} = 0$	M1	Solve	
	(as limit non-zero) limiting value = 12000	A1		
				3
	(B) Growth rate max when			
	$\mathrm{f}(P) = P\left(1 - \frac{P}{12\,000}\right)^{\frac{1}{2}}$ max	M1	Recognise expression to maximise	
	$\mathrm{f}'(P) = \left(1 - \frac{P}{12\,000}\right)^{\frac{1}{2}} - \frac{1}{2\times 12000}P\left(1 - \frac{P}{12\,000}\right)^{-\frac{1}{2}}$	M1	Reasonable attempt at derivative	
	$\mathrm{f}'(P) = 0 \Leftrightarrow \left(1 - \frac{P}{12\,000}\right) - \frac{1}{2\times 12000}P = 0$	M1	Set derivative to zero	
	$\Leftrightarrow P = 8000$	A1		
				4

4(i)	$\ddot{x} = -3\dot{x} + \dot{y}$	M1	Differentiate first equation	
	$= -3\dot{x} + (-5x + y + 15)$	M1	Substitute for $\dot{y}$	
	$y = 3x - 9 + \dot{x}$	M1	y in terms of $x, \dot{x}$	
	$\ddot{x} = -3\dot{x} - 5x + (3x - 9 + \dot{x}) + 15$	M1	Substitute for y	
	$\ddot{x} + 2\dot{x} + 2x = 6$	E1		
				5
(ii)	$\lambda^2 + 2\lambda + 2 = 0$	M1	Auxiliary equation	
	$\lambda = -1 \pm j$	A1		
	CF $x = e^{-t}(A\cos t + B\sin t)$	M1	CF for complex roots	
		F1	CF for their roots	
	PI $x = a$	B1	Constant PI	
	$2a = 6 \Rightarrow a = 3$	B1	PI correct	
	GS $x = 3 + e^{-t}(A\cos t + B\sin t)$	F1	Their CF + PI (with two arbitrary constants)	
				7
(iii)	$y = 3x - 9 + \dot{x}$	M1	y in terms of $x, \dot{x}$	
	$= 9 + 3e^{-t}(A\cos t + B\sin t) - 9$ $- e^{-t}(A\cos t + B\sin t) + e^{-t}(-A\sin t + B\cos t)$	M1	Differentiate x and substitute	
	$y = e^{-t}((2A + B)\cos t + (2B - A)\sin t)$	A1	Constants must correspond with those in x	
				3
(iv)	$0 = 3 + A \Rightarrow A = -3$	M1	Condition on x	
	$0 = 2A + B \Rightarrow B = 6$	M1	Condition on y	
	$x = 3 + 3e^{-t}(2\sin t - \cos t)$	F1	Follow their GS	
	$y = 15e^{-t}\sin t$	F1	Follow their GS	
				4
(v)		B1	Sketch of x starts at origin	
		B1	Asymptote $x = 3$	
		B1	Sketch of y starts at origin	
		B1	Decaying oscillations (may decay rapidly)	
		B1	Asymptote $y = 0$	
				5