RECOGNISING ACHIEVEMENT

ADVANCED GCE UNIT

MATHEMATICS (MEI)

4754(A)/01

Applications of Advanced Mathematics (C4)

Paper A

THURSDAY 14 JUNE 2007

Afternoon
Time: 1 hour 30 minutes

Additional materials:
 Answer booklet (8 pages)
 Graph paper
 MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.

ADVICE TO CANDIDATES

- Read each question carefully and make sure you know what you have to do before starting your answer.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

NOTE

- This paper will be followed by **Paper B: Comprehension**.

This document consists of **6** printed pages and **2** blank pages.

Section A (36 marks)

1 Express $\sin\theta - 3\cos\theta$ in the form $R\sin(\theta - \alpha)$, where R and α are constants to be determined, and $0° < \alpha < 90°$.

Hence solve the equation $\sin\theta - 3\cos\theta = 1$ for $0° \leqslant \theta \leqslant 360°$. [7]

2 Write down normal vectors to the planes $2x + 3y + 4z = 10$ and $x - 2y + z = 5$.

Hence show that these planes are perpendicular to each other. [4]

3 Fig. 3 shows the curve $y = \ln x$ and part of the line $y = 2$.

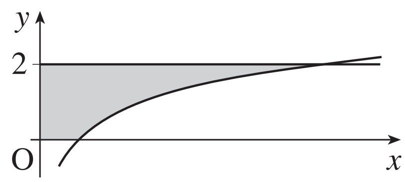

Fig. 3

The shaded region is rotated through 360° about the y-axis.

(i) Show that the volume of the solid of revolution formed is given by $\int_0^2 \pi e^{2y}\,dy$. [3]

(ii) Evaluate this, leaving your answer in an exact form. [3]

4 A curve is defined by parametric equations

$$x = \frac{1}{t} - 1, \; y = \frac{2+t}{1+t}.$$

Show that the cartesian equation of the curve is $y = \dfrac{3+2x}{2+x}$. [4]

5 Verify that the point $(-1, 6, 5)$ lies on both the lines

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} 0 \\ 6 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}.$$

Find the acute angle between the lines. [7]

6 Two students are trying to evaluate the integral $\int_1^2 \sqrt{1+e^{-x}}\,dx$.

Sarah uses the trapezium rule with 2 strips, and starts by constructing the following table.

x	1	1.5	2
$\sqrt{1+e^{-x}}$	1.1696	1.1060	1.0655

(i) Complete the calculation, giving your answer to 3 significant figures. [2]

Anish uses a binomial approximation for $\sqrt{1+e^{-x}}$ and then integrates this.

(ii) Show that, provided e^{-x} is suitably small, $\left(1+e^{-x}\right)^{\frac{1}{2}} \approx 1+\frac{1}{2}e^{-x}-\frac{1}{8}e^{-2x}$. [3]

(iii) Use this result to evaluate $\int_1^2 \sqrt{1+e^{-x}}\,dx$ approximately, giving your answer to 3 significant figures. [3]

Section B (36 marks)

7 Data suggest that the number of cases of infection from a particular disease tends to oscillate between two values over a period of approximately 6 months.

(a) Suppose that the number of cases, P thousand, after time t months is modelled by the equation $P = \dfrac{2}{2 - \sin t}$. Thus, when $t = 0, P = 1$.

(i) By considering the greatest and least values of $\sin t$, write down the greatest and least values of P predicted by this model. [2]

(ii) Verify that P satisfies the differential equation $\dfrac{dP}{dt} = \frac{1}{2} P^2 \cos t$. [5]

(b) An alternative model is proposed, with differential equation

$$\frac{dP}{dt} = \tfrac{1}{2}(2P^2 - P)\cos t. \qquad (*)$$

As before, $P = 1$ when $t = 0$.

(i) Express $\dfrac{1}{P(2P-1)}$ in partial fractions. [4]

(ii) Solve the differential equation (*) to show that

$$\ln\left(\frac{2P-1}{P}\right) = \tfrac{1}{2}\sin t. \qquad [5]$$

This equation can be rearranged to give $P = \dfrac{1}{2 - e^{\frac{1}{2}\sin t}}$.

(iii) Find the greatest and least values of P predicted by this model. [4]

8

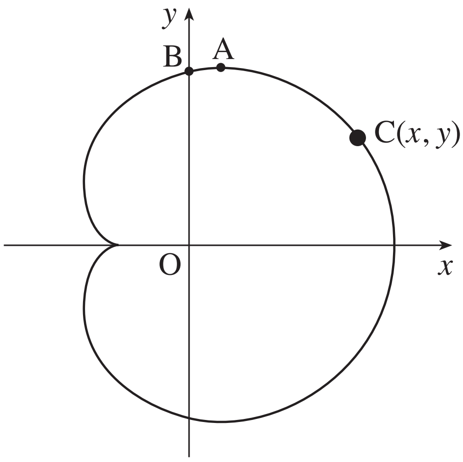

Fig. 8

In a theme park ride, a capsule C moves in a vertical plane (see Fig. 8). With respect to the axes shown, the path of C is modelled by the parametric equations

$$x = 10\cos\theta + 5\cos 2\theta, \quad y = 10\sin\theta + 5\sin 2\theta, \qquad (0 \leqslant \theta < 2\pi),$$

where x and y are in metres.

(i) Show that $\dfrac{dy}{dx} = -\dfrac{\cos\theta + \cos 2\theta}{\sin\theta + \sin 2\theta}$.

