OCR RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2605

Pure Mathematics 5

Wednesday **14 JANUARY 2004** Morning 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

This question paper consists of 3 printed pages and 1 blank page.

1 **(a)** **(i)** Show that, in general, when a polynomial $\mathrm{p}(x)$ is divided by $(x-a)$, the remainder is $\mathrm{p}(a)$. [2]

When the polynomial $\mathrm{f}(x)$ is divided by $(x-2)$, the remainder is 23.

When $\mathrm{f}(x)$ is divided by $(x+3)$, the remainder is 8.

When $\mathrm{f}(x)$ is divided by $(x-2)(x+3)$, the quotient is $\mathrm{g}(x)$ and the remainder is $px+q$, so that

$$\mathrm{f}(x) \equiv (x-2)(x+3)\mathrm{g}(x) + px + q.$$

(ii) Find p and q. [6]

(iii) Given that $\mathrm{f}(x) = 5x^2 + rx + s$, find r and s. [4]

(b) The equation $x^4 + 4x^2 + 3x - 5 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

Using a substitution, or otherwise, find a quartic equation with integer coefficients which has roots

$$\alpha^2 - 1, \quad \beta^2 - 1, \quad \gamma^2 - 1, \quad \delta^2 - 1.$$ [8]

2 In this question, $z = \cos\theta + \mathrm{j}\sin\theta$, where θ is real.

(i) Simplify $z^n + \dfrac{1}{z^n}$ and $z^n - \dfrac{1}{z^n}$. [3]

(ii) By considering $\left(z - \dfrac{1}{z}\right)^6$, find p, q, r and s such that

$$\sin^6\theta = p\cos 6\theta + q\cos 4\theta + r\cos 2\theta + s.$$ [7]

(iii) Given that x and y are real, show that $\mathrm{Re}(\mathrm{e}^{x+y\mathrm{j}}) = \mathrm{e}^x\cos y$, and find $\mathrm{Im}(\mathrm{e}^{x+y\mathrm{j}})$. [3]

(iv) Write down the first four terms of the Maclaurin series for e^z. [1]

(v) Two infinite series C and S are given by

$$C = 1 + \cos\theta + \frac{\cos 2\theta}{2!} + \frac{\cos 3\theta}{3!} + \dots,$$

$$S = \sin\theta + \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} + \dots.$$

Find C and S. [6]

3 **(i)** Starting from $\cosh u = \frac{1}{2}(e^u + e^{-u})$ and $\sinh u = \frac{1}{2}(e^u - e^{-u})$, prove that

(*A*) $\cosh 2u = 1 + 2\sinh^2 u$,

(*B*) $\operatorname{arcosh} x = \ln(x + \sqrt{x^2 - 1})$. [8]

(ii) Show that $\int_0^{\ln 5} \sinh^2 u \, du = \frac{78}{25} - \frac{1}{2}\ln 5$. [5]

(iii) Show that $\int_{0.625}^{1.3} \frac{1}{\sqrt{4x^2 - 1}} \, dx = \frac{1}{2}(\ln 5 - \ln 2)$. [4]

(iv) Find $\int_{0.5}^{1.3} \sqrt{4x^2 - 1} \, dx$, giving your answer in an exact form. [3]

4 **(a)** A curve has polar equation $r = a(1 - \cos\theta)$, for $0 \leqslant \theta \leqslant 2\pi$, where a is a positive constant.

(i) Sketch the curve. [2]

(ii) Find the area of the region enclosed by the curve. [6]

(b) A parabola has cartesian equation $y^2 = 4ax$.

(i) Find the equation of the normal to the parabola at the point $(at^2, 2at)$. [3]

(ii) Verify that the normal at $(9a, 6a)$ passes through the point P $(15a, -12a)$, and find the coordinates of the other two points on the parabola for which the normal passes through P. [7]

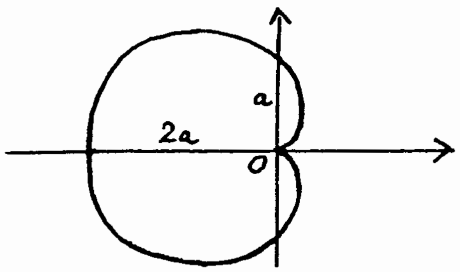
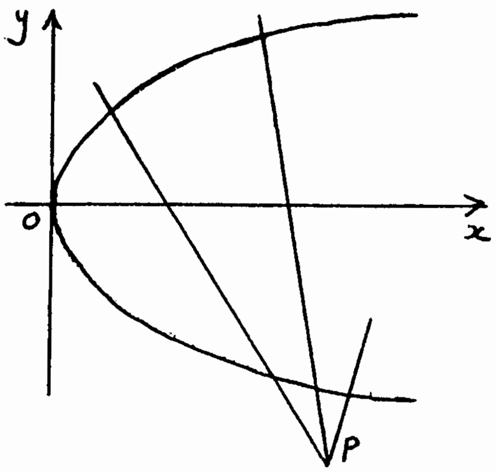
(iii) Draw a sketch showing the parabola and the three normals passing through P. [2]

