OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE **4753/01**

MATHEMATICS (MEI)

Methods for Advanced Mathematics (C3)

FRIDAY 11 JANUARY 2008 Morning

Time: 1 hour 30 minutes

Additional materials: Answer Booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages.

[Turn over

Section A (36 marks)

1 Differentiate $\sqrt[3]{1+6x^2}$. [4]

2 The functions $\mathrm{f}(x)$ and $\mathrm{g}(x)$ are defined for all real numbers x by

$$\mathrm{f}(x) = x^2, \qquad \mathrm{g}(x) = x - 2.$$

(i) Find the composite functions $\mathrm{fg}(x)$ and $\mathrm{gf}(x)$. [3]

(ii) Sketch the curves $y = \mathrm{f}(x)$, $y = \mathrm{fg}(x)$ and $y = \mathrm{gf}(x)$, indicating clearly which is which. [2]

3 The profit £P made by a company in its nth year is modelled by the exponential function

$$P = A\mathrm{e}^{bn}.$$

In the first year (when $n = 1$), the profit was £10 000. In the second year, the profit was £16 000.

(i) Show that $\mathrm{e}^b = 1.6$, and find b and A. [6]

(ii) What does this model predict the profit to be in the 20th year? [2]

4 When the gas in a balloon is kept at a constant temperature, the pressure P in atmospheres and the volume V m^3 are related by the equation

$$P = \frac{k}{V},$$

where k is a constant. [This is known as Boyle's Law.]

When the volume is 100 m^3, the pressure is 5 atmospheres, and the volume is increasing at a rate of 10 m^3 per second.

(i) Show that $k = 500$. [1]

(ii) Find $\frac{\mathrm{d}P}{\mathrm{d}V}$ in terms of V. [2]

(iii) Find the rate at which the pressure is decreasing when $V = 100$. [4]

5 **(i)** Verify the following statement:

'$2^p - 1$ is a prime number for all prime numbers p less than 11'. [2]

(ii) Calculate 23×89, and hence disprove this statement:

'$2^p - 1$ is a prime number for all prime numbers p'. [2]

6 Fig. 6 shows the curve $e^{2y} = x^2 + y$.

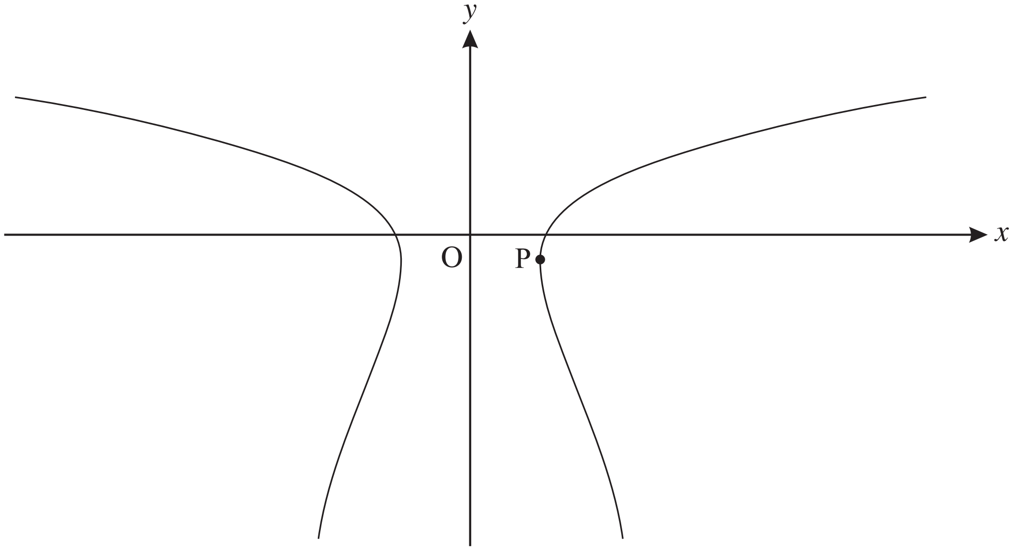


Fig. 6

(i) Show that $\frac{dy}{dx} = \frac{2x}{2e^{2y} - 1}$. [4]

(ii) Hence find to 3 significant figures the coordinates of the point P, shown in Fig. 6, where the curve has infinite gradient. [4]

Section B (36 marks)

7 A curve is defined by the equation $y = 2x \ln(1 + x)$.

(i) Find $\frac{dy}{dx}$ and hence verify that the origin is a stationary point of the curve. [4]

(ii) Find $\frac{d^2y}{dx^2}$, and use this to verify that the origin is a minimum point. [5]

(iii) Using the substitution $u = 1 + x$, show that $\int \frac{x^2}{1+x}\,dx = \int \left(u - 2 + \frac{1}{u}\right) du$.

