RECOGNISING ACHIEVEMENT

ADVANCED GCE UNIT
MATHEMATICS (MEI)

4753/01

Methods for Advanced Mathematics (C3)

MONDAY 11 JUNE 2007

Afternoon
Time: 1 hour 30 minutes

Additional materials:
Answer booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.

ADVICE TO CANDIDATES

- Read each question carefully and make sure you know what you have to do before starting your answer.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages.

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Section A (36 marks)

1 **(i)** Differentiate $\sqrt{1+2x}$. [3]

(ii) Show that the derivative of $\ln(1 - e^{-x})$ is $\dfrac{1}{e^x - 1}$. [4]

2 Given that $f(x) = 1 - x$ and $g(x) = |x|$, write down the composite function $gf(x)$.

On separate diagrams, sketch the graphs of $y = f(x)$ and $y = gf(x)$. [3]

3 A curve has equation $2y^2 + y = 9x^2 + 1$.

(i) Find $\dfrac{dy}{dx}$ in terms of x and y. Hence find the gradient of the curve at the point A $(1, 2)$. [4]

(ii) Find the coordinates of the points on the curve at which $\dfrac{dy}{dx} = 0$. [4]

4 A cup of water is cooling. Its initial temperature is 100°C. After 3 minutes, its temperature is 80°C.

(i) Given that $T = 25 + ae^{-kt}$, where T is the temperature in °C, t is the time in minutes and a and k are constants, find the values of a and k. [5]

(ii) What is the temperature of the water

(A) after 5 minutes,

(B) in the long term? [3]

5 Prove that the following statement is false.

For all integers n greater than or equal to 1, $n^2 + 3n + 1$ is a prime number. [2]

6 Fig. 6 shows the curve $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \frac{1}{2}\arctan x$.

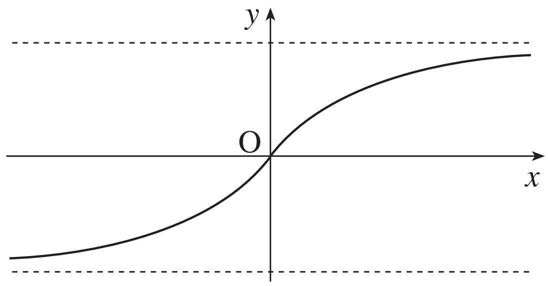

Fig. 6

(i) Find the range of the function $\mathrm{f}(x)$, giving your answer in terms of π. [2]

(ii) Find the inverse function $\mathrm{f}^{-1}(x)$. Find the gradient of the curve $y = \mathrm{f}^{-1}(x)$ at the origin. [5]

(iii) Hence write down the gradient of $y = \frac{1}{2}\arctan x$ at the origin. [1]

Section B (36 marks)

7 Fig. 7 shows the curve $y = \dfrac{x^2}{1+2x^3}$. It is undefined at $x = a$; the line $x = a$ is a vertical asymptote.

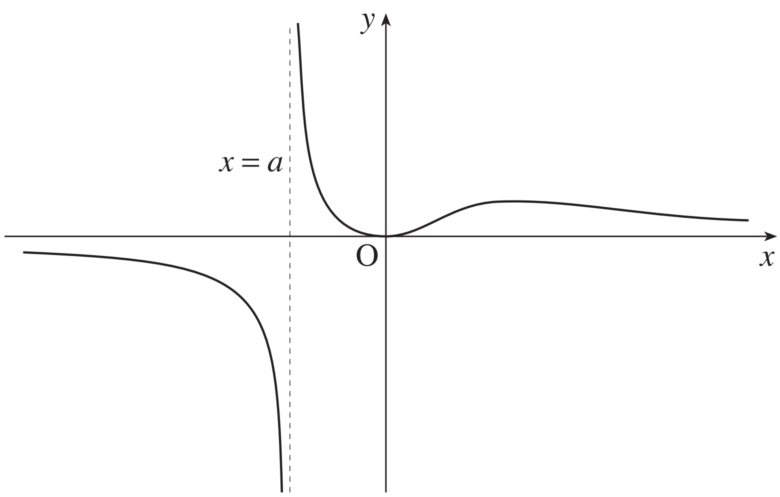

Fig. 7

(i) Calculate the value of a, giving your answer correct to 3 significant figures. [3]