Mark Scheme

1 (a)(i)	$p(x) = (x-a)q(x) + R$ Putting $x = a$, $p(a) = R$	M1 A1 (ag) **2**	
(ii)	Putting $x = 2$, $2p + q = 23$ Putting $x = -3$, $-3p + q = 8$ $p = 3$, $q = 17$	M1 A1 A1 M1 A1A1 **6**	Substituting $x = 2$ *or* $x = -3$ Solving to obtain p or q
(iii)	$g(x) = 5$, so $f(x) = 5(x-2)(x+3) + 3x + 17$ $= 5x^2 + 5x - 30 + 3x + 17$ $= 5x^2 + 8x - 13$ $r = 8$, $s = -13$	M1 A1 M1 A1 **4**	For $g(x)$ is a constant
	OR $f(2) = 23 \Rightarrow 20 + 2r + s = 23$ M1 $f(-3) = 8 \Rightarrow 45 - 3r + s = 8$ A1 M1 $r = 8$, $s = -13$ A1		Forming one equation Both equations correct Solving to obtain r or s
(b)	Substituting $y = x^2 - 1$ $x = (\pm)\sqrt{y+1}$ $(y+1)^2 + 4(y+1) + 3\sqrt{y+1} - 5 = 0$ $y^2 + 6y = -3\sqrt{y+1}$ $(y^2 + 6y)^2 = 9(y+1)$ $y^4 + 12y^3 + 36y^2 - 9y - 9 = 0$	M1 A1 M1 A1 M1 A1A1 ft A1 **8**	Substituting into equation Obtaining a quartic equation
	OR $\sum\alpha = 0$, $\sum\alpha\beta = 4$, $\sum\alpha\beta\gamma = -3$, $\alpha\beta\gamma\delta = -5$ $\sum\alpha^2 = -8$, $\sum\alpha^2\beta^2 = 6$ $\sum\alpha^2\beta^2\gamma^2 = 49$ B1 $\sum(\alpha^2 - 1) = \sum\alpha^2 - 4 = -12$ B1 $\sum(\alpha^2-1)(\beta^2-1) = \sum\alpha^2\beta^2 - 3\sum\alpha^2 + 6$ M1 $= 36$ A1 $\sum(\alpha^2-1)(\beta^2-1)(\gamma^2-1)$ $= \sum\alpha^2\beta^2\gamma^2 - 2\sum\alpha^2\beta^2 + 3\sum\alpha^2 - 4$ M1 $= 9$ A1 $(\alpha^2-1)(\beta^2-1)(\gamma^2-1)(\delta^2-1)$ $= (\alpha\beta\gamma\delta)^2 - \sum\alpha^2\beta^2\gamma^2 + \sum\alpha^2\beta^2 - \sum\alpha^2 + 1$ M1 $= -9$ Eqn is $y^4 + 12y^3 + 36y^2 - 9y - 9 = 0$ A1		May be implied

2 (i)	$z^n = \cos n\theta + j\sin n\theta$, $\dfrac{1}{z^n} = \cos n\theta - j\sin n\theta$	M1	For either
	$z^n + \dfrac{1}{z^n} = 2\cos n\theta$, $z^n - \dfrac{1}{z^n} = 2j\sin n\theta$	A1A1 **3**	
(ii)	$\left(z - \dfrac{1}{z}\right)^6 = (2j\sin\theta)^6 = -64\sin^6\theta$	B1	
	$\left(z - \dfrac{1}{z}\right)^6 = z^6 - 6z^4 + 15z^2 - 20 + \dfrac{15}{z^2} - \dfrac{6}{z^4} + \dfrac{1}{z^6}$	M1A1	
	$= 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$	A3	Give A1 for one cos term correct A2 for two cos terms correct
	$\sin^6\theta = -\frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta - \frac{15}{32}\cos 2\theta + \frac{5}{16}$	A1 **7**	
(iii)	$e^{x+yj} = e^x e^{yj} = e^x(\cos y + j\sin y)$	M1	
	$\text{Re}(e^{x+yj}) = e^x \cos y$, $\text{Im}(e^{x+yj}) = e^x \sin y$	A1(ag)A1 **3**	
(iv)	$e^z = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \ldots$	B1 **1**	
(v)	$C + jS = 1 + z + \dfrac{z^2}{2!} + \ldots$	M1	
	$= e^z$	A1	
	$= e^{\cos\theta + j\sin\theta} = e^{\cos\theta}[\cos(\sin\theta) + j\sin(\sin\theta)]$	A1	
		M1	Equating real or imaginary parts
	$C = e^{\cos\theta}\cos(\sin\theta)$	A1	
	$S = e^{\cos\theta}\sin(\sin\theta)$	A1 **6**	