Verify that $\dfrac{dy}{dx} = 0$ when $\theta = \frac{1}{3}\pi$. Hence find the exact coordinates of the highest point A on the path of C. [6]

(ii) Express $x^2 + y^2$ in terms of θ. Hence show that

$$x^2 + y^2 = 125 + 100\cos\theta. \qquad [4]$$

(iii) Using this result, or otherwise, find the greatest and least distances of C from O. [2]

You are given that, at the point B on the path vertically above O,

$$2\cos^2\theta + 2\cos\theta - 1 = 0.$$

(iv) Using this result, and the result in part (**ii**), find the distance OB. Give your answer to 3 significant figures. [4]

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RECOGNISING ACHIEVEMENT

ADVANCED GCE UNIT **4754(B)/01**

MATHEMATICS (MEI)

Applications of Advanced Mathematics (C4)

Paper B: Comprehension

THURSDAY 14 JUNE 2007

Afternoon
Time: Up to 1 hour

Additional materials:
Rough paper
MEI Examination Formulae and Tables (MF2)

Candidate Name	

Centre Number						Candidate Number				

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces above.
- Answer **all** the questions.
- Write your answers in the spaces provided on the question paper.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 18.
- The insert contains the text for use with the questions.
- You may find it helpful to make notes and do some calculations as you read the passage.
- You are **not** required to hand in these notes with the question paper.

ADVICE TO CANDIDATES

- Read each question carefully and make sure you know what you have to do before starting your answer.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

For Examiner's Use	
Qu.	Mark
1	
2	
3	
4	
5	
Total	

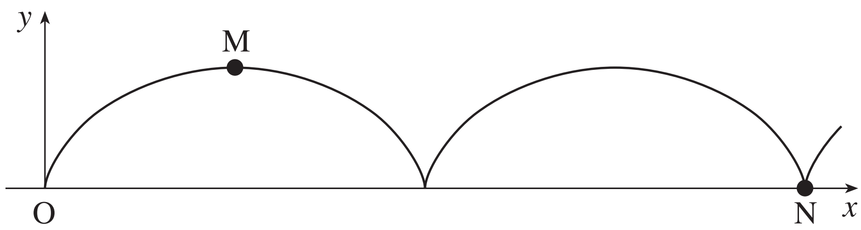
This document consists of **5** printed pages, **3** blank pages and an insert.

[Turn over

For Examiner's Use

1 This basic cycloid has parametric equations

$$x = a\theta - a\sin\theta, \quad y = a - a\cos\theta.$$



Find the coordinates of the points M and N, stating the value of θ at each of them. [2]

Point M ....

Point N ....

2 A sea wave has parametric equations (in suitable units)

$$x = 7\theta - 0.25\sin\theta, \quad y = 0.25\cos\theta.$$

Find the wavelength and height of the wave. [3]

....

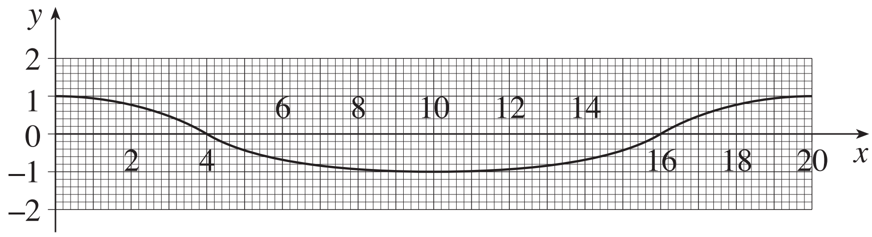
....

....

3 The graph below shows the profile of a wave.

(i) Assuming that it has parametric equations of the form given on line 68, find the values of a and b. [2]

(ii) Investigate whether the ratio of the trough length to the crest length is consistent with this shape. [3]



(i) ....

....

(ii) ....

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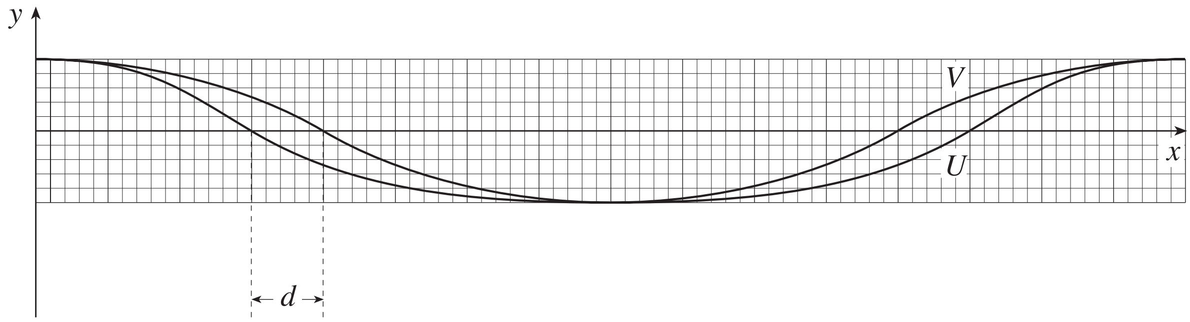
....

4 This diagram illustrates two wave shapes U and V. They have the same wavelength and the same height.



One of the curves is a sine wave, the other is a curtate cycloid.

(i) State which is which, justifying your answer. [1]

(i) ....

....

The parametric equations for the curves are:

$$x = a\theta, \quad y = b\cos\theta,$$

and $$x = a\theta - b\sin\theta, \quad y = b\cos\theta.$$

(ii) Show that the distance marked d on the diagram is equal to b. [2]

(iii) Hence justify the statement in lines 109 to 111: "In such cases, the curtate cycloid and the sine curve with the same wavelength and height are very similar and so the sine curve is also a good model." [2]

(ii) ....

....

....

....

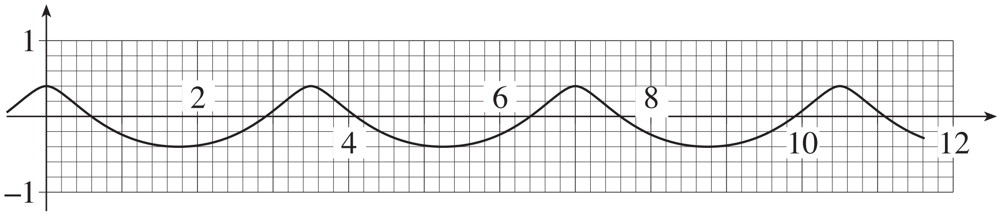
(iii) ....