(ii) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2x - 2x^4}{(1+2x^3)^2}$. Hence determine the coordinates of the turning points of the curve. [8]

(iii) Show that the area of the region between the curve and the x-axis from $x = 0$ to $x = 1$ is $\frac{1}{6}\ln 3$. [5]

8 Fig. 8 shows part of the curve $y = x\cos 2x$, together with a point P at which the curve crosses the x-axis.

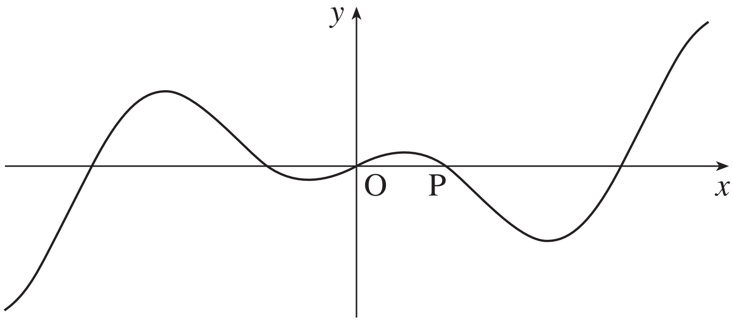

Fig. 8

(i) Find the exact coordinates of P. [3]

(ii) Show algebraically that $x\cos 2x$ is an odd function, and interpret this result graphically. [3]

(iii) Find $\frac{dy}{dx}$. [2]

(iv) Show that turning points occur on the curve for values of x which satisfy the equation $x\tan 2x = \frac{1}{2}$. [2]

(v) Find the gradient of the curve at the origin.

Show that the second derivative of $x\cos 2x$ is zero when $x = 0$. [4]

(vi) Evaluate $\int_0^{\frac{1}{4}\pi} x\cos 2x\,dx$, giving your answer in terms of π. Interpret this result graphically. [6]

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Mark Scheme 4753
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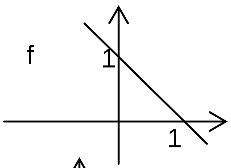
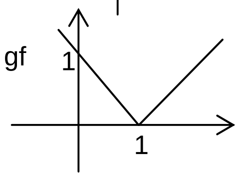
Section A

1 (i) ½ $(1 + 2x)^{-1/2} \times 2$ $= \dfrac{1}{\sqrt{1+2x}}$	M1 B1 A1 [3]	chain rule ½ $u^{-1/2}$ or ½ $(1 + 2x)^{-1/2}$ oe, but must resolve ½ × 2 = 1
(ii) $y = \ln(1-e^{-x})$ $\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{1-e^{-x}}.(-e^{-x})(-1)$ $= \dfrac{e^{-x}}{1-e^{-x}}$ $= \dfrac{1}{e^{x}-1}$ *	M1 B1 A1 E1 [4]	chain rule $\dfrac{1}{1-e^{-x}}$ or $\dfrac{1}{u}$ if substituting $u = 1-e^{-x}$ $\times(-e^{-x})(-1)$ or e^{-x} www (may imply $\times e^{x}$ top and bottom)
2 $gf(x) = \|1-x\|$  	B1 B1 B1 [3]	 intercepts must be labelled line must extend either side of each axis condone no labels, but line must extend to left of y axis
3(i) Differentiating implicitly: $(4y+1)\dfrac{dy}{dx} = 18x$ $\Rightarrow \dfrac{dy}{dx} = \dfrac{18x}{4y+1}$ When $x = 1$, $y = 2$, $\dfrac{dy}{dx} = \dfrac{18}{9} = 2$	M1 A1 M1 A1cao [4]	$(4y+1)\dfrac{dy}{dx} = \ldots$ allow $4y+1\dfrac{dy}{dx} = \ldots$ condone omitted bracket if intention implied by following line. $4y\dfrac{dy}{dx}+1$ M1 A0 substituting $x = 1$, $y = 2$ into their derivative (provided it contains x's and y's). Allow unsupported answers.
(ii) $\dfrac{dy}{dx} = 0$ when $x = 0$ $\Rightarrow 2y^2 + y = 1$ $\Rightarrow 2y^2 + y - 1 = 0$ $\Rightarrow (2y - 1)(y + 1) = 0$ $\Rightarrow y = ½$ or $y = -1$ So coords are (0, ½) and (0, –1)	B1 M1 A1 A1 [4]	$x = 0$ from their numerator = 0 (must have a denominator) Obtaining correct quadratic and attempt to factorise or use quadratic formula $y = \dfrac{-1 \pm \sqrt{1-4\times -2}}{4}$ cao allow unsupported answers provided quadratic is shown

4(i) $T = 25 + ae^{-kt}$. When $t = 0$, $T = 100$ $\Rightarrow$ $100 = 25 + ae^0$ $\Rightarrow$ $a = 75$ When $t = 3$, $T = 80$ $\Rightarrow$ $80 = 25 + 75e^{-3k}$ $\Rightarrow$ $e^{-3k} = 55/75$ $\Rightarrow$ $-3k = \ln(55/75)$, $k = -\ln(55/75) / 3$ $= 0.1034$	M1 A1 M1 M1 A1cao [5]	substituting $t = 0$ and $T = 100$ into their equation (even if this is an incorrect version of the given equation) substituting $t = 3$ and $T = 80$ into (their) equation taking lns correctly at any stage 0.1 or better or $-\frac{1}{3}\ln(\frac{55}{75})$ o.e. if final answer
(ii) (A) $T = 25 + 75e^{-0.1034\times5}$ $= 69.72$ (B) 25°C	M1 A1 B1cao [3]	substituting $t = 5$ into their equation 69.5 to 70.5, condone inaccurate rounding due to value of k.
5 $n = 1$, $n^2 + 3n + 1 = 5$ prime $n = 2$, $n^2 + 3n + 1 = 11$ prime $n = 3$, $n^2 + 3n + 1 = 19$ prime $n = 4$, $n^2 + 3n + 1 = 29$ prime $n = 5$, $n^2 + 3n + 1 = 41$ prime $n = 6$, $n^2 + 3n + 1 = 55$ not prime so statement is false	M1 E1 [2]	One or more trials shown finding a counter-example – must state that it is not prime.
6 (i) $-\pi/2 < \arctan x < \pi/2$ $\Rightarrow$ $-\pi/4 < f(x) < \pi/4$ $\Rightarrow$ range is $-\pi/4$ to $\pi/4$	M1 A1cao [2]	$\pi/4$ or $-\pi/4$ or 45 seen not $\leq$
(ii) $y = ½ \arctan x$ $x \leftrightarrow y$ $x = ½ \arctan y$ $\Rightarrow$ $2x = \arctan y$ $\Rightarrow$ $\tan 2x = y$ $\Rightarrow$ $y = \tan 2x$ *either* $\frac{dy}{dx} = 2\sec^2 2x$	M1 A1cao M1 A1cao	$\tan(\arctan y \text{ or } x) = y \text{ or } x$ derivative of tan is $\sec^2$ used
or $y = \frac{\sin 2x}{\cos 2x} \Rightarrow \frac{dy}{dx} = \frac{2\cos^2 2x + 2\sin^2 2x}{\cos^2 2x}$ $= \frac{2}{\cos^2 2x}$	M1 A1cao	quotient rule (need not be simplified but mark final answer)
When $x = 0$, $dy/dx = 2$	B1 [5]	www
(iii) So gradient of $y = ½ \arctan x$ is ½ .	B1ft [1]	ft their '2', but not 1 or 0 or ∞

Section B

7(i) Asymptote when $1 + 2x^3 = 0$ $\Rightarrow$ $2x^3 = -1$ $\Rightarrow$ $x = -\frac{1}{\sqrt[3]{2}}$ $= -0.794$	M1 A1 A1cao [3]	 oe, condone $\pm\frac{1}{\sqrt[3]{2}}$ if positive root is rejected must be to 3 s.f.
(ii) $\frac{dy}{dx} = \frac{(1+2x^3).2x - x^2.6x^2}{(1+2x^3)^2}$ $= \frac{2x+4x^4-6x^4}{(1+2x^3)^2}$ $= \frac{2x-2x^4}{(1+2x^3)^2}$ * $dy/dx = 0$ when $2x(1 - x^3) = 0$ $\Rightarrow$ $x = 0, y = 0$ or $x = 1$, $y = 1/3$	M1 A1 E1 M1 B1 B1 B1 B1 [8]	Quotient or product rule: (udv–vdu M0) $2x(1+2x^3)^{-1} + x^2(-1)(1+2x^3)^{-2}.6x^2$ allow one slip on derivatives correct expression – condone missing bracket if if intention implied by following line derivative = 0 $x = 0$ or 1 – allow unsupported answers $y = 0$ and 1/3 SC–1 for setting denom = 0 or extra solutions (e.g. $x = -1$)
(iii) $A = \int_0^1 \frac{x^2}{1+2x^3}dx$	M1	Correct integral and limits – allow $\int_1^0$
either $= \left[\frac{1}{6}\ln(1+2x^3)\right]_0^1$ $= \frac{1}{6}\ln 3$ *	M1 A1 M1 E1	$k\ln(1 + 2x^3)$ $k = 1/6$ substituting limits dep previous M1 www
or let $u = 1 + 2x^3 \Rightarrow du = 6x^2dx$ $\Rightarrow$ $A = \int_1^3 \frac{1}{6}.\frac{1}{u}du$ $= \left[\frac{1}{6}\ln u\right]_1^3$ $= \frac{1}{6}\ln 3$ *	 M1 A1 M1 E1 [5]	 $\frac{1}{6u}$ $\frac{1}{6}\ln u$ substituting correct limits (but must have used substitution) www

8 (i) $x \cos 2x = 0$ when $x = 0$ or $\cos 2x = 0$ $\Rightarrow$ $2x = \pi/2$ $\Rightarrow$ $x = \frac{1}{4}\pi$ $\Rightarrow$ P is $(\pi/4, 0)$	M1 M1 A1 [3]	$\cos 2x = 0$ or $x = \frac{1}{2}\cos^{-1}0$ $x = 0.785..$ or 45 is M1 M1 A0
(ii) $f(-x) = -x\cos(-2x)$ $= -x\cos 2x$ $= -f(x)$ Half turn symmetry about O.	M1 E1 B1 [3]	$-x\cos(-2x)$ $= -x\cos 2x$ Must have two of: rotational, order 2, about O, (half turn = rotational order 2)
(iii) $f'(x) = \cos 2x - 2x\sin 2x$	M1 A1 [2]	product rule
(iv) $f'(x) = 0 \Rightarrow \cos 2x = 2x\sin 2x$ $\Rightarrow$ $2x\dfrac{\sin 2x}{\cos 2x} = 1$ $\Rightarrow$ $x\tan 2x = \frac{1}{2}$ *	 M1 E1 [2]	 $\dfrac{\sin}{\cos} = \tan$ www
(v) $f'(0) = \cos 0 - 2.0.\sin 0 = 1$ $f''(x) = -2\sin 2x - 2\sin 2x - 4x\cos 2x$ $= -4\sin 2x - 4x\cos 2x$ $\Rightarrow$ $f''(0) = -4\sin 0 - 4.0.\cos 0 = 0$	B1ft M1 A1 E1 [4]	allow ft on (their) product rule expression product rule on $(2)x \sin 2x$ correct expression – mark final expression www
(vi) Let $u = x$, $dv/dx = \cos 2x$ $\Rightarrow v = \frac{1}{2}\sin 2x$ $\int_0^{\pi/4} x\cos 2x\,dx = \left[\frac{1}{2}x\sin 2x\right]_0^{\pi/4} - \int_0^{\pi/4}\frac{1}{2}\sin 2x\,dx$ $= \frac{\pi}{8} + \left[\frac{1}{4}\cos 2x\right]_0^{\pi/4}$ $= \frac{\pi}{8} - \frac{1}{4}$ Area of region enclosed by curve and x-axis between $x = 0$ and $x = \pi/4$	M1 A1 A1 M1 A1 B1 [6]	Integration by parts with $u = x$, $dv/dx = \cos 2x$ $\left[\frac{1}{4}\cos 2x\right]$ - sign consistent with their previous line substituting limits – dep using parts www or graph showing correct area – condone P for $\pi/4$.

4753: Concepts in Pure Mathematics (C3)

General Comments

This paper proved to be accessible to the full range of candidates, many of whom scored over 50 marks. There were plenty of relatively straightforward marks available, and scores of less than 20 were uncommon. Candidates rarely had trouble completing the paper. In general, the core calculus topics appear to be well known by candidates. The topics which seem to cause problems are modulus, inverse trig functions, proof and disproof, implicit differentiation, and function language (domain and range etc). This paper seems to have had relatively accessible questions on most of these topics.

The general standard of presentation was mixed, and the usual issues of poor notation and algebra are still evident in some scripts. In particular, omitting brackets in expressions like the quotient rule remains common, and the conventions of calculus are quite often inaccurately applied.

One particular issue with regard to this paper is the use of graph paper. We have considered removing this from the rubric: it is almost always the case that using graph paper in this paper for sketches is less efficient than not. However, we realise that some candidates like to be able to draw accurate sketches. We recommend, however, that centres do not automatically issue graph paper to all candidates, as this encourages its use when it is unnecessary.

Comments on Individual Questions

Section A

1 Part (i) was very well answered, with the majority of candidates scoring three marks. Weaker candidates often achieved the 'B' mark for ½ $u^{-1/2}$ without multiplying this by 2.

Part (ii) proved to be a harder test, with many candidates tempted into 'fudges' to get from $1 - e^{-x}$ to $e^x - 1$ in the denominator. Again, weaker candidates scored the 'B' mark for differentiating $\ln(1 - e^{-x})$ to get $\frac{1}{1-e^{-x}}$. Only the better candidates achieved the final 'E' mark for showing that $\frac{e^{-x}}{1-e^{-x}} = \frac{1}{e^x - 1}$ (although this statement without working was allowed).

2 This question was well done generally. The composite function was usually correct, though some candidates did 'f' and 'g' in the wrong order to get $1 - |x|$. We needed some evidence of the values of the intercepts with the axes for the first sketch, though condoned their omission from the sketch of gf(x). Many candidates used graph paper here – we would encourage candidates not to do this for sketches, and therefore discourage centres from routinely handing out graph paper for this paper.

3 Most candidates achieved full marks for part (i). There were instances of sloppy notation, for example $\frac{dy}{dx} = 4y\frac{dy}{dx} + \frac{dy}{dx} = 18x$, which were usually condoned. However, implicit differentiation does of course require accurate deployment of d*y*'s and d*x*'s.

Part (ii) was less successful, with weaker candidates unable to derive or solve the quadratic in *y*. Nevertheless, most candidates scored the first five marks, and many scored full marks.

4 This was another well answered question. There were some generous 'M' marks here, and even poor solutions often achieved 4 or 5 marks out of 8. It is pleasing to note that the large majority of candidates handled the use of logarithms to solve the equations successfully, and the modal mark was 8 out of 8.

