OCR RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2603(A)

Pure Mathematics 3

Section A

Monday **12 JANUARY 2004** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer **all** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

NOTE

- This paper will be followed by **Section B: Comprehension.**

This question paper consists of 4 printed pages.

1 (a) Given that $x = t^3$ and $y = \sqrt{1+t}$, find $\frac{dy}{dx}$ in terms of t. [4]

(b) Find the first four terms in ascending powers of x of the binomial expansion of $\frac{1}{(1+2x)^3}$.

State the range of values of x for which the expansion is valid. [5]

(c) Express $\tan\left(x + \frac{\pi}{4}\right)$ in terms of $\tan x$.

Hence show that $\tan\left(x + \frac{\pi}{4}\right)\tan\left(x - \frac{\pi}{4}\right) = -1$. [4]

(d) Using small angle approximations for sine and cosine, show that, for small values of x,

$$\frac{1 - \cos x}{x \sin 2x} \approx \frac{1}{4}.$$ [3]

[Total 16]

2 (i) Given that $\frac{1}{x(1+x)^2} \equiv \frac{A}{x} + \frac{B}{1+x} + \frac{C}{(1+x)^2}$, find A, B and C. [5]

(ii) Hence show by integration that the differential equation

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x(1+x)^2}$$

has the general solution

$$\ln y = \ln\left(\frac{x}{1+x}\right) + \frac{1}{1+x} + c.$$ [5]

(iii) Given that $y = \frac{1}{2}$ when $x = 1$, find the exact value of c. Find also the value of y when $x = 2$, giving your answer to 3 significant figures. [4]

[Total 14]

3 Fig. 3 shows part of the curve $y = x - 2\sin x$. P is a minimum point on the curve.

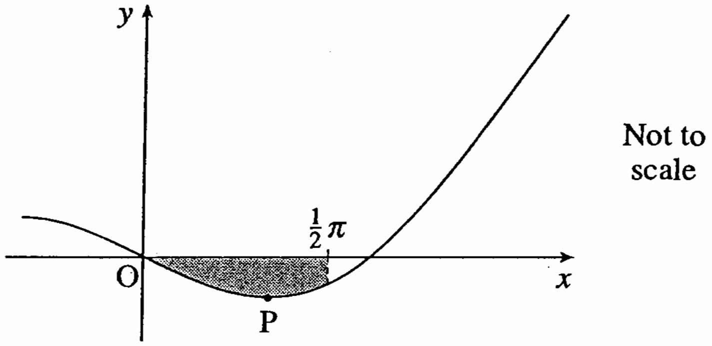

Fig. 3

(i) Show that the x-coordinate of P is $\frac{\pi}{3}$. [3]

(ii) Evaluate $\int_0^{\frac{1}{2}\pi} x\sin x\,dx$. [4]

(iii) Show that $\int_0^{\frac{1}{2}\pi} \sin^2 x\,dx = \frac{\pi}{4}$. [4]

(iv) The shaded region bounded by the curve, the x-axis and the line $x = \frac{1}{2}\pi$ is rotated through 360° about the x-axis. Using your results from parts **(ii)** and **(iii)**, find the volume of the solid of revolution formed, giving your answer in terms of π. [4]

[Total 15]

4 Fig. 4 shows a tetrahedron ABCD with vertices $A(1,0,0)$, $B(0,1,0)$, $C(0,0,2)$ and $D(-2,0,0)$.

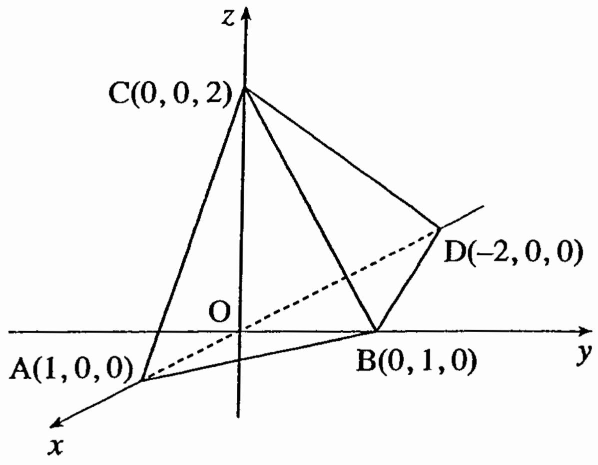

Fig. 4

(i) Find the scalar product $\overrightarrow{BA}.\overrightarrow{BC}$. Hence or otherwise find the angle ABC. [4]

(ii) Verify that the equation of the plane ABC is $2x + 2y + z = 2$. Write down a vector normal to the plane ABC. [4]

(iii) Write down a vector equation of the perpendicular l from the point D to the plane ABC. [2]

(iv) By finding where l meets the plane, find the distance from D to the plane ABC. [5]

[Total 15]

Mark Scheme

1 (a) $x = t^3 \Rightarrow \frac{dx}{dt} = 3t^2$ $y = \sqrt{1+t} \Rightarrow \frac{dy}{dt} = \frac{1}{2}(1+t)^{-1/2}$ $\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ $= \frac{\frac{1}{2}(1+t)^{-1/2}}{3t^2}$ $= \frac{1}{6t^2\sqrt{1+t}}$	B1 B1 M1 A1 [4]	$\frac{\text{their } dy/dt}{\text{their } dx/dt}$ soi Any correct form of the answer in which the 3 and the two have been combined as 6. Allow negative indices.
(b) $\frac{1}{(1+2x)^3} = (1+2x)^{-3}$ $= 1 + (-3)(2x) + \frac{(-3)(-4)}{2!}(2x)^2 + \frac{(-3)(-4)(-5)}{3!}(2x)^3 + \ldots$ $= 1 - 6x + 24x^2 - 80x^3 + \ldots$ Valid for $-1 < 2x < 1 \Rightarrow -½ < x < ½$ or $\lvert x \rvert < 1/2$	M1 B1 B1 B1 B1 [5]	All the binomial coeffs correct $-6x$ $+24x^2$ $-80x^3$ $\le$ is B0
(c) $\tan(x + \frac{\pi}{4}) = \frac{\tan x + \tan\frac{\pi}{4}}{1 - \tan x \tan\frac{\pi}{4}}$ $= \frac{\tan x + 1}{1 - \tan x}$ $\tan(x+\frac{\pi}{4})\tan(x-\frac{\pi}{4}) = \frac{\tan x + 1}{1 - \tan x}.\frac{\tan x - 1}{1 + \tan x}$ $= -1$*	M1 A1 M1 E1 [4]	Correct formula for tan $(A + B)$ used Allow retrospectively tan $(A - B)$ formula used www
(d) $\frac{1-\cos x}{x \sin 2x} \approx \frac{1-(1-\frac{x^2}{2})}{x.2x}$ $= \frac{\frac{x^2}{2}}{2x^2}$ $= 1/4$*	M1 M1 E1 [3] [Total 16]	$\cos x \approx 1 - x^2/2$ used $\sin 2x \approx 2x$ or $\sin x \approx x$ used Condone the absence of brackets if the subsequent working is correct.

2 (i) $\frac{1}{x(1+x)^2} \equiv \frac{A}{x} + \frac{B}{1+x} + \frac{C}{(1+x)^2}$ $\Rightarrow 1 \equiv A(1+x)^2 + Bx(1+x) + Cx$ $x = 0 \Rightarrow 1 = A$ $x = -1 \Rightarrow 1 = -C \Rightarrow C = -1$ coeff of x^2: $0 = A + B \Rightarrow B = -1$	M1 M1 A1 A1 A1 [5]	Correct identity. Any correct substitution or correct equating of coefficients or any equivalent valid method. All the coefficients correct implies both Ms $A = 1$ The accuracy marks are $B = -1$ dependent on the first M1 $C = -1$
(ii) $\frac{1}{y}\frac{dy}{dx} = \frac{1}{x(1+x)^2}$ $\Rightarrow \int \frac{1}{y}dy = \int \frac{1}{x(1+x)^2}dx$ $= \int [\frac{1}{x} - \frac{1}{1+x} - \frac{1}{(1+x)^2}]dx$ $\Rightarrow \ln y = \ln x - \ln(1+x) + \frac{1}{1+x} + c$ $= \ln\frac{x}{1+x} + \frac{1}{1+x} + c$ *	M1 M1 B1ft B1ft E1 [5]	Separating the variables. Condone poor notation providing that the intention is clear. substituting their partial fractions $\ln x - \ln(1+x)$ $\frac{1}{1+x}$ condone the absence of c follow thro' their integral providing that the M1s have been earned. www. $\ln y$ and c must be seen
(iii) substituting $x = 1$ and $y = ½$: $\Rightarrow \ln ½ = \ln ½ + ½ + c$ $\Rightarrow c = -½$ when $x = 2$ $\ln y = \ln 2/3 + 1/3 - ½$ $\Rightarrow y = 0.564$ (3 s.f.)	M1 A1 M1 A1 [4] [Total 14]	Allow both M marks if the candidate uses his own equation providing the work is comparable cao

3 (i) $y = x - 2\sin x$ $\Rightarrow \frac{dy}{dx} = 1 - 2\cos x$ $= 0$ $\Rightarrow \cos x = ½$ $\Rightarrow x = \pi/3$*	B1 M1 E1 [3]	$\frac{dy}{dx} = 1 - 2\cos x$ their $\frac{dy}{dx} = 0$ cos x = ½ must be seen, but allow verification
(ii) $\int_0^{\frac{\pi}{2}} x\sin x\,dx$ Let $u = x$, $dv/dx = \sin x$ $\Rightarrow v = -\cos x$ $= [-x\cos x]_0^{\pi/2} + \int_0^{\frac{\pi}{2}} \cos x.1dx$ $= 0 + [\sin x]_0^{\pi/2}$ $= 1$.	M1 A1 A1 A1 [4]	Correct choice of parts and an attempt at $[-x\cos x] + \int$ ..... $-x\cos x$ cao $\sin x$ cao cao
(iii) $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$ $\cos 2x = 1 - 2\sin^2 x$ $\Rightarrow \sin^2 x = ½ (1 - \cos 2x)$ $= \int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos 2x)\,dx$ $= \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi/2}$ $= \pi/4$*	M1 A1 A1ft E1 [4]	A reasonable attempt to use cos $2x$ $\int 1/2(1-\cos 2x)dx$ $\left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]$ www
(iv) $V = \int_0^{\pi/2} \pi y^2 dx$ $= \int_0^{\pi/2} \pi(x - 2\sin x)^2 dx$ $= \pi \int_0^{\pi/2} (x^2 - 4x\sin x + 4\sin^2 x)dx$ $= \pi\left[\frac{x^3}{3}\right]_0^{\pi/2} - 4\pi + 4\pi.\frac{\pi}{4}$ $= \frac{\pi^4}{24} - 4\pi + \pi^2$	M1 B1 M1 A1 [4] Total [15]	$\int_0^{\pi/2} \pi y^2 dx$ Correct limits and π seen Correct expansion of the bracket $\left[\frac{x^3}{3}\right]$ + their result from (ii) and the above result from (iii) substituted. or equivalent.

4(i) $\vec{BA} = \begin{pmatrix}1\\0\\0\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$ $\vec{BC} = \begin{pmatrix}0\\0\\2\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\\2\end{pmatrix}$ $\Rightarrow \vec{BA}.\vec{BC} = 1\times0 + (-1)\times(-1) + 0\times2$ $= 1$ $\cos ABC = \frac{\vec{BA}.\vec{BC}}{\lvert BA\rvert\lvert BC\rvert} = \frac{1}{\sqrt{2}.\sqrt{5}}$ $\Rightarrow ABC = 71.6°$	M1 A1 M1 A1[4]	Allow any two vectors Accept omission of working or correct scalar product seen in the formula for cos ABC
(ii) $2x + 2y + z = 2$ At A (1,0,0) $2\times1 + 2\times0 + 0 = 2$ at B (0,1,0) $2\times0 + 2\times1 + 0 = 2$ at C (0,0,2) $2\times0 + 2\times0 + 2 = 2$ $\Rightarrow$ Equation of the plane ABC is $2x + 2y + z = 2$ Normal vector is $\begin{pmatrix}2\\2\\1\end{pmatrix}$ Alternative schemes <u>Vector equation :</u> M1 for the correct form of a vector equation of the plane. M1 for the complete elimination of two parameters. A1 for deducing the correct equation of the plane B1 for the normal vector.	M1 B2,1.0 B1 [4]	Substituting the coordinates of one point into the eq'n. Accept eg 2+0+0 = 2 but not 2=2. $k\begin{pmatrix}2\\2\\1\end{pmatrix}$ Accept a row vector, but not the equation of a particular normal <u>The use of a vector product</u>: M1 A1 for the correct vector product of two vectors in the plane. B1 for identifying the normal A1 for the RHS of the equation <u>The use of scalar products:</u> B1 for the normal vector. M1 for the scalar product of the normal and one vector in the plane. A1 for two such products = 0 A1 for the RHS of the equation
(iii) $\mathbf{r} = \begin{pmatrix}-2\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\2\\1\end{pmatrix}$	B1 B1ft [2]	$\begin{pmatrix}-2\\0\\0\end{pmatrix} + \ldots$ $\ldots + \lambda\begin{pmatrix}2\\2\\1\end{pmatrix}$ Condone the omission of $\mathbf{r} =$
(iv) $\mathbf{r} = \begin{pmatrix}-2+2\lambda\\2\lambda\\\lambda\end{pmatrix}$: $2(-2+2\lambda) + 4\lambda + \lambda = 2$ $\Rightarrow 9\lambda = 6$ $\Rightarrow \lambda = 2/3$ So line meets plane at $(-\frac{2}{3}, \frac{4}{3}, \frac{2}{3})$. Distance $= \sqrt{(-2+\frac{2}{3})^2 + (-\frac{4}{3})^2 + (-\frac{2}{3})^2} = 2$	M1 A1 A1ft M1 A1 [5]	ft their equation in (iii) ft their λ and their equation Or $\left\lvert \frac{2\times(-2) + 2\times0 + 1\times0 - 2}{\sqrt{2^2 + 2^2 + 1^2}} \right\rvert = 2$

Examiner’s Report

2603 Pure Mathematics 3

General Comments

There was a very good response to this paper with a high proportion of candidates scoring very good marks in the 60+ range. Of particular note was the number of candidates who obtained full marks on both sections A and B. There was a small number of candidates, scoring less than 20 marks, who perhaps were not yet ready to take this unit. The presentation of work was generally good and the work of many of the high scoring candidates was immaculate. Weaker candidates continue to suffer from poor notation, particularly in integration and work with vectors.

The response to the Section B comprehension was good. The questions that required arithmetic or algebra to answer them were very well answered, but where candidates needed to express their understanding of parts of the passage some were unable to do so adequately.

There was no evidence that any candidates were short of time in either section, nor were there many misinterpretations of questions.

Comments on Individual Questions

Question 1 (Various topics)

Most candidates attempted most of this question and, on average, it was the highest scoring question on the paper.

(a) A few candidates attempted to eliminate the parameter t and then find dy/dx directly in terms of x, but most made some error along the way and few returned to t. The great majority proceeded in the accepted way by differentiating with respect to t. The resulting negative index in dy/dt caused a problem for a few, eg $\frac{\frac{1}{2}(1+t)^{-\frac{1}{2}}}{3t^2} = \frac{\frac{1}{2\sqrt{1+t}}}{3t^2} = \frac{3t^2}{2\sqrt{1+t}}$, but perhaps the most common loss of a mark was in not combining the 2 and the 3 in the denominator of the result.

(b) The binomial expansion was very well done, just a few candidates confusing $(2x)^2$ with $2x^2$; but many candidates were unable to write down the correct range of validity, and $\leq$ was often seen.

(c) Most candidates were able to obtain the correct expressions for $\tan(x + \pi/4)$ and $\tan(x - \pi/4)$, although a few failed to make the substitution $\tan \pi/4 = 1$. However, a number of candidates spoiled their solution by clearly incorrect cancelling.

(d) Here again although most candidates knew the approximations for $\sin x$ and $\cos x$, some candidates lost the final mark because of incorrect simplification. One such error was $1 - (1 - \frac{x^2}{2}) = 1 - 1 - \frac{x^2}{2}$ or even $\frac{1-1-x^2}{2}$. Quite a large number of candidates used $\sin 2x = 2\sin x \cos x$ and needed to disregard x^4.

(a) $\frac{1}{6t^2\sqrt{1+t}}$; (b) $1 - 6x + 24x^2 - 80x^3 + \ldots$ Valid for $-\frac{1}{2} < x < \frac{1}{2}$

Question 2 (Partial fractions and differential equations)

Part (i) The work here was generally sound and many candidates obtained full marks. However a significant number gave the wrong identity, most commonly

$$1 \equiv A(1+x)(1+x)^2 + Bx(1+x)^2 + Cx(1+x),$$

but also, $1 \equiv A(1+x^2) + Bx(1+x) + Cx$.

Part (ii) Again there were many good, completely correct solutions showing full details of the separation of the variables, the substitution of the partial fractions and the integration. However many candidates gave the impression that they had worked backwards from the given solution; ln y appeared without any $\int \frac{1}{y}dy$, and integral signs were placed in front of the partial fractions without any dx or other indication as to why they should be integrated. The final fraction $\frac{1}{1+x}$ was quite often seen in the solution without any further explanation even, sometimes, when there was an error in the sign of the corresponding partial fraction.