....

....

For Examiner's Use

5 The diagram shows a curtate cycloid with scales given. Show that this curve could not be a scale drawing of the shape of a stable sea wave. [3]



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ADVANCED GCE UNIT

4754(B)/01

MATHEMATICS (MEI)

Applications of Advanced Mathematics (C4)

Paper B: Comprehension

INSERT

THURSDAY 14 JUNE 2007

Afternoon

Time: Up to 1 hour

INSTRUCTIONS TO CANDIDATES

- This insert contains the text for use with the questions.

This document consists of **8** printed pages.

Modelling sea waves

Introduction

There are many situations in which waves and oscillations occur in nature and often they are accurately modelled by the sine curve. However, this is not the case for sea waves as these come in a variety of shapes. The photograph in Fig. 1 shows an extreme form of sea wave being ridden by a surfer.

Fig. 1

At any time many parts of the world's oceans are experiencing storms. The strong winds create irregular *wind waves*. However, once a storm has passed, the waves form into a regular pattern, called *swell*. Swell waves are very stable; those resulting from a big storm would travel several times round the earth if they were not stopped by the land.

Fig. 2 illustrates a typical swell wave, but with the vertical scale exaggerated. The horizontal distance between successive *peaks* is the *wavelength*; the vertical distance from the lowest point in a *trough* to a peak is called the *height*. The height is twice the *amplitude* which is measured from the horizontal *line of mid-height*. The upper part is the *crest*. These terms are illustrated in Fig. 2.

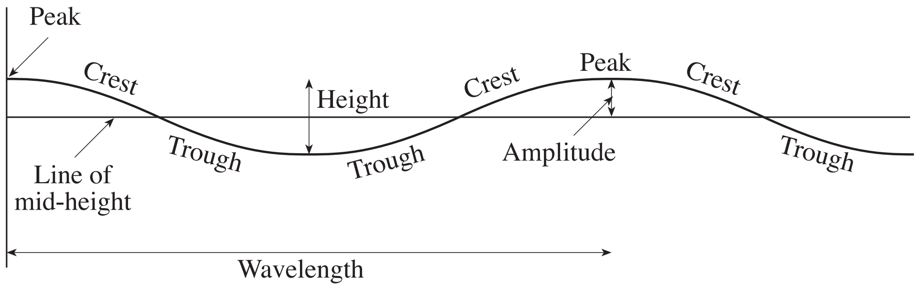


Fig. 2

The speed of a wave depends on the depth of the water; the deeper the water, the faster the wave. (This is, however, not true for very deep water, where the wave speed is independent of the depth.) This has a number of consequences for waves as they come into shallow water.

- Their speed decreases.
- Their wavelength shortens.
- Their height increases.

Observations show that, as their height increases, the waves become less symmetrical. The troughs become relatively long and the crests short and more pointed.

The profile of a wave approaching land is illustrated in Fig. 3. Eventually the top curls over and the wave "breaks".

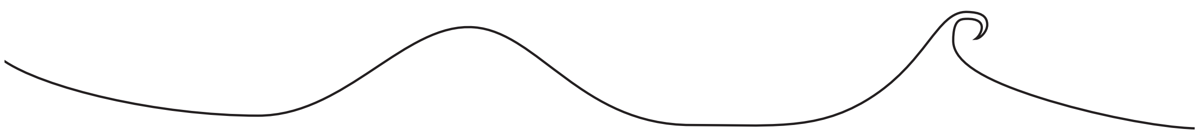


Fig. 3

If you stand at the edge of the sea you will see the water from each wave running up the shore towards you. You might think that this water had just travelled across the ocean. That would be wrong. When a wave travels across deep water, it is the shape that moves across the surface and not the water itself. It is only when the wave finally reaches land that the actual water moves any significant distance.

Experiments in wave tanks have shown that, except near the shore, each drop of water near the surface undergoes circular motion (or very nearly so). This has led people to investigate the possiblility that a form of cycloid would provide a better model than a sine curve for a sea wave.

Cycloids

There are several types of cycloid. In this article, the name *cycloid* refers to one of the family of curves which form the locus of a point on a circle rolling along a straight horizontal path.

Fig. 4 illustrates the basic cycloid; in this case the point is on the circumference of the circle.

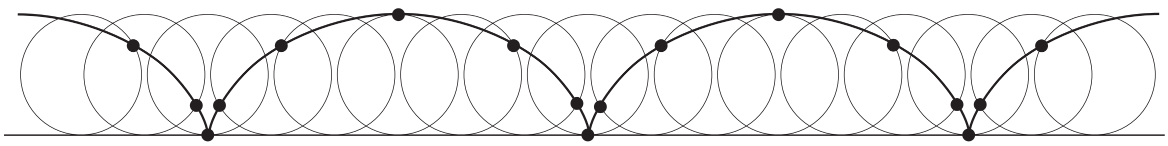
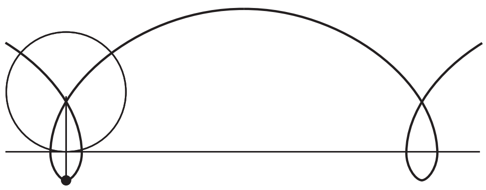


Fig. 4

Two variations on this basic cycloid are the prolate cycloid, illustrated in Fig. 5, and the curtate cycloid illustrated in Fig. 6. The prolate cycloid is the locus of a point attached to the circle but outside the circumference (like a point on the flange of a railway train's wheel); the curtate cycloid is the locus of a point inside the circumference of the circle.



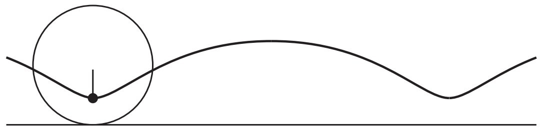


Fig. 5 **Fig. 6**

When several cycles of the curtate cycloid are drawn "upside down", as in Fig. 7, the curve does indeed look like the profile of a wave in shallow water.