5 'Proof' questions have been found difficult in recent papers, but this two-marker proved to be very accessible, with most understanding the concept of counter-example. Some arithmetic errors were made in evaluating the quadratic expression. The most common counter-example was naturally $n = 6$, but $n = 11$ also proved quite popular. Occasionally, candidates went off the rails by trying to do some algebra with the quadratic expression.

6 This proved to be the most demanding of the section A questions – in general, candidates do not find questions on inverse trigonometric functions easy. The first part, in particular, was not well done – many could not find the range of arctan *x* and, even if successful with this, failed to handle arctan ½ *x*. Some then lost marks for using ≤ rather than < in the inequality.

In part (ii), there are still candidates who mix up $\tan^{-1} x$ with 1/tan *x*. In a substantial number of solutions we found $f^{-1}(x) = 2 \tan x$ instead of tan 2*x* – we awarded an M1 mark for this for reversing the arctan. The derivative of tan 2*x* was often not known, with many candidates starting from scratch using a quotient rule on sin 2*x* /cos 2*x*.

In awarding the mark in part (iii), we allowed where possible follow-through on their answers from part (ii). Many changed the sign of the reciprocal, confusing this situation with the condition for perpendicularity of gradients.

Section B

7 There were plenty of easy marks available here, and many weaker candidates scored reasonably on this question.

(i) Most candidates realised that the asymptote occurs when the denominator is zero, and there were many fully correct solutions. However, there was some carelessness with signs, either in re-arranging the equation $1 + 2x^3 = 0$, or in dropping the minus sign subsequently.

(ii) This was a straightforward quotient rule, made easier by the answer being given. We condoned omitted brackets in the numerator (but reserve the right not to do so in future!). However, quotient rules starting $u\,\mathrm{d}v/\mathrm{d}x - v\,\mathrm{d}u/\mathrm{d}x$ achieved M0.

There were plenty of algebraic errors in finding the turning points. The most common mistake was to find an extra turning point, either by setting the denominator to zero, or from $x^3 = 1 \Rightarrow x = \pm 1$. Other errors were $2x - 2x^4 = 2x(1 - x^2)$ or $2x - 2x^4 = 2x(1 - 2x^3)$.

(iii) This proved to be an easy 5 marks for most candidates, either by inspection or substituting $u = 1 + 2x^3$. However, there was quite a lot of sloppy notation, including omitting dx or du, limits incompatible with variable, etc. We would encourage teachers to advise students that this can cause marks to be penalised, although on this paper the mark scheme was generous.

A minority of candidates tried integration by parts here, losing time in the process.

8 This was less well done overall than question 7. Weak candidates still managed to 'cherry-pick' a few marks here and there, but completely correct solutions were relatively few.

(i) This was not as well done as might have been predicted. Using degrees as the default instead of radians was quite common – 45° scored M1M1A0.

(ii) Most scored one mark for $f(-x) = -x\cos(-2x)$. To achieve the 'E' mark required clearly equating this to an expression for $-f(x)$. Attempts based on individual points achieved no marks. The 'B' mark was better done, though quite a few candidates took 'odd' to mean 'not even'.

(iii) The product rule was well done, though some candidates omitted the '2' from the derivative of cos 2x.

(iv) Spotting the connection $\tan 2x = \dfrac{\sin 2x}{\cos 2x}$ was common, and many candidates got the algebra correct.

(v) Most candidates substituted $x = 0$ into their derivative to achieve the first mark – follow-through was allowed on product rule expressions here. However, many missed the further application of the product rule required for the second derivative, or made errors in expanding the bracket, and thereby lost the final 'E' mark, which required them to use the correct second derivative.

(vi) Integration by parts proved to be familiar territory for candidates, even if they had struggled with the differentiation earlier. Sign errors in integrating sin and cos were quite common, and only the better candidates managed to obtain a correct, exact answer. The interpretation of the result as an area, either described verbally or by sketch, was well done, though some candidates tried to link this to the ‘oddness’ of the function.