3(i)(*A*)	$1+2\sinh^2 u = 1+\frac{1}{2}(e^{2u}-2+e^{-2u})$	B1	For $e^{2u}-2+e^{-2u}$
	$=\frac{1}{2}(e^{2u}+e^{-2u})$	B1	Using $\cosh 2u=\frac{1}{2}(e^{2u}+e^{-2u})$
	$=\cosh 2u$	B1 (ag)	For completion
(*B*)	Let $y=\operatorname{arcosh} x$, $x=\cosh y=\frac{1}{2}(e^{y}+e^{-y})$	M1	
	$e^{2y}-2xe^{y}+1=0$	M1	
	$e^{y}=x\pm\sqrt{x^2-1}$	M1	
	$y=\ln(x\pm\sqrt{x^2-1})$	A1	
	Since $\operatorname{arcosh} x>0$, $\operatorname{arcosh} x=y=\ln(x+\sqrt{x^2-1})$	A1 (ag) **8**	
(ii)	$\int_0^{\ln 5}\sinh^2 u\,du=\int_0^{\ln 5}\frac{1}{2}(\cosh 2u-1)\,du$	M1	
	$=\left[\frac{1}{4}\sinh 2u-\frac{1}{2}u\right]_0^{\ln 5}$	A1A1	
	$=\frac{1}{4}\sinh(2\ln 5)-\frac{1}{2}\ln 5$	M1	Substitution of limits
	$=\frac{1}{8}(25-\frac{1}{25})-\frac{1}{2}\ln 5$		
	$=\frac{78}{25}-\frac{1}{2}\ln 5$	A1 (ag) **5**	Exact evaluation of $\sinh(2\ln 5)$ must be shown
(iii)	$\int_{0.625}^{1.3}\frac{1}{\sqrt{4x^2-1}}dx=\left[\frac{1}{2}\operatorname{arcosh} 2x\right]_{0.625}^{1.3}$	B1B1	For $\frac{1}{2}$ and $\operatorname{arcosh} 2x$ *or* $\frac{1}{2}\ln(2x+\sqrt{4x^2-1})$ *or* $\frac{1}{2}\ln(x+\sqrt{x^2-\frac{1}{4}})$
	$=\frac{1}{2}(\operatorname{arcosh} 2.6-\operatorname{arcosh} 1.25)$		
	$=\frac{1}{2}[\ln(2.6+\sqrt{2.6^2-1})-\ln(1.25+\sqrt{1.25^2-1})]$	M1	
	$=\frac{1}{2}(\ln 5-\ln 2)$	A1 (ag) **4**	
(iv)	Putting $2x=\cosh u$,	M1	
	$\int_{0.5}^{1.3}\sqrt{4x^2-1}\,dx=\int_0^{\ln 5}\sinh u(\frac{1}{2}\sinh u)\,du$	A1	For $\int\frac{1}{2}\sinh^2 u\,du$
	$=\frac{39}{25}-\frac{1}{4}\ln 5$	A1 **3**	