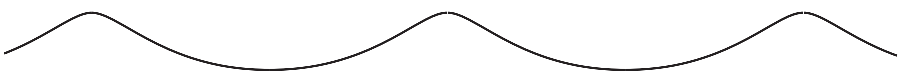


Fig. 7

The equation of a cycloid

The equation of a cycloid is usually given in parametric form.

Fig. 8.1 and Fig. 8.2 illustrate a circle rolling along the x-axis. The circle has centre Q and radius a. P and R are points on its circumference and angle PQR $= \theta$, measured in radians. Fig. 8.1 shows the initial position of the circle with P at its lowest point; this is the same point as the origin, O. Some time later the circle has rolled to the position shown in Fig. 8.2 with R at its lowest point.

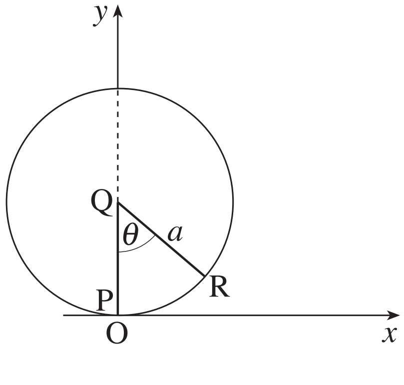


Fig. 8.1

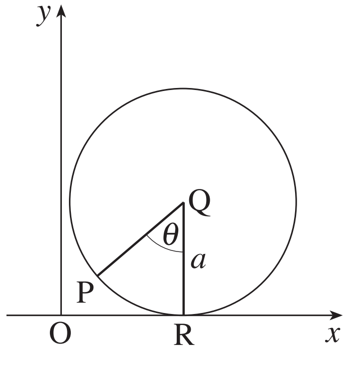


Fig. 8.2

In travelling to its new position, the circle has rolled the distance OR in Fig. 8.2. Since it has rolled along its circumference, this distance is the same as the arc length PR, and so is $a\theta$. Thus the coordinates of the centre, Q, in Fig. 8.2 are $(a\theta, a)$. To find the coordinates of the point P in Fig. 8.2, look at triangle QPZ in Fig. 9.

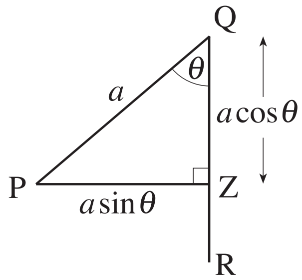


Fig. 9

You can see that

$$PZ = a\sin\theta \qquad \text{and} \qquad QZ = a\cos\theta.$$

Hence the coordinates of P are $(a\theta - a\sin\theta, a - a\cos\theta)$, and so the locus of the point P is described by the curve with parametric equations

$$x = a\theta - a\sin\theta, \quad y = a - a\cos\theta.$$

This is the basic cycloid.

These parametric equations can be generalised to

$$x = a\theta - b\sin\theta, \quad y = a - b\cos\theta,$$

where b is the distance of the moving point from the centre of the circle.

For	$b < a$	the curve is a *curtate cycloid*,
	$b = a$	the curve is a *basic cycloid*,
	$b > a$	the curve is a *prolate cycloid*.

The equivalent equations with the curve turned "upside down", and with the mid-height of the curve now on the x-axis, are

$$x = a\theta - b\sin\theta, \quad y = b\cos\theta.$$

(Notice that positive values of y are still measured vertically upwards.)

Modelling a particular wave

A question that now arises is how to fit an equation to a particular wave profile.

If you assume that the wave is a cycloid, there are two parameters to be found, a and b.

Since $y = b\cos\theta$,

- the maximum value of y is b and this occurs when $\theta = 0, 2\pi, 4\pi, \dots$,
- the minimum value of y is $-b$ and this occurs when $\theta = \pi, 3\pi, 5\pi, \dots$,
- the height of the wave is $2b$.

The wavelength is the horizontal distance between successive maximum values of y.

- A maximum occurs when $\theta = 0$ and $x = 0$.
- The next maximum occurs when $\theta = 2\pi$ and in that case $x = a \times (2\pi) - b \times \sin(2\pi) = 2\pi a$.
- The wavelength is $2\pi a$.

Thus if a wave has a cycloid form, $a = \dfrac{\text{wavelength}}{2\pi}$ and $b = \dfrac{\text{height}}{2}$.

The profile of a possible wave shape is illustrated in Fig. 10. You can see that the wavelength is 16 units and the height is 2 units. So if it is a cycloid shape, the values of a and b would be 2.54… and 1; it is actually easier to work with πa, which would have the value 8 in this case.

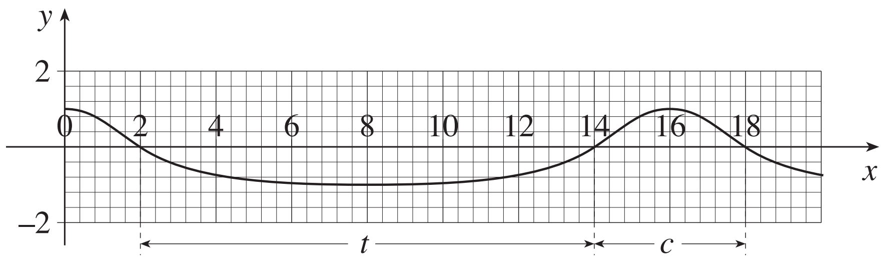


Fig. 10

However, finding values for a and b does not in itself show that the form of a wave is indeed a cycloid. One way of checking whether this could be a good model is to measure the length of the trough of the wave (the distance for which it is below mid-height, and so $y < 0$) and the length of its crest (the distance for which it is above mid-height, and so $y > 0$.) These distances are marked as t and c in Fig. 10.

The ratio of the measured distances t and c is then compared with the equivalent ratio for a cycloid.

In Fig. 10, $t = 12$ and $c = 4$, so the ratio $t : c$ is $3 : 1$.

To find the equivalent ratio for a cycloid, start by finding the values of θ for which the wave is at mid-height.

When $y = 0$, the values of θ are $\dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \dots$.

The corresponding values of x are $\left(\frac{\pi a}{2}-b\right)$, $\left(\frac{3\pi a}{2}+b\right)$, $\left(\frac{5\pi a}{2}-b\right)$, ...

So
$$t=\left(\frac{3\pi a}{2}+b\right)-\left(\frac{\pi a}{2}-b\right)=\pi a+2b,$$
$$c=\left(\frac{5\pi a}{2}-b\right)-\left(\frac{3\pi a}{2}+b\right)=\pi a-2b,$$

and the ratio $t:c$ is $(\pi a+2b):(\pi a-2b)$.

Using $\pi a = 8$ and $b = 1$, the values of t and c would be 10 and 6 and the ratio $t:c$ would be $10:6$ or $1.67:1$.

As this ratio is quite different from $3:1$, the curve in Fig. 10 is not a cycloid. In this case, the troughs are too long and the crests too short.

Sea waves

In fact, observations of real swell waves show that they are well modelled as curtate cycloids.

In the deep ocean, the wavelength may be hundreds of metres and the height less than 5 metres. This corresponds to a large value of a and a small value of b. For a wavelength of 200 metres and a height of 2 metres, the ratio of height to wavelength is $1:100$. In such cases, the curtate cycloid and the sine curve with the same wavelength and height are very similar and so the sine curve is also a good model.

As the wave comes into shallower water, the ratio of height to wavelength increases. The curtate cycloid remains a good model but the sine curve becomes increasingly unsuitable.

During this phase the value of a decreases and the value of b increases. Thought of in terms of the locus of a point on a rolling circle, the point moves away from the centre while the radius decreases.

Eventually, however, the wave breaks with its top curling over. It is tempting to imagine that this would correspond to the case $b = a$, when the cycloid changes from curtate to prolate and develops a loop. That would correspond to a height to wavelength ratio of $1:\pi$. However, observation shows that breaking occurs well before that, when the height to wavelength ratio is about $1:7$.

This observation is not surprising when you remember that the motion of the drops of water in a wave is circular. Such circular motion cannot occur at the sharp point at the peak of a basic cycloid. The wave becomes unstable when the circular motion can no longer be sustained within the wave. At this point the cycloid ceases to provide a model for sea waves.

Other sea waves

In addition to swell and wind waves, there are several other types of sea wave.

Internal waves are formed when a current runs over an uneven seabed. Because the currents are often caused by the tides, these waves are often called tidal rips. They often form off headlands, and can also take the form of whirlpools like the famous Corrievrechan off the west of Scotland.

Tidal waves are caused by the combined gravitational pull of the moon and sun; they have a period of just over 12 hours and a wavelength of half the circumference of the earth. In mid-ocean their height is extremely small but in coastal waters it can be over 10 metres.

Tsunamis are caused by events such as earthquakes and volcanoes. Compared with swell, a tsunami has a very long wavelength, typically at least 100 kilometres. In mid-ocean their height is small and so they are not usually noticed by sailors. However, because of their long wavelength, they build up to a great height when they come into land and cause devastation to coastal areas.

The study of sea waves has been given a boost recently by satellite imaging. This allows the profiles of waves to be determined accurately. One discovery is that *very high waves* are much more common than had been expected. These giant waves are not well understood and it may well be that they require a new model.

Mark Scheme 4754
June 2007

Section A

1 $\sin\theta - 3\cos\theta = R\sin(\theta - \alpha)$ $= R(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$ $\Rightarrow R\cos\alpha = 1$, $R\sin\alpha = 3$ $\Rightarrow R^2 = 1^2 + 3^2 = 10 \Rightarrow R = \sqrt{10}$ $\tan\alpha = 3 \Rightarrow \alpha = 71.57°$ $\sqrt{10}\sin(\theta - 71.57°) = 1$ $\Rightarrow \theta - 71.57° = \sin^{-1}(1/\sqrt{10})$ $\theta - 71.57° = 18.43°, 161.57°$ $\Rightarrow \theta = 90°$, $233.1°$	M1 B1 M1 A1 M1 B1 A1 [7]	equating correct pairs oe ft www cao (71.6°or better) oe ft R, α www and no others in range (MR-1 for radians)
2 Normal vectors are $\begin{pmatrix}2\\3\\4\end{pmatrix}$ and $\begin{pmatrix}1\\-2\\1\end{pmatrix}$ $\Rightarrow \begin{pmatrix}2\\3\\4\end{pmatrix}.\begin{pmatrix}1\\-2\\1\end{pmatrix} = 2-6+4=0$ $\Rightarrow$ planes are perpendicular.	B1 B1 M1 E1 [4]	
3 **(i)** $y = \ln x \Rightarrow x = e^y$ $\Rightarrow V = \int_0^2 \pi x^2 dy$ $= \int_0^2 \pi(e^y)^2 dy = \int_0^2 \pi e^{2y} dy$ *	B1 M1 E1 [3]	
(ii) $\int_0^2 \pi e^{2y} dy = \pi\left[\frac{1}{2}e^{2y}\right]_0^2$ $= \frac{1}{2}\pi(e^4 - 1)$	B1 M1 A1 [3]	$\frac{1}{2} e^{2y}$ substituting limits in $k\pi e^{2y}$ or equivalent, but must be exact and evaluate e^0 as 1.
4 $x = \frac{1}{t} - 1 \Rightarrow \frac{1}{t} = x+1$ $\Rightarrow t = \frac{1}{x+1}$ $\Rightarrow y = \dfrac{2+\frac{1}{x+1}}{1+\frac{1}{x+1}} = \dfrac{2x+2+1}{x+1+1} = \dfrac{2x+3}{x+2}$	M1 A1 M1 E1	Solving for t in terms of x or y Subst their t which must include a fraction, clearing subsidiary fractions/ changing the subject oe www
or $\dfrac{3+2x}{2+x} = \dfrac{3+\frac{2-2t}{t}}{2+\frac{1-t}{t}}$ $= \dfrac{3t+2-2t}{2t+1-t}$ $= \dfrac{t+2}{t+1} = y$	M1 A1 M1 E1 [4]	substituting for x or y in terms of t clearing subsidiary fractions/changing the subject

5 $\mathbf{r}=\begin{pmatrix}1\\2\\-1\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\3\end{pmatrix} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1-\lambda\\2+2\lambda\\-1+3\lambda\end{pmatrix}$		
When $x=-1$, $1-\lambda=-1$, $\Rightarrow \lambda=2$	M1	Finding λ or μ
$\Rightarrow y=2+2\lambda=6$, $z=-1+3\lambda=5$ $\Rightarrow$ point lies on first line	E1	checking other two coordinates
$\mathbf{r}=\begin{pmatrix}0\\6\\3\end{pmatrix}+\mu\begin{pmatrix}1\\0\\-2\end{pmatrix} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\mu\\6\\3-2\mu\end{pmatrix}$		
When $x=-1$, $\mu=-1$, $\Rightarrow y=6$, $z=3-2\mu=5$ $\Rightarrow$ point lies on second line	E1	checking other two co-ordinates
Angle between $\begin{pmatrix}-1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}1\\0\\-2\end{pmatrix}$ is θ, where	M1	Finding angle between correct vectors
$\cos\theta=\dfrac{-1\times1+2\times0+3\times-2}{\sqrt{14}.\sqrt{5}}$	M1	use of formula
$=-\dfrac{7}{\sqrt{70}}$	A1	$\pm\dfrac{7}{\sqrt{70}}$
$\Rightarrow \theta=146.8°$ $\Rightarrow$ acute angle is 33.2°	A1cao [7]	Final answer must be acute angle
6(i) $A\approx0.5[\dfrac{(1.1696+1.0655}{2}+1.1060]$	M1	Correct expression for trapezium rule
$=1.11$ (3 s.f.)	A1 cao [2]	
(ii) $(1+e^{-x})^{1/2}=1+\dfrac{1}{2}e^{-x}+\dfrac{\frac{1}{2}.-\frac{1}{2}}{2!}(e^{-x})^2+\dots$	M1 A1	Binomial expansion with $p=½$ Correct coeffs
$\approx1+\dfrac{1}{2}e^{-x}-\dfrac{1}{8}e^{-2x}$ *	E1 [3]	
(iii) $I=\int_1^2(1+\dfrac{1}{2}e^{-x}-\dfrac{1}{8}e^{-2x})dx$		
$=\left[x-\dfrac{1}{2}e^{-x}+\dfrac{1}{16}e^{-2x}\right]_1^2$	M1	integration
$=(2-\dfrac{1}{2}e^{-2}+\dfrac{1}{16}e^{-4})-(1-\dfrac{1}{2}e^{-1}+\dfrac{1}{16}e^{-2})$	A1	substituting limits into correct expression
$=1.9335-0.8245$ $=1.11$ (3 s.f.)	A1 [3]	

Section B

7 (a) (i) $P_{max} = \frac{2}{2-1} = 2$ $P_{min} = \frac{2}{2+1} = 2/3.$	B1 B1 [2]	
(ii) $P = \frac{2}{2-\sin t} = 2(2-\sin t)^{-1}$ $\Rightarrow \frac{dP}{dt} = -2(2-\sin t)^{-2}.-\cos t$ $= \frac{2\cos t}{(2-\sin t)^2}$ $\frac{1}{2}P^2\cos t = \frac{1}{2}\frac{4}{(2-\sin t)^2}\cos t$ $= \frac{2\cos t}{(2-\sin t)^2} = \frac{dP}{dt}$	M1 B1 A1 DM1 E1 [5]	chain rule $-1(\ldots)^{-2}$ soi (or quotient rule M1,numerator A1,denominator A1) attempt to verify or by integration as in (b)(ii)
(b)(i) $\frac{1}{P(2P-1)} = \frac{A}{P} + \frac{B}{2P-1}$ $= \frac{A(2P-1)+BP}{P(2P-1)}$ $\Rightarrow \quad 1 = A(2P-1) + BP$ $P = 0 \Rightarrow 1 = -A \Rightarrow A = -1$ $P = ½ \Rightarrow 1 = A.0 + ½ B \Rightarrow B = 2$ So $\frac{1}{P(2P-1)} = -\frac{1}{P} + \frac{2}{2P-1}$	M1 M1 A1 A1 [4]	correct partial fractions substituting values, equating coeffs or cover up rule $A = -1$ $B = 2$
(ii) $\frac{dP}{dt} = \frac{1}{2}(2P-P^2)\cos t$ $\Rightarrow \int \frac{1}{2P^2-P}dP = \int \frac{1}{2}\cos t\,dt$ $\Rightarrow \int (\frac{2}{2P-1} - \frac{1}{P})dP = \int \frac{1}{2}\cos t\,dt$ $\Rightarrow \quad \ln(2P-1) - \ln P = ½ \sin t + c$ When $t = 0$, $P = 1$ $\Rightarrow \quad \ln 1 - \ln 1 = ½ \sin 0 + c \Rightarrow c = 0$ $\Rightarrow \ln(\frac{2P-1}{P}) = \frac{1}{2}\sin t$ *	M1 A1 A1 B1 E1 [5]	separating variables $\ln(2P-1) - \ln P$ ft their A,B from (i) ½ sin t finding constant = 0
(iii) $P_{max} = \frac{1}{2-e^{1/2}} = 2.847$ $P_{min} = \frac{1}{2-e^{-1/2}} = 0.718$	M1A1 M1A1 [4]	www www

8 (i) $\frac{dy}{dx}=\frac{10\cos\theta+10\cos 2\theta}{-10\sin\theta-10\sin 2\theta}$	M1	$dy/d\theta \div dx/d\theta$
$= -\frac{\cos\theta+\cos 2\theta}{\sin\theta+\sin 2\theta}$ *	E1	
When $\theta=\pi/3$, $\frac{dy}{dx} = -\frac{\cos\pi/3+\cos 2\pi/3}{\sin\pi/3+\sin 2\pi/3}$ $= 0$ as $\cos \pi/3 = ½$, $\cos 2\pi/3 = -½$	B1	or solving $\cos\theta+\cos 2\theta=0$
At A $x = 10\cos \pi/3 + 5\cos 2\pi/3$	M1	substituting $\pi/3$ into x or y
$= 2½$	A1	2½
$y = 10\sin \pi/3 + 5\sin 2\pi/3 = 15\sqrt{3}/2$	A1 [6]	$15\sqrt{3}/2$ (condone 13 or better)
(ii) $x^2+y^2=(10\cos\theta+5\cos 2\theta)^2+(10\sin\theta+5\sin 2\theta)^2$	B1	
$= 100\cos^2\theta+100\cos\theta\cos 2\theta+25\cos^2 2\theta$ $+100\sin^2\theta+100\sin\theta\sin 2\theta+25\sin^2 2\theta$	M1	expanding
$= 100 + 100\cos(2\theta-\theta) + 25$	DM1	$\cos 2\theta\cos\theta+\sin 2\theta\sin\theta=\cos(2\theta-\theta)$ or substituting for $\sin 2\theta$ and $\cos 2\theta$
$= 125 + 100\cos\theta$ *	E1 [4]	
(iii) Max $\sqrt{125+100} = 15$	B1	
min $\sqrt{125-100} = 5$	B1 [2]	
(iv) $2\cos^2\theta+2\cos\theta-1=0$ $\cos\theta=\frac{-2\pm\sqrt{12}}{4}=\frac{-2\pm 2\sqrt{3}}{4}$	M1	quadratic formula
At B, $\cos\theta=\frac{-1+\sqrt{3}}{2}$	A1	or θ=68.53° or 1.20radians, correct root selected
$OB^2 = 125 + 50(-1+\sqrt{3}) = 75 + 50\sqrt{3} = 161.6\ldots$	M1	or $OB=10\sin\theta+5\sin 2\theta$ ft their $\theta/\cos\theta$
$\Rightarrow$ $OB = \sqrt{161.6\ldots} = 12.7$ (m)	A1 [4]	oe cao

Paper B Comprehension

1)	M $(a\pi, 2a)$, $\theta=\pi$ N $(4a\pi, 0)$, $\theta=4\pi$	B1 B1	
2)	Compare the equations with equations given in text, $x = a\theta - b\sin\theta$, $y = b\cos\theta$	M1	Seeing a=7, b=0.25
	Wavelength = $2\pi a = 14\pi$ (≈ 44) Height = $2b = 0.5$	A1 B1	
3i)	Wavelength = 20 $\Rightarrow a = \frac{10}{\pi}$ (=3.18…) Height = 2 $\Rightarrow b = 1$	B1 B1	
ii)	In this case, the ratio is observed to be 12:8 Trough length : Peak length = $\pi a + 2b : \pi a - 2b$ and this is $(10 + 2 \times 1) : (10 - 2 \times 1)$ So the curve is consistent with the parametric equations	B1 M1 A1	substituting
4i)	$x = a\theta$, $y = b\cos\theta$ is the sine curve V and $x = a\theta - b\sin\theta$, $y = b\cos\theta$ is the curtate cycloid U. The sine curve is above mid-height for half its wavelength (or equivalent)	B1	
ii)	$d = a\theta-(a\theta-b\sin\theta)$ $\theta=\pi/2$, $d=\left(\frac{\pi a}{2}\right)-\left(\frac{\pi a}{2}-b\right)=b$	M1 E1	Subtraction Using $\theta=\pi/2$
iii)	Because b is small compared to a, the two curves are close together.	M1 E1	Comparison attempted Conclusion
5)	Measurements on the diagram give Wavelength ≈ 3.5cm, Height ≈ 0.8cm $\frac{\text{Wavelength}}{\text{Height}} \approx \frac{3.5}{0.8} = 4.375$ Since 4.375 < 7, the wave will have become unstable and broken.	B1 M1 E1	measurements/reading ratio [18]

4754: Applications of Advanced Mathematics (C4)

General Comments

This summer the questions in Paper A proved to be accessible to almost all students with correct solutions to all questions seen. Some candidates failed to complete question 8 so there may have been a shortage of time. All candidates scored well on Section A and found parts of the Section B questions more challenging. The Comprehension was the least well answered question and scored low marks in general compared with the rest of the paper.

It was pleasing to see that candidates did not make some of the mistakes that they did in earlier papers. The correct formula for the trapezium rule was used most of the time, unlike in the January 2007 paper, and efficient methods were used generally except in 8(ii). The most disappointing factors were:-

- failing to put a constant of integration in indefinite integrals
- failing to give full stages of verification in order to establish given answers
- poor algebraic skills, including sign errors and the absence of brackets.

Comments on Individual Questions

Paper A

Section A

1 The first part- using the 'R' method- was well answered. Most errors arose from attempts to quote results rather than working from first principles.
In the second part most candidates used the correct method but most only found the solution 90° and omitted the final solution. Those that realised an extra solution was expected often gave 270° instead of 233.1°. It was disappointing to note that in the second part some candidates still incorrectly expanded sin(A+B) as sinA+sinB.

2 Most candidates wrote down the expected normal vectors - although some used them without clearly stating them. The scalar product was usually well answered with full working shown. The most common error was in not stating that they had shown the planes were in fact perpendicular. There were a large proportion of fully correct solutions to this question.

