OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2610/1

Differential Equations (Mechanics 4)

Friday **16 JANUARY 2004** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- Take $g = 9.8 \text{ m s}^{-2}$ unless otherwise instructed.
- The total number of marks for this paper is 60.

This question paper consists of 4 printed pages.

1 A student is conducting an experiment on air resistance. She drops a ball of mass m kg from rest from a height of 1 m above a bench.

She first models the air resistance, R N, as proportional to the velocity of the ball, v m s^{-1}, using the formula $R = mkv$, where k is a constant. The time after the ball is released is t seconds.

(i) Show that $v = \frac{g}{k}(1 - e^{-kt})$. [6]

(ii) Find an expression for the displacement of the ball from the point of release. [4]

(iii) Use the approximation $e^x \approx 1 + x + \frac{1}{2}x^2$ for small values of x to show that the expressions for velocity and displacement for small values of k are approximately the same as those obtained from the formulae for constant acceleration. [4]

The student records that the ball hits the bench 0.46 seconds after being released.

(iv) Verify that $k = 0.24$ is consistent with this reading. For this value of k calculate the velocity with which the ball hits the bench. [2]

The calculated value of the velocity is slightly different from the experimental data, so the student refines the model to $R = m(0.25v + 0.02v^2)$, giving the differential equation

$$\frac{dv}{dt} = g - 0.25v - 0.02v^2.$$

(v) She uses Euler's method from $t = 0, v = 0$ with a step length of 0.01, obtaining $v = 4.0413$ when $t = 0.44$. The algorithm is given by

$$v_{r+1} = v_r + h\dot{v}_r, t_{r+1} = t_r + h.$$

Carry out two more steps to estimate the velocity at $t = 0.46$. [4]

2 The simultaneous differential equations

$$\frac{dx}{dt} = 11x - 18y + e^{-2t} \qquad (1)$$

$$\frac{dy}{dt} = 6x - 10y \qquad (2)$$

are to be solved.

(i) Eliminate y to show that $\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 8e^{-2t}$. [5]

(ii) Find the general solution for x. Use your solution and equation (1) to find the corresponding general solution for y. [9]

(iii) Show that if either of x or y tends to zero as t tends to infinity, then so does the other. [1]

(iv) Show that if the solutions do not tend to zero, $\frac{y}{x} \longrightarrow \frac{1}{2}$ as $t \to \infty$. What are the possible values of this limit if the solutions for x and y tend to zero? [5]

3 The differential equation $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 10y = 3\sin t$ is to be solved.

(i) Find the general solution. [8]

When $t = 0, y = \frac{dy}{dt} = 0$.

(ii) Find the particular solution. [4]

(iii) Describe the behaviour of y for large values of t. Sketch the solution for large values of t, indicating the range of values of y. [5]

(iv) Describe the difference between the behaviour of the solution in part **(ii)** and the solution to the equation $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 10y = 3\sin t$, for large values of t. [3]

4 The differential equation

$$x\frac{dy}{dx} + 2y = x^r \sin 2x$$

is to be investigated for $x > 0$ and for different values of r.

Consider first the case $r = -1$.

(i) Show that the general solution is $y = \frac{A}{x^2} - \frac{\cos 2x}{2x^2}$, where A is an arbitrary constant. [6]

(ii) When $A = 1$, what are the limiting values of y as x tends to infinity and as x tends to zero? [2]

(iii) Find the particular solution for which $y = \frac{8}{\pi^2}$ when $x = \frac{1}{4}\pi$. Use the small angle approximation $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ to show that y tends to 1 as x tends to zero. [5]

Now consider the case $r = 0$.

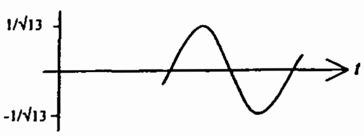
(iv) Find the general solution of the differential equation. Find the solution for which y tends to zero as x tends to zero. [7]

[You are given that, for small θ, $\sin\theta \approx \theta - \frac{1}{6}\theta^3$.]

Mark Scheme

1(i)	$m\frac{dv}{dt} = mg - mkv$	M1	N2L
	$\int \frac{dv}{g-kv} = \int dt$	M1	separate variables and integrate
	$-\frac{1}{k}\ln\lvert g-kv\rvert = t + c$	A1	LHS
	$v = \frac{1}{k}(g - Ae^{-kt})$	M1	rearranging
	$t = 0, v = 0 \Rightarrow A = g$	M1	conditions
	$v = \frac{g}{k}\left(1 - e^{-kt}\right)$	E1	
			6
(ii)	$\frac{dx}{dt} = \frac{g}{k}\left(1 - e^{-kt}\right)$	M1	attempt to integrate
	$x = \frac{g}{k}\left(t + \frac{1}{k}e^{-kt}\right) + B$	A1	
	$t = 0, x = 0 \Rightarrow B = -\frac{g}{k^2}$	M1	use $t = 0, x = 0$ to calculate constant
	$x = \frac{g}{k^2}\left(kt - 1 + e^{-kt}\right)$	A1	
			4
(iii)	$e^{-kt} \approx 1 - kt + \frac{1}{2}k^2t^2$		
	$v \approx \frac{g}{k}(kt - \frac{1}{2}k^2t^2)$	M1	substitute approximation for e^{-kt} in v
	$\approx gt$ (ignoring terms in k)		
	$u = 0, a = g \Rightarrow v = u + at = gt$	E1	both results
	$x \approx \frac{g}{k^2}(kt - 1 + 1 - kt + \frac{1}{2}k^2t^2) = \frac{1}{2}gt^2$	M1	substitute approximation for e^{-kt} in x
	$u = 0, a = g \Rightarrow x = ut + \frac{1}{2}at^2 = \frac{1}{2}gt^2$	E1	both results
			4
(iv)	$x = \frac{9.8}{0.24^2}(0.24\times0.46 - 1 + e^{-0.24\times0.46}) = 0.9997 \approx 1$	B1	must be shown, not stated
	$v = \frac{9.8}{0.24}(1 - e^{-0.24\times0.46}) = 4.268$	B1	
			2
(v)	t v $\dot{v}$	M1	use of algorithm
	0.44 4.0413 8.4630	A1	$v(0.45)$ agrees to 3dp
	0.45 4.1259 8.4281	M1	use of algorithm
	0.46 4.2102	A1	$v(0.46)$ agrees to 3dp
			4

2(i)	$\ddot{x} = 11\dot{x} - 18\dot{y} - 2e^{-2t}$	M1 A1	differentiate $\ddot{x}$	
	$= 11\dot{x} - 18(6x - 10y) - 2e^{-2t}$	M1	substitute for $\dot{y}$	
	$= 11\dot{x} - 108x - 10(\dot{x} - 11x - e^{-2t}) - 2e^{-2t}$	M1	substitute for y	
	$= \dot{x} + 2x + 8e^{-2t} \Rightarrow \ddot{x} - \dot{x} - 2x = 8e^{-2t}$	E1		
				5
(ii)	$\lambda^2 - \lambda - 2 = 0$	M1	auxiliary equation	
	$\lambda = -1$ or 2	A1		
	$x = Ae^{-t} + Be^{2t}$	F1	CF for their roots with two arbitrary constants	
	PI $x = ae^{-2t}$	B1		
	$\dot{x} = -2ae - 2t, \ddot{x} = 4ae - 2t \Rightarrow 4a + 2a - 2a = 8$	M1	differentiate twice, substitute and compare	
	$a = 2$	A1		
	$x = Ae^{-t} + Be^{2t} + 2e^{-2t}$	F1	their CF + their PI	
	$y = \frac{1}{18}(11x + e^{-2t} - \dot{x})$			
	$y = \frac{1}{18}(11(Ae^{-t} + Be^{2t} + 2e^{-2t}) + e^{-2t} - (-Ae^{-t} + 2Be^{2t} - 4e^{-2t}))$	M1	substitute x and $\dot{x}$ to find y	
	$y = \frac{2}{3}Ae^{-t} + \frac{1}{2}Be^{2t} + \frac{3}{2}e^{-2t}$	A1	cao	
				9
(iii)	$x \to 0 \Leftrightarrow B = 0 \Leftrightarrow y \to 0$	E1		
				1
(iv)	if $B \neq 0$, for large t, $\frac{y}{x} = \frac{\frac{1}{2}Be^{2t}(1 + \text{decaying terms})}{Be^{2t}(1 + \text{decaying terms})}$	M1	consider ratio and recognise dominant terms	
	$\to \frac{1}{2}$ as $t \to \infty$	E1		
	if $B = 0, A \neq 0$, $\frac{y}{x} = \frac{\frac{2}{3}Ae^{-t}(1 + \text{decaying terms})}{Ae^{-t}(1 + \text{decaying terms})}$	M1	consider effect on all terms	
	$\to \frac{2}{3}$ as $t \to \infty$	A1		
	if $B = 0, A = 0$, $\frac{y}{x} = \frac{3}{4}$ for all t	B1	value and some reasoning given	
				5

3(i)	$\alpha^2+6\alpha+10=0$	M1	
	$\alpha=-3\pm i$	A1	
	CF $y=e^{-3t}(A\cos t+B\sin t)$	F1	CF for their roots
	PI $y=a\sin t+b\cos t$	B1	
	$(-a\sin t-b\cos t)+6(a\cos t-b\sin t)+10(a\sin t+b\cos t)=3\sin t$	M1	differentiate twice and substitute
	$-a-6b+10a=3$, $-b+6a+10b=0$	M1	compare coefficients and solve
	$a=\frac{3}{13}, b=-\frac{2}{13}$	A1	
	$y=\frac{1}{13}(3\sin t-2\cos t)+e^{-3t}(A\cos t+B\sin t)$	F1	their CF + their PI
			8
(ii)	$-\frac{2}{13}+A=0$	M1	equation from their y
	$\dot{y}=\frac{1}{13}(3\cos t+2\sin t)-3e^{-3t}(A\cos t+B\sin t)+e^{-3t}(-A\sin t+B\cos t)$	M1	differentiate (product rule)
	$\frac{3}{13}-3A+B=0$ and so $A=\frac{2}{13}, B=\frac{3}{13}$	M1	substitute $t=0$ and solve
	$y=\frac{1}{13}(3\sin t-2\cos t)+\frac{1}{13}e^{-3t}(2\cos t+3\sin t)$	A1	cao
			4
(iii)	oscillatory motion with constant amplitude	B1	consistent with their y
	for large t, $y=\frac{1}{13}(3\sin t-2\cos t)$	M1	ignore negative exponentials
		A1	oscillatory sketch consistent with their y
		M1	finding amplitude (for $a\sin kt+b\cos kt$)
		A1	± amplitude marked consistent with their y
			5
(iv)	$\alpha^2-6\alpha+10=0 \Rightarrow \alpha=3\pm i$	B1	e^{3t} seen (or equivalent)
	CF $y=e^{3t}(A\cos t+B\sin t)$	B1	full CF considered
	oscillations with exponentially growing (or unbounded) oscillations	B1	
			3

4(i)	$\frac{dy}{dx}+\frac{2}{x}y=x^{-2}\sin 2x$	M1	rearrange	
	$I=\exp(\int\frac{2}{x}dx)=\exp(2\ln x)=x^2$	M1 A1	integrating factor	
	$x^2\frac{dy}{dx}+2xy=\sin 2x$			
	$x^2y=\int\sin 2x\,dx$	M1	multiply and integrate	
	$=-\frac{1}{2}\cos 2x+A$	A1		
	$y=\frac{A}{x^2}-\frac{\cos 2x}{2x^2}$	E1		6
(ii)	as $x\to\infty, y\to 0$	B1		
	as $x\to 0, y\to\infty$	B1		2
(iii)	$\frac{8}{\pi^2}=\frac{16A}{\pi^2}-\frac{8}{\pi^2}\cos\frac{1}{2}\pi$	M1	use conditions to calculate constant	
	$\Rightarrow A=\frac{1}{2}\Rightarrow y=\frac{1-\cos 2x}{2x^2}$	A1		
	for small x, $\cos 2x\approx 1-\frac{1}{2}(2x)^2=1-2x^2$	B1	accept unsimplified	
	so $y\approx\frac{1-(1-2x^2)}{2x^2}=1$	M1 E1	use in y	5
(iv)	$x^2y=\int x\sin 2x\,dx$	M1	use integrating factor with $r=0$	
	$=-\frac{1}{2}x\cos 2x+\int\frac{1}{2}\cos 2x\,dx=-\frac{1}{2}x\cos 2x+\frac{1}{4}\sin 2x+B$	M1	integration by parts	
	$y=-\frac{\cos 2x}{2x}+\frac{\sin 2x}{4x^2}+\frac{B}{x^2}$	A1		
	for small x, $y\approx-\frac{1-2x^2}{2x}+\frac{2x-\frac{1}{6}(2x)^3}{4x^2}+\frac{B}{x^2}$	M1	approximation for y	
	$=\frac{2}{3}x+\frac{B}{x^2}$	A1	cao	
	so finite limit $\Rightarrow B=0$	M1	valid attempt to deduce constant	
	$y=-\frac{\cos 2x}{2x}+\frac{\sin 2x}{4x^2}=\frac{\sin 2x-2x\cos x}{4x^2}$	A1		7

Examiner’s Report

2610 Mechanics 4

General Comments

The standard of solution of differential equations was very high and most candidates were able to work with reasonably high levels of accuracy. However, candidates found aspects of modelling and interpretation harder.

Comments on Individual Questions

Question 1 (Vertical motion by separating variables and Euler's method)

This was the least popular question by a considerable margin. However, candidates who attempted it usually did relatively well. The first two parts were often done well, although some candidates then forgot to include an arbitrary constant when integrating the velocity to get the displacement. Some candidates used their own origin for displacement rather than that given in the question. Many candidates made some progress in considering the solutions for small k but only a minority completed the argument clearly. The calculations in the last two parts were often done well.

(ii) $x = \frac{g}{k^2}\left(kt - 1 + e^{-kt}\right)$; (iv) 4.27 m s^{-1}; (v) 4.2102 m s^{-1}

Question 2 (Simultaneous differential equations)

This question was very popular and usually well done. Most candