OCR

RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2623/1

Numerical Methods

Wednesday **21 JANUARY 2004** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer **all** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

This question paper consists of 3 printed pages and 1 blank page.

1 The expression $(1+p)^n$ is sometimes approximated by e^{np} when n is large and p is close to zero.

(i) Copy and complete the following table.

n	p	$\frac{e^{np}}{(1+p)^n}$	relative error
10	0.05	1.012172	
50	0.05		
10	0.02		
50	0.02		

State how the accuracy of the approximation appears to change as

(*A*) n increases but p is held constant,

(*B*) p gets closer to zero but n is held constant. [7]

(ii) Now suppose that n and p are both varied but the product, np, is kept equal to 0.1. Starting with $n = 10$ and $p = 0.01$, determine how the accuracy changes as n increases. [4]

(iii) It is known that $(1+p)^n \approx 1 + np + \frac{1}{2}n(n-1)p^2$ and $e^{np} \approx 1 + np + \frac{1}{2}n^2p^2$. Obtain an approximate expression for the absolute error in the approximation and show that it will tend to zero if np is kept constant and n increases. [4]

2 (i) Show that the equation

$$\frac{x(x^{10}-1)}{x-1} = 14$$

has a root in the interval (1.01, 1.09). [2]

(ii) Use the bisection method to find an interval (a, b) that contains the root and for which $b - a = 0.01$.

Give the best possible estimate of the root at this stage, and state the maximum possible error in that estimate. [7]

(iii) Use a single application of linear interpolation on the interval (a, b) to obtain a further estimate of the root.

Determine whether or not this estimate is accurate to 5 decimal places. [6]

3 A function $f(x)$ has values correct to 3 decimal places as shown in the table.

x	0	2	4
$f(x)$	7.389	11.023	16.445

The value of $\int_0^4 f(x)\,dx$ is denoted by I.

(i) Obtain estimates of I using

(*A*) the trapezium rule and the ordinates $f(0)$ and $f(4)$ only,

(*B*) the mid-point rule,

(*C*) Simpson's rule. [6]

(ii) You are now given further values of the function, also correct to 3 decimal places, as shown.

x	0.5	1	1.5	2.5	3	3.5
$f(x)$	8.166	9.025	9.974	12.182	13.464	14.880

Find two further Simpson's rule estimates of I. [6]

(iii) Hence give the best estimate of I you can. Justify the number of significant figures to which you give your answer. [3]

4 In this question, $f(x) = x^x$.

(i) Use your calculator to evaluate $f(x)$ at $x = 10^{-3}, 10^{-5}, 10^{-7}, 10^{-9}$. [3]

You are now *given* that $f(0) = 1$.

(ii) Use the forward difference method with $h = 10^{-3}$ to estimate $f'(0)$.

Obtain further estimates of $f'(0)$ by taking $h = 10^{-5}, 10^{-7}, 10^{-9}$.

Comment on the sequence of estimates in relation to the likely value of $f'(0)$. [7]

(iii) A cheap calculator gives powers (such as x^x) rounded to 5 significant figures. What conclusions might be drawn by someone using such a calculator to carry out the processes in part **(i)** and part **(ii)**?

What is the relevance of this result for more accurate calculators and computers? [5]

Mark Scheme

1 (i)

n	p	$e^{np}/(1+p)^n$	rel error		
10	0.05	1.012172	**0.012172**		**[M1A1]**
50	0.05	**1.062359**	**0.062359**	*sc: [2] for 3rd*	**[A1]**
10	0.02	**1.001976**	**0.001976**	*column only*	**[A1]**
50	0.02	**1.009917**	**0.009917**		**[A1]**

(A) relative error increases or accuracy decreases **[E1]**

(B) relative error decreases or accuracy increases **[E1]**

[subtotal 7]

(ii) EG:

n	p	$e^{np}/(1+p)^n$	rel error		
10	0.01	1.000497	0.000497		
20	0.005	1.000249	0.000249	1st comparison	**[M1A1]**
40	0.0025	1.000125	0.000125	confirmation	**[A1]**

relative error decreases (accuracy increases) with increasing n **[E1]**

[subtotal 4]

(iii) Error approximately $0.5np^2$ **[M1A1]**

As n tends to infinity, p tends to zero (may be implied) **[M1]**

This is 0.5 x constant x p, which tends to zero as p tends to zero **[A1]**

Allow a sequence of evaluations (with explanation/comment) for last [2] ***[subtotal 4]***

[TOTAL 15]

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2 (i)

x	f(x)	
1.01	10.56683	< 14
1.09	16.56029	> 14

[M1A1]

[subtotal 2]

(ii)

a	b	x	f(x)	
1.01	1.09	1.05	13.20679	**[M1A1]**
1.05	1.09	1.07	14.7836	**[M1A1]**
1.05	1.07	1.06	13.97164	
1.06	1.07	– which is the required interval		**[A1]**

Best estimate 1.065 with mpe 0.005 **[A1A1]**

[subtotal 7]

(iii)

a	b	f(a) - 14	f(b) - 14	
1.06	1.07	-0.02836	0.783599	**[A1]**

linear interpolation: (af(b)-b(f(a))/(f(b)-f(a))= 1.060349 I.e. 1.06035 to 5 dp **[M1A1]**

x	f(x)		
1.060345	13.99885	< 14	
1.060355	13.99964	< 14	not accurate to 5 dp

[M1A1E1]

[subtotal 6]

[TOTAL 15]

3 (i)

F0	F2	F4
7.389	11.023	16.445

T1 = (4/2)(F0+F4) =	47.668	[M1A1]
M1 = 4F2 =	44.092	[M1A1]
S1 = (2M1+T1)/3 =	45.284	[M1A1]

[subtotal 6]

(ii)

F0.5	F1	F1.5	F2.5	F3	F3.5
8.166	9.025	9.974	12.182	13.464	14.88

T2 = (M1+T1)/2 =	45.88		[M1]
M2 = 2(F1+F3) =	44.978	(or [M1M1A1] for direct	[M1]
S2 = (2M2+T2)/3 =	45.27867	Simpson's rule)	[A1]
T4 = (M2+T2)/2 =	45.429		[M1]
M4 = (F0.5...F3.5) =	45.202	(or [M1A1A1] for direct	[A1]
S4 = (2M4+T4)/3 =	45.27767	Simpson's rule)	[A1]

[subtotal 6]

(iii)

S1	45.284	diffs		
S2	45.27867	-0.00533	3 dp looks secure BUT the data are to 3 dp	[M1M1]
S4	45.27767	-0.001	so only 2 dp: 45.28	[B1]

Allow informal comparison of S values for first [M1] *[subtotal 3]*

[TOTAL 15]

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4 (i)

x	1.E-03	1.E-05	1E-07	1E-09
x^x	0.993116	0.999885	0.999998	0.999999979

[B1B1B1]
[subtotal 3]

(ii) (x^x-1)/x -6.88395 -11.5123 -16.1181 -20.7233 **[M1A1A1A1]**
The estimates are getting larger in magnitude (with large increases). **[E1]**
Not converging. Can't estimate f'(0) (or tending to minus infinity). **[E1E1]**
[subtotal 7]

(iii)

x	1.E-03	1.E-05	1E-07	1E-09	
x^x	0.99312	0.99988	1.0000	1.0000	to 5 sf
(x^x-1)/x	-6.88	-12	0	0	

[M1A1] [A1]

This calculator would indicate a converging sequence beyond x=10^-7 **[E1]**
The [M1] is for considering what happens to x^x AND (x^x-1)/x when x is small.
The table is not required, but must consider more than one vaklue of x.
ALL calculators and computers have this problem at the limits of their accuracy. **[E1]**
[subtotal 5]

[TOTAL 15]

Examiner’s Report

Numerical Methods (2623)

Generally, the work seen was of a high standard with candidates attempting suitable tasks. A few candidates, however, did work which gave them little scope for appropriate development in the error analysis section (usually on polynomial interpolation) and in some cases a small downward adjustment of marks reflecting this was necessary. Most candidates chose numerical integration as the most fertile area for project work, and many were able to demonstrate a clear understanding of when to use certain algorithms and how to implement them on a spreadsheet.

There are a number of minor points to make about the assessment of these tasks.

- A substantial application of the selected algorithm is expected – usually a systematic reduction in strip width, as far as say 128 strips may be sufficient, depending on the integral selected
- An annotated print-out of the formulae used in the spreadsheet is sufficient to explain the use of technology

- Error analysis should not consist of comparing calculated values with the "real" value
- A number of candidates spoilt their work by finding relative errors from either known values (such as *p*) or more accurate values found from a graphical calculator
- It is expected that candidates will use iteration or extrapolation to achieve a particular level of accuracy which they can justify solely from their own results
- A brief explanation of how the mark for oral communication is arrived at is expected.