<table>
<tr><td>4(a)(i)</td><td></td><td>B1
B1
2</td><td>Correct shape in correct position
Sharp cusp and indication of scale</td></tr>
<tr><td>(ii)</td><td>Area is $\int_0^{2\pi} \frac{1}{2}a^2(1-\cos\theta)^2\,d\theta$
$= \frac{1}{2}a^2\int_0^{2\pi}(1-2\cos\theta+\frac{1}{2}(1+\cos 2\theta))\,d\theta$
$= \frac{1}{2}a^2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi}$
$= \frac{3}{2}\pi a^2$</td><td>M1
A1
B1
B1B1 ft
A1
6</td><td>For $\int(1-\cos\theta)^2\,d\theta$
Use of $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$
Integration of $a + b\cos\theta$ and $c\cos 2\theta$</td></tr>
<tr><td>(b)(i)</td><td>$\frac{dy}{dx} = \frac{2a}{2at}$
Gradient of normal is $-t$
Equation of normal is $y - 2at = -t(x - at^2)$
$tx + y = at^3 + 2at$</td><td>M1
A1
A1
3</td><td>Or $2y\frac{dy}{dx} = 4a$, etc</td></tr>
<tr><td>(ii)</td><td>Normal passes through P if
$15at - 12a = at^3 + 2at$
$t^3 - 13t + 12 = 0$
$t = 1,\ 3,\ -4$
$t = 3$ gives $(9a,\ 6a)$
$t = 1$ gives $(a,\ 2a)$
$t = -4$ gives $(16a,\ -8a)$</td><td>M1A1
M1
M1
B1 (ag)
A1
A1
7</td><td>Obtaining at least one value of t
Using a value of t to obtain a point
Can also be awarded for showing that $t = 3,\ x = 15a,\ y = -12a$ satisfies equation of normal</td></tr>
<tr><td>(iii)</td><td></td><td>B1
B1
2</td><td>Parabola, and point P below it
Three normals</td></tr>
</table>

Examiner’s Report

2605 Pure Mathematics 5

General Comments

There were many excellent scripts, with about a quarter of the candidates scoring 50 marks or more (out of 60). On the other hand, about a quarter scored 30 marks or fewer. Question 4 was considerably less popular than the other three questions.

Comments on Individual Questions

Question 1 (Algebra)

This was the best answered question, with half the attempts scoring 16 marks or more (out of 20) and about 20% scoring full marks. The remainder theorem was applied accurately in parts (a)(ii) and (iii), although many could not provide the proof in part (a)(i), often because they assumed a linear remainder instead of a constant. Only a few answered part (a)(iii) by putting $g(x) = 5$ into the previous result; most started afresh, using the remainder theorem to produce simultaneous equations. In part (b), the method of substitution was generally well understood, although many used incorrect methods for removing the fractional powers, such as squaring each term in the equation. There were also several attempts to use sums of products of roots, but few of these made substantial progress.

(a)(ii) $p = 3$, $q = 17$; (iii) $r = 8$, $s = -13$; (b) $y^4 + 12y^3 + 36y^2 - 9y - 9 = 0$.

Question 2 (Complex Numbers)

All parts of this question were generally well understood. In part (ii) there were many sign errors, in the binomial expansion, or writing the given expression as $64\sin^6\theta$ instead of $-64\sin^6\theta$. Factors of 2 were sometimes missing from one or more of the cosine terms. A few candidates insisted on considering $(\cos\theta + j\sin\theta)^6$, but these were unable to make any progress. In part (v), a surprising number thought that the series was geometric, despite the strong hint in part (iv).

(i) $2\cos n\theta$, $2j\sin n\theta$; (ii) $p = -\frac{1}{32}$, $q = \frac{3}{16}$, $r = -\frac{15}{32}$, $s = \frac{5}{16}$; (iii) $e^x\sin y$;

(iv) $1 + z + \frac{z^2}{2!} + \frac{z^3}{3!}$; (v) $C = e^{\cos\theta}\cos(\sin\theta)$, $S = e^{\cos\theta}\sin(\sin\theta)$.

Question 3 (Hyperbolic functions)

Part (i) was answered very well, except that the reason for rejecting the unwanted root was often omitted, or given as 'you cannot have the logarithm of a negative number', instead of $\text{arcosh}x \geq 0$. Most candidates obtained the results in parts (ii) and (iii), but since the answers were given, several lost marks for showing insufficient working (for example, $\sinh(2\ln 5) = 12.48$ needs to be justified as an exact result). In part (iv), those who made the substitution $2x = \cosh u$ and obtained half the integral in part (ii) were able to complete the question quickly, but other methods often led to very lengthy work which was only occasionally successful.

(iv) $\frac{39}{25} - \frac{1}{4}\ln 5$.

Question 4 (Conics)

This question was attempted by only about one third of the candidates, and it was the worst answered question, with a mean mark of about 11. Parts (a) and (b)(i) were answered quite well, but in part (b)(ii) few candidates realised how to use the equation of the general normal to find the other two points for which the normal passes through P.

(a)(ii) $\frac{3}{2}\pi a^2$; (b)(i) $tx + y = at^3 + 2at$; (ii) $(a, 2a)$, $(16a, -8a)$.