Part (iii) was well done even by many weaker candidates. There were occasional sign errors, such as $\ln \frac{1}{2} = \ln \frac{1}{2} + \frac{1}{2} + c \Rightarrow c = \frac{1}{2}$, but a more common error was in the next line when substituting the value $x = 2$, $\ln y = \ln \frac{2}{3} + \frac{1}{3} - \frac{1}{2} \Rightarrow y = \frac{2}{3} + e^{\frac{1}{3}} - e^{\frac{1}{2}}$.

(i) $A = 1$, $B = -1$, $C = -1$; (iii) 0.564

Question 3 (Integration)

Part (i) This was very well answered by almost all candidates. Just an occasional candidate gave $\frac{dy}{dx}$ as $1 + 2\cos x$. A small number of candidates verified the result, which was acceptable.

Part (ii) Also well done. Again just a few candidates had a sign error, this time integrating $\sin x$ to $\cos x$.

Part (iii) This was not well answered except by the most able candidates. Some had sign errors in the use of the double angle formula; some, starting from $\cos 2x = 1 - 2\sin^2 x$, had algebraic errors in obtaining $\sin^2 x$ and some attempted integration by substitution obtaining $\frac{1}{3}\sin^3 x$.

Part (iv) Only a few good candidates were able to complete this. Candidates either did not know the formula for the volume of a solid of revolution, or made an error in squaring y or omitted the π.

(ii) 1; (iv) $\frac{\pi^4}{24} - 4\pi + \pi^2$

Question 4 (Vectors)

This question was mostly well answered.

Part (i) A few candidates got **BA** or **BC** or both the wrong way round, and occasionally notation was poor, **BA** . **BC** = $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 1$.

Part (ii) A variety of methods was used in this part, most of them correctly. Writing down a vector equation of the plane ABC and eliminating the two parameters was very popular. In this case the elimination was quite easy and therefore nearly always correctly done. The straightforward substitution of the coordinates of points A, B and C was also done correctly by many. A variation of this using scalar products, eg $\begin{pmatrix}1\\0\\0\end{pmatrix}.\begin{pmatrix}2\\2\\1\end{pmatrix}=2$, was not always successful because it was not repeated for the other two points. Similarly a few candidates showing that $\mathbf{BA}.\begin{pmatrix}2\\2\\1\end{pmatrix}=0$ and therefore BA is perpendicular to the normal to the plane, failed to repeat the process for **BC** or **CA**.

Some further mathematics candidates used the vector product of, eg **BA** and **BC**, to find the normal to the plane and then calculated *d*. Occasionally such candidates failed to identify the normal as such.

Part (iii) was an easy write down for most candidates.

Part (iv) Well done by many candidates. Having obtained the point of intersection of the perpendicular from D to the plane some candidates found the distance of this point from the origin instead of the length of the perpendicular. Others made an error in subtracting the coordinates of this point from D.

(i) 71.6°; (ii) $\begin{pmatrix}2\\2\\1\end{pmatrix}$; (iii) $\mathbf{r}=\begin{pmatrix}-2\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\2\\1\end{pmatrix}$; (iv) 2

Section B (Comprehension)

Q1 Almost all candidates read the graph correctly but some failed to change the percentage of patients with rubella into the *number* of 15-19 year olds.

Q2 This question was usually well answered. Just a few candidates were not convincing with their argument for the final inequality.

Q3 This proved to be the most difficult question. A significant number of candidates failed to understand that if the population of animals does not have herd immunity there is a minimum number of animals which are susceptible to the disease. These candidates wrote generally about how the number of susceptible animals changes over the course of an epidemic. However, those candidates who started, correctly, from the statement that for herd immunity $sR_0 < 1$ usually made some progress. A few of these candidates referred back to question 2 and used $i > 0.9$, and a few confused s and i. Others reached the correct conclusion that $s \geq 1/15$ and then failed to convert from the proportion of the population to the number of animals.

Q4 was very well answered indeed, almost all candidates scoring the four marks available.

Q5 This question required accurate use of language and quite a large number of candidates were not up to it. Also, it was clear that many candidates were thinking in terms of just two periods instead of a number of periods in an iterative model. Nevertheless there were many good answers.

1. 108 (104-111); 3. $S \geq 400$; 4. 4 years