ADVANCED GCE UNIT **4777/01**

MATHEMATICS (MEI)

Numerical Computation

FRIDAY 22 JUNE 2007

Morning

Time: 2 hours 30 minutes

Additional materials:
- Answer booklet (8 pages)
- Graph paper
- MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- Additional sheets, including computer print-outs, should be fastened securely to the answer booklet.

COMPUTER RESOURCES

- Candidates will require access to a computer with a spreadsheet program and suitable printing facilities during the examination.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- In each of the questions you are required to write spreadsheet routines to carry out various numerical analysis processes. You should note the following points.
- You will not receive credit for using any numerical analysis functions which are provided within the spreadsheet. For example, many spreadsheets provide a solver routine; you will not receive credit for using this routine when asked to write your own procedure for solving an equation.

  You may use the following built-in mathematical functions: square root, sin, cos, tan, arcsin, arccos, arctan, ln, exp.
- For each question you attempt, you should submit print-outs showing the spreadsheet routine you have written and the output it generates. It will be necessary to print out the *formulae* in the cells as well as the *values* in the cells.

  You are not expected to print out and submit everything your routine produces, but you are required to submit sufficient evidence to convince the examiner that a correct procedure has been used.
- The total number of marks for this paper is 72.

ADVICE TO CANDIDATES

- Read each question carefully and make sure you know what you have to do before starting your answer.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

1 **(i)** The iterative sequence $x_0, x_1, x_2, \ldots$, where $x_{r+1} = \mathrm{g}(x_r)$, has a fixed point α, so that $\alpha = \mathrm{g}(\alpha)$.

Given that $x_{r+1} - \alpha \approx k(x_r - \alpha)$, show that

$$k \approx \frac{(x_2 - x_1)}{(x_1 - x_0)},$$

and obtain an approximate equation for α in terms of x_2, x_1 and k. [5]

(ii) Use a spreadsheet to demonstrate graphically that the equation $x = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \mathrm{e}^{\frac{1}{3}x} + \mathrm{e}^{-\frac{1}{3}x} - 0.5$$

has two roots. Let these roots be α and β where $\alpha < \beta$.

Show that the iteration $x_{r+1} = \mathrm{f}(x_r)$ will converge only slowly to α and that it will not converge to β at all. [7]

(iii) Use the acceleration technique developed in part **(i)** to speed up the convergence to α. Find α correct to 6 decimal places.

Show that, with a carefully chosen starting point, the acceleration technique may be used to produce convergence to β. Find β correct to 6 decimal places.

Determine, correct to 1 decimal place, the range of starting values for which convergence to β is assured within 5 iterations of the acceleration technique. [12]

2 The Gaussian 4-point integration formula has the form

$$\int_{-h}^{h} \mathrm{f}(x)\,\mathrm{d}x = a\mathrm{f}(-\alpha) + b\mathrm{f}(-\beta) + b\mathrm{f}(\beta) + a\mathrm{f}(\alpha).$$

(i) Obtain the four equations that determine a, b, α and β, showing that one of them is

$$a\alpha^6 + b\beta^6 = \tfrac{1}{7}h^7.$$ [7]

You are now *given* the following values, correct to 8 decimal places.

$$\begin{aligned} a &= 0.347\,854\,85h \\ b &= 0.652\,145\,15h \\ \alpha &= 0.861\,136\,31h \\ \beta &= 0.339\,981\,04h \end{aligned}$$

(ii) Use a spreadsheet to show that, for x in radians, $\dfrac{\sin x}{x}$ tends to 1 as x tends to 0.

Use a spreadsheet to obtain a sketch of the function $\mathrm{f}(x) = \dfrac{\sin x}{x}$ for $0 \leqslant x \leqslant \pi$.

Taking $h = \frac{1}{2}\pi$ initially, use the Gaussian 4-point rule to estimate the value of

$$\int_0^{\pi} \frac{\sin x}{x}\,\mathrm{d}x.$$

Repeat the process, halving h as necessary, in order to establish the value of the integral correct to 6 decimal places. [13]

(iii) Modify the routines used in part **(ii)** to determine the value of t, correct to 3 decimal places, such that

$$\int_0^{t} \frac{\sin x}{x}\,\mathrm{d}x = 1.$$ [4]

3 The differential equation

$$\frac{dy}{dx} = x + 0.1e^{y},$$

where $y = 0$ when $x = 0$, is to be solved in order to estimate y when $x = 1$.

(i) Use Euler's method with $h = 0.2, 0.1, 0.05, 0.025$ to obtain a sequence of estimates of y when $x = 1$. Hence demonstrate that Euler's method has first order convergence. [7]

(ii) Show similarly that the modified Euler method has second order convergence. [6]

(iii) Develop a solution to the differential equation using a predictor-corrector method. Use Euler's method as the predictor and the modified Euler method as the corrector. Apply the corrector 3 times at each step.

Compare the accuracy of this method with that of the modified Euler method. [8]

(iv) Obtain a sequence of estimates of y when $x = 1$ by averaging the estimates found in parts **(ii)** and **(iii)**. Show that this sequence appears to have approximately third order convergence. [3]

4 The augmented matrix given below is denoted by $\mathbf{M} \mid \mathbf{c}$.

$$\left(\begin{array}{cccc|c} 0 & 1 & 2 & 3 & 1 \\ 3 & 0 & 1 & 2 & 2 \\ 2 & 3 & 0 & 1 & 3 \\ 1 & 2 & 3 & 0 & 4 \end{array}\right)$$

(i) Set up a spreadsheet using Gaussian elimination to solve the system of equations represented by $\mathbf{M} \mid \mathbf{c}$. Make clear at each stage which element is used for pivoting and explain why. Show how to check the accuracy of your solution. [13]

(ii) Apply the routine developed in part **(i)** to systems of the form $\mathbf{M} \mid \mathbf{v}$, for appropriate vectors $\mathbf{v}$ so as to find the inverse of the matrix $\mathbf{M}$. [6]

(iii) Use part **(i)** to obtain the determinant of $\mathbf{M}$, making it clear how you establish its sign. [5]

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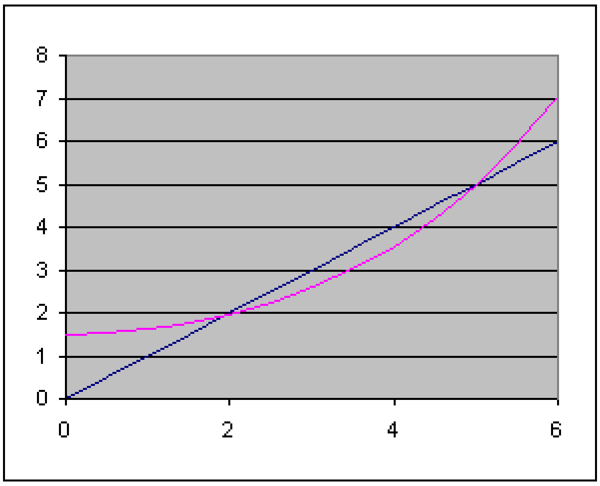
1(i) Convincing algebra to $k = (x_2 - x_1)/(x_1 - x_0)$ **[M1A1]**

Convincing algebra to $\alpha = (x_2 - k\,x_1)/(1 - k)$ or equivalent **[M1A1A1]**

[subtotal 5]

(ii)

x	y=x	y=f(x)
0	0	1.5
0.5	0.5	1.527842
1	1	1.612144
1.5	1.5	1.755252
2	2	1.961151
2.5	2.5	2.235574
3	3	2.586161
3.5	3.5	3.022674
4	4	3.557265
4.5	4.5	4.204819
5	5	4.983366
5.5	5.5	5.914581
6	6	7.024391



[G2]

converges	2	*diverges*	4.5	5	5.5	*set up*	
slowly	1.961151	*from*	4.204819	4.983366	5.914581	*iteration*	**[M1A1]**
to	1.942783	*root*	3.807921	4.95514	6.820878		
root	1.934241	*near*	3.339412	4.90763	9.3175	*near 2*	**[A1]**
near	1.9303	*5*	2.872419	4.828739	21.8726		
2	1.928489		2.488967	4.70068	1466.344	*near 5*	**[A1A1]**
	1.927657		2.228729	4.500432	1.9E+212	(theoretical arguments	
	1.927276		2.07777	4.205432	#NUM!	involving f 'acceptable)	

[subtotal 7]

(iii)

x0	x1	x2	k	new x0			
2	1.961151	1.942783	0.472807	1.92631		k	**[M1A1]**
1.92631	1.926659	1.926818	0.458143	1.926953		est of root	**[M1A1]**
1.926953	1.926953	1.926953	0.45827	**1.926953**	**=alpha**	use as x0	**[M1]**
						iterate	**[M1A1]**

x0	x1	x2	k	new x0			
5	4.983366	4.95514	1.696813	5.023872		alpha	**[A1]**
5.023872	5.024167	5.024673	1.71656	5.023461			
5.023461	5.023461	5.023461	1.716217	**5.023461**	**= beta**	beta	**[A1]**

x0	x1	x2	k	new x0		
4.6	4.349412	3.996895	1.406756	5.216066		
5.216066	5.365628	5.647933	1.887551	5.047555	**range**	
5.047555	5.064991	5.095267	1.73646	5.02388	**4.6 to 5.7**	**[M1A1A1]**
5.02388	5.024181	5.024697	1.716567	5.023461		
5.023461	5.023461	5.023461	1.716217	**5.023461**		

[subtotal 12]

[TOTAL 24]

2 (i) Substitute $f(x) = 1, x^2, x^4, x^6$ into the integration fomula **[M1M1M1M1]**

Obtain $a + b = h$ **[A1]**

$a\alpha^2 + b\beta^2 = h^3/3$ **[A1]**

$a\alpha^4 + b\beta^4 = h^5/5$ **[A1]**

$(a\alpha^6 + b\beta^6 = h^7/7)$

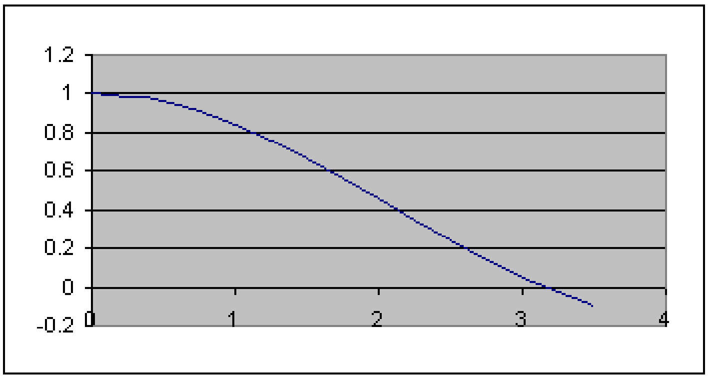
[subtotal 7]

(ii) E.g.

x	0.1	0.01	0.001	
sin(x) / x	0.998334	0.999983	1	**[B1]**

(ii)

x	sin(x) / x
0	1
0.5	0.958851
1	0.841471
1.5	0.664997
2	0.454649
2.5	0.239389
3	0.04704
3.5	-0.10022



[G2]

Single application of Gaussian 4-pt rule

m=	1.570796	h=	1.570796		
α, β	x	f(x)	a, b		
-0.86114	0.218127	0.992089	0.347855	0.345103	**set up**
-0.33998	1.036755	0.830241	0.652145	0.541438	**[M4]**
0.339981	2.104837	0.408942	0.652145	0.26669	
0.861136	2.923466	0.074022	0.347855	0.025749	
			sum:	1.17898	
			integral:	**1.851937**	**[A1]**

Subdividing the interval

m=	0.785398	h=	0.785398		
			gives	1.370762	**[M1A1]**
m=	2.356194	h=	0.785398		
			gives	0.481175	**[M1A1]**
			sum	**1.851937** (= 6dp)	**[A1]**

[subtotal 13]

(iii) By trial and error

m=	0.53242	h=	0.53242		
α, β	x	f(x)	a, b		
-0.86114	0.073934	0.999089	0.347855	0.347538	**trial**
-0.33998	0.351407	0.979546	0.652145	0.638806	**and**
0.339981	0.713433	0.917302	0.652145	0.598214	**error**
0.861136	0.990906	0.8442	0.347855	0.293659	**[M1A1]**
				1	

Hence t = 2m = 1.065 (1.06484) **[M1A1]**

[subtotal 4]

[TOTAL 24]

3 (i) Euler

h	x	y	y '	new y	
0.2	0	0	0.1	0.02	
0.2	0.2	0.02	0.30202	0.080404	**setup**
0.2	0.4	0.080404	0.508372	0.182079	**[M2]**
0.2	0.6	0.182079	0.719971	0.326073	
0.2	0.8	0.326073	0.938552	0.513783	**estimates**
0.2	1	0.513783			**[A1A1]**

h	y(1)	diffs	ratio of diffs		
0.2	0.513783				**differences**
0.1	0.569802	0.056019			**[M1A1]**
0.05	0.598337	0.028535	0.509387		
0.025	0.612748	0.014411	0.505038	approx 0.5, so first order	**[E1]**

[subtotal 7]

(ii) Modified Euler

h	x	y	k1	k2	new y	
0.2	0	0	0.02	0.060404	0.040202	
0.2	0.2	0.040202	0.06082	0.102126	0.121675	**setup**
0.2	0.4	0.121675	0.102588	0.145028	0.245483	**[M2]**
0.2	0.6	0.245483	0.145565	0.189571	0.413051	
0.2	0.8	0.413051	0.190228	0.236562	0.626446	**estimates**
0.2	1	0.626446				**[A1A1]**

h	y(1)	diffs	ratio of diffs		
0.2	0.626446				**differences**
0.1	0.627065	0.000619			**[A1]**
0.05	0.627213	0.000147	0.238113		
0.025	0.627249	3.58E-05	0.242993	approx 0.25, so second order	**[E1]**

[subtotal 6]

(iii) predictor-corrector

h	x	y	y '	pred	corr1	corr2	corr3	
0.2	0	0	0.1	0.02	0.040202	0.04041	0.040412	
0.2	0.2	0.040412	0.304124	0.101237	0.12189	0.122121	0.122124	**setup**
0.2	0.4	0.122124	0.512989	0.224722	0.245942	0.246211	0.246214	**[M3]**
0.2	0.6	0.246214	0.727917	0.391798	0.413802	0.414132	0.414137	
0.2	0.8	0.414137	0.951306	0.604398	0.627569	0.627998	0.628006	
0.2	1	0.628006						**estimates** **[A1A1]**

h	y(1)				
0.2	0.628006				**differences**
0.1	0.627447	-0.00056			**[A1]**
0.05	0.627307	-0.00014	0.250462	Still second order. Differences very	
0.025	0.627272	-3.5E-05	0.250113	similar in magnitude to modified Euler.	**[E1E1]**

[subtotal 8]

(iv)

mod Euler	pre-corr	average	diffs	ratio of diffs		
						values
0.626446	0.628006	0.627226				**[A1]**
0.627065	0.627447	0.627256	3.04E-05			**differences**
0.627213	0.627307	0.62726	3.78E-06	0.124555	approx 0.125	**[A1]**
0.627249	0.627272	0.627261	4.21E-07	0.111332	so third order	**[E1]**

[subtotal 3]

[TOTAL 24]

4 (i)	0	1	2	3	1			
	3	0	1	2	2	x1 =	0.666667	**elimin'n**
	2	3	0	1	3			**[M1M1M1]**
	1	2	3	0	4			**[A1A1]**
		1	2	3	1			
		3	-0.66667	-0.33333	1.666667	x2 =	0.666667	**back sub**
		2	2.666667	-0.66667	3.333333			**[M1]**
			2.222222	3.111111	0.444444			**solutions**
			3.111111	-0.44444	2.222222	x3 =	0.666667	**[A1A1A1A1]**
				3.428571	-1.14286	x4 =	-0.33333	

pivot (shaded) is element **[M1]**
of largest magnitude in column **[E1]**
Demonstrate check by substituting values back into equations. **[B1]**
[subtotal 13]

(ii)	Apply to	1	0	0	0		**at least one v**
	v =	0	1	0	0		**[M1]**
		0	0	1	0	*NB: clear*	**other three**
		0	0	0	1	*evidence*	**[M1]**
						required	
	To get	-0.20833	0.29167	0.04167	0.04167	*that own*	
	M^{-1} =	0.04167	-0.20833	0.29167	0.04167	*routine*	
		0.04167	0.04167	-0.20833	0.29167	*is used*	**columns**
		0.29167	0.04167	0.04167	-0.20833		**[A1A1A1A1]**

[subtotal 6]

(iii) The product of the pivots is 96 **[M1A1]**
In each of the first three cases, the pivot is in the second row
of the reduced matrix. This is equivalent to three row
interchanges. Hence multiply by $(-1)^3$. **[M1E1]**
i.e. determinant is -96 **[A1]**
[subtotal 5]

[TOTAL 24]

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General Comments

The candidature for this paper was, once again, small and so generalisations are difficult. Several candidates scored high marks, but the rest scored poorly, showing little knowledge of the necessary theory and little familiarity with the techniques. The poorest candidates seemed not fully at home in the use of a spreadsheet.

Comments on Individual Questions

1 **Solution of an equation; acceleration**

In part (i) the algebra was sometimes unconvincing. Parts (ii) and (iii) were done better, though some candidates did not appreciate that, when the acceleration formula is used to produce an improved estimate, that value should be used to re-start the process.

2 **Gaussian 4-point rule**

There was just one good attempt at this question, with the candidate scoring highly. The other attempts were poor, with candidates unable to make much progress in the theoretical or practical parts of the question.

3 **Predictor-corrector method**

This was the least popular question, and also the least well done with no candidate achieving more than half marks. Euler's method was known, but the modified Euler and the predictor-corrector extensions were beyond all candidates.

4 **Gaussian elimination**

Those who talked this question did well, with no candidate scoring less than half marks. The fundamental ideas of Gaussian elimination to solve equations, find inverses and find determinants were well understood and successfully implemented on a spreadsheet.