Hence evaluate $\int_0^1 \frac{x^2}{1+x}\,dx$, giving your answer in an exact form. [6]

(iv) Using integration by parts and your answer to part **(iii)**, evaluate $\int_0^1 2x \ln(1 + x)\,dx$. [4]

8 Fig. 8 shows the curve $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = 1 + \sin 2x$ for $-\frac{1}{4}\pi \leqslant x \leqslant \frac{1}{4}\pi$.

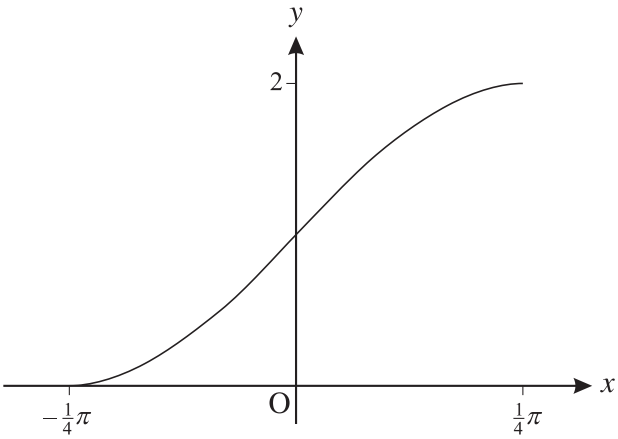


Fig. 8

(i) State a sequence of two transformations that would map part of the curve $y = \sin x$ onto the curve $y = \mathrm{f}(x)$. [4]

(ii) Find the area of the region enclosed by the curve $y = \mathrm{f}(x)$, the x-axis and the line $x = \frac{1}{4}\pi$. [4]

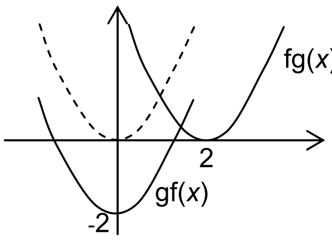
(iii) Find the gradient of the curve $y = \mathrm{f}(x)$ at the point $(0, 1)$. Hence write down the gradient of the curve $y = \mathrm{f}^{-1}(x)$ at the point $(1, 0)$. [4]

(iv) State the domain of $\mathrm{f}^{-1}(x)$. Add a sketch of $y = \mathrm{f}^{-1}(x)$ to a copy of Fig. 8. [3]

(v) Find an expression for $\mathrm{f}^{-1}(x)$. [2]

4753 (C3) Methods for Advanced Mathematics

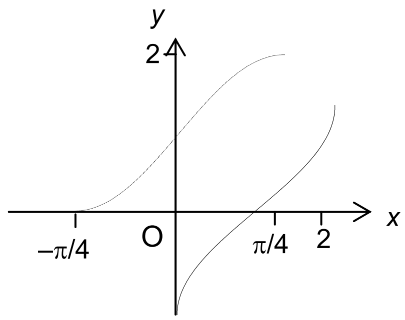
Section A

1 $y=(1+6x^2)^{1/3}$ $\Rightarrow \dfrac{dy}{dx}=\dfrac{1}{3}(1+6x^2)^{-2/3}.12x$ $=4x(1+6x^2)^{-2/3}$	M1 B1 A1 A1 [4]	chain rule used $\frac{1}{3}u^{-2/3}$ ×12*x* cao (must resolve 1/3 × 12) Mark final answer
2 (i) fg(x) = f(x – 2) = $(x-2)^2$ gf(x) = g(x^2) = $x^2 - 2$.	M1 A1 A1 [3]	forming a composite function mark final answer If fg and gf the wrong way round, M1A0A0
(ii) 	B1ft B1ft [2]	fg – must have (2, 0)labelled (or inferable from scale). Condone no y-intercept, unless wrong gf – must have (0, –2) labelled (or inferable from scale) Condone no x-intercepts, unless wrong Allow ft only if fg and gf are correct but wrong way round.
3 (i) When n = 1, 10 000 = A e^b when n = 2, 16 000 = A e^{2b} $\Rightarrow \dfrac{16000}{10000}=\dfrac{Ae^{2b}}{Ae^b}=e^b$ $\Rightarrow$ e^b = 1.6 $\Rightarrow$ b = ln 1.6 = 0.470 A = 10000/1.6 = 6250.	B1 B1 M1 E1 B1 B1 [6]	soi soi eliminating A (do not allow verification) SCB2 if initial 'B's are missing, and ratio of years = 1.6 = e^b ln 1.6 or 0.47 or better (mark final answer) cao – allow recovery from inexact b's
(ii) When n = 20, P = 6250×e$^{0.470×20}$ = £75,550,000	M1 A1 [2]	substituting n = 20 into their equation with their A and b Allow answers from £75 000 000 to £76 000 000.
4 (i) 5 = k/100 ⇒ k = 500*	E1 [1]	NB answer given
(ii) $\dfrac{dP}{dV}=-500V^{-2}=-\dfrac{500}{V^2}$	M1 A1 [2]	$(-1)V^{-2}$ o.e. – allow $-k/V^2$
(iii) $\dfrac{dP}{dt}=\dfrac{dP}{dV}.\dfrac{dV}{dt}$ When V = 100, dP/dV = –500/10000 = –0.05 dV/dt = 10 $\Rightarrow$ dP/dt = –0.05 × 10 = –0.5 So P is decreasing at 0.5 Atm/s	M1 B1ft B1 A1 [4]	chain rule (any correct version) (soi) (soi) –0.5 cao

5(i) $p = 2$, $2^p - 1 = 3$, prime $p = 3$, $2^p - 1 = 7$, prime $p = 5$, $2^p - 1 = 31$, prime $p = 7$, $2^p - 1 = 127$, prime	M1 E1 [2]	Testing at least one prime testing all 4 primes (correctly) Must comment on answers being prime (allow ticks) Testing $p = 1$ is E0
(ii) $23\times89 = 2047 = 2^{11} - 1$ 11 is prime, 2047 is not So statement is false.	M1 E1 [2]	$2^{11} - 1$ must state or imply that 11 is prime ($p = 11$ is sufficient)
6 (i) $e^{2y} = x^2 + y$ $\Rightarrow \quad 2e^{2y}\dfrac{dy}{dx} = 2x + \dfrac{dy}{dx}$ $\Rightarrow \quad (2e^{2y} - 1)\dfrac{dy}{dx} = 2x$ $\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{2x}{2e^{2y} - 1}$ *	M1 A1 M1 E1 [4]	Implicit differentiation – allow one slip (but with dy/dx both sides) collecting terms
(ii) Gradient is infinite when $2e^{2y} - 1 = 0$ $\Rightarrow \quad e^{2y} = ½$ $\Rightarrow \quad 2y = \ln ½$ $\Rightarrow \quad y = ½ \ln ½ = -0.347$ (3 s.f.) $x^2 = e^{2y} - y = ½ - (-0.347)$ $= 0.8465$ $\Rightarrow \quad x = 0.920$	M1 A1 M1 A1 [4]	 must be to 3 s.f. substituting their y and solving for x cao – must be to 3 s.f., but penalise accuracy once only.

Section B

7(i) $y = 2x \ln(1+x)$ $\Rightarrow \frac{dy}{dx} = \frac{2x}{1+x} + 2\ln(1+x)$ When $x = 0$, $dy/dx = 0 + 2 \ln 1 = 0$ $\Rightarrow$ origin is a stationary point.	M1 B1 A1 E1 [4]	product rule $d/dx(\ln(1+x)) = 1/(1+x)$ soi www (i.e. from correct derivative)
(ii) $\frac{d^2y}{dx^2} = \frac{(1+x).2 - 2x.1}{(1+x)^2} + \frac{2}{1+x}$ $= \frac{2}{(1+x)^2} + \frac{2}{1+x}$ When $x = 0$, $d^2y/dx^2 = 2 + 2 = 4 > 0$ $\Rightarrow$ (0, 0) is a min point	M1 A1ft A1 M1 E1 [5]	Quotient or product rule on their $2x/(1+x)$ correctly applied to their $2x/(1+x)$ o.e., e.g. $\frac{4+2x}{(1+x)^2}$ cao substituting $x = 0$ into their d^2y/dx^2 www – dep previous A1
(iii) Let $u = 1 + x \Rightarrow du = dx$ $\Rightarrow \int \frac{x^2}{1+x}dx = \int \frac{(u-1)^2}{u}du$ $= \int \frac{(u^2 - 2u + 1)}{u}du$ $= \int (u - 2 + \frac{1}{u})du$ * $\Rightarrow \int_0^1 \frac{x^2}{1+x}dx = \int_1^2 (u - 2 + \frac{1}{u})du$ $= \left[\frac{1}{2}u^2 - 2u + \ln u\right]_1^2$ $= 2 - 4 + \ln 2 - (\frac{1}{2} - 2 + \ln 1)$ $= \ln 2 - \frac{1}{2}$	M1 E1 B1 B1 M1 A1 [6]	$\frac{(u-1)^2}{u}$ www (but condone du omitted except in final answer) changing limits (or substituting back for x and using 0 and 1) $\left[\frac{1}{2}u^2 - 2u + \ln u\right]$ substituting limits (consistent with u or x) cao
(iv) $A = \int_0^1 2x\ln(1+x)dx$ Parts: $u = \ln(1+x)$, $du/dx = 1/(1+x)$ $dv/dx = 2x \Rightarrow v = x^2$ $= \left[x^2 \ln(1+x)\right]_0^1 - \int_0^1 \frac{x^2}{1+x}dx$ $= \ln 2 - \ln 2 + \frac{1}{2} = \frac{1}{2}$	 M1 A1 M1 A1 [4]	 soi substituting their $\ln 2 - \frac{1}{2}$ for $\int_0^1 \frac{x^2}{1+x}dx$ cao

8 (i) Stretch in x-direction s.f. ½ translation in y-direction 1 unit up	M1 A1 M1 A1 [4]	(in either order) – allow 'contraction' dep 'stretch' allow 'move', 'shift', etc – direction can be inferred from $\begin{pmatrix}0\\1\end{pmatrix}$ or $\begin{pmatrix}0\\1\end{pmatrix}$ dep 'translation'. $\begin{pmatrix}0\\1\end{pmatrix}$ alone is M1 A0
(ii) $A=\int_{-\pi/4}^{\pi/4}(1+\sin 2x)dx$ $=\left[x-\frac{1}{2}\cos 2x\right]_{-\pi/4}^{\pi/4}$ $= \pi/4 - ½ \cos \pi/2 + \pi/4 + ½ \cos (-\pi/2)$ $= \pi/2$	M1 B1 M1 A1 [4]	correct integral and limits. Condone dx missing; limits may be implied from subsequent working. substituting their limits (if zero lower limit used, must show evidence of substitution) or 1.57 or better – cao (www)
(iii) $y = 1 + \sin 2x$ $\Rightarrow$ $dy/dx = 2\cos 2x$ When $x = 0$, $dy/dx = 2$ So gradient at (0, 1) on f(x) is 2 $\Rightarrow$ gradient at (1, 0) on $f^{-1}(x) = ½$	M1 A1 A1ft B1ft [4]	differentiating – allow 1 error (but not $x + 2\cos 2x$) If 1, then must show evidence of using reciprocal, e.g. 1/1
(iv) Domain is $0 \le x \le 2$. 	B1 M1 A1 [3]	Allow 0 to 2, but not $0 < x < 2$ or y instead of x clear attempt to reflect in $y = x$ correct domain indicated (0 to 2), and reasonable shape
(v) $y = 1 + \sin 2x$ $x \leftrightarrow y$ $x = 1 + \sin 2y$ $\Rightarrow$ $\sin 2y = x - 1$ $\Rightarrow$ $2y = \arcsin(x - 1)$ $\Rightarrow$ $y = ½ \arcsin(x - 1)$	M1 A1 [2]	or $\sin 2x = y - 1$ cao