3 (i) Candidates did not always give clear reasons why the given formula was the required volume of revolution. Stages were too often missed out and this caused marks to be lost as the answer was given in the question. For example, some did not clearly state that they were starting from $\int \pi x^2 dy$ and others omitted the limits.

(ii) This part was more successful although candidates sometimes multiplied by two rather than dividing when integrating. Others failed to give their answer in exact form or failed to evaluate e^0.

4 Most candidates rearranged the equation to give *t* in terms of *x* and then substituted this in the equation for *y*. The first part was usually completed successfully but there were some confused attempts at eliminating the subsidiary fractions. Their basic algebra was disappointing. For example 2+1/(*x*+1) was too often changed to 2+*x*+1.Other methods were possible and successful but much less common.

5 The verification was attempted in a variety of ways and often successfully. The most common involved finding the values of λ and μ first and then substituting to show that these satisfied all the coordinates. The most common mistake was to not show the full verification of these coordinates.
Although the method for the second part was well understood many candidates chose the incorrect vectors- usually the position vectors-in order to find the required angle. Another common error was to omit the negative sign in the vector 1i+0j-2k. Surprisingly many candidates found only the obtuse angle-either ignoring the need for the acute angle or not realising what was required.

6 In part (i) most candidates used the correct trapezium rule formula and the correct answer was usually found and given with appropriate accuracy.
In part (ii), the binomial expansion was well known and was almost always used correctly.
Part (iii) was less successful. A surprisingly large number failed to integrate the expression and substituted the limits into the integrand. Others made errors in the integration, often $\int -1/8e^{-2x}$ =1/4e ^{2x}, or in their evaluation of the terms in e.

Section B

7 (a) The first part was usually correct with most candidates realising that *sin t* took maximum and minimum values at ±1. The most frequent error was the substitution of *t* = 0 and *t* = 6 since 0 < *t* < 6.
The second part was approached in a variety of ways. Use of the quotient rule for differentiation was quite common and was often successful. There were sign errors and many made the error (*2-sin t).0= 2-sin t*. For those that progressed beyond this stage, explanations of the substitution for *P* were not always clear and they had been asked to verify.
Some candidates used the chain rule. There were often sign errors in the differentiation of *2-sin t*. *Cos t* was often given as the differentiation - possibly influenced by the given answer.
Perhaps the most common approach was to separate the variables and integrate. Again there were sign errors but the most common mistake was to omit the constant of integration and thus not be able to achieve the required result.
A less common, but often successful approach - particularly from good candidates - was to use implicit differentiation. Different starting points were seen but *2/P = 2-sin t* leading to *-2/P²dP/dt = -cos t* and then rearranging was an efficient way of achieving the result directly.

(b) Almost all candidates were able to gain all the marks for the partial fractions.
In part (ii) most candidates separated the variables correctly and integrated both sides. There were, however, some poor attempts at the separation with $2P^2 - P$ appearing as a numerator. For those that did integrate correctly - and there were many - it was once again disappointing to note the frequent absence of the constant of integration. In some cases it seemed that candidates had tried to work backwards from the given answer.
In the third part good marks were scored although some candidates prematurely approximated their working and achieved inaccurate answers.

8 (i) Many candidates scored well in this part. Some made the error of quoting $dy/d\theta = \cos\theta + \cos 2\theta$ and similarly for x – possibly working backwards from the given answer. Others made sign errors or differentiated $5\cos 2\theta$ as $5\sin 2\theta$.
When verifying $dy/dx = 0$ in the next part it was necessary to see the evaluation of each part of $\cos \pi/3$ and $\cos 2\pi/3$. This was often omitted. The answer, zero, was given and so it needed to be established. The final part, finding the coordinates of A, was usually correct.

(ii) Although most candidates realised that they needed to square both the expressions in x and y and add them together, the squaring of the terms was often incorrect. Many omitted the middle term completely or wrote incorrectly that $\cos\theta \,.\, \cos 2\theta = \cos 3\theta$ or that $\cos 2\theta \,.\, \cos 2\theta = \cos 4\theta$. There were also some very long methods in this part. Some of them were successful. Full marks usually followed substitution for $\sin 2\theta$ and $\cos 2\theta$.

(iii) & (iv) A number of candidates did not attempt this or the final part and were perhaps short of time at this stage. The common error here was to fail to square root at the final stage.
Some felt that this quadratic equation could be factorised. Others used the quadratic equation formula correctly to solve for $\cos\theta$ but then used this value as θ in the final stage. Some candidates found the distance OB^2, thinking it was OB, as they forgot to square root their final value.

Paper B
The Comprehension

1 Although there were some completely correct solutions many candidates found this question difficult. Common incorrect answers were $M(a\pi/2-a,\ a)$ and $N\ (2\pi a, 0)$. There also seemed to be a lack of understanding of when to use radians.

2 This question was the most successful in the Comprehension and most candidates found both the wavelength and height correctly.

3 Many found the value of a correctly although a common mistake was to cancel $20/2\pi$ to 10π. The value for b was usually correct. In the second part most candidates found the values of 12 and 8 from the given figure although 12 and 4 were often seen. In the final part many did not realise they needed to substitute in the formula $\pi a+2b:\pi a-2b$ but those that did were usually successful.

4 Answers to question 4 were often wordy and missed the point.
Among the best explanations in the first part involved comparing the ratios of the lengths of the troughs and crests in both graphs. Some did not fully answer the question either indicating which was which without justification or explaining without indicating which was which. Explanations were often insufficient. For example, saying that the sine curve was symmetrical.
In the second part counting the squares on the graph was often given as the explanation of why $d = b$. Others started by saying $a\theta = a\theta - b\sin\theta$.
In the final part few realised that *b* needed to be small in comparison to *a*.
The last two parts were often omitted.

5 Few candidates tried to work with the ratios for wavelength and height. Attempts using the ratio for troughs and crests were common. Those who attempted the correct ratio often made numerical errors - often height = 0.4. When the correct final ratio was obtained it was often compared with 1:100 rather than the required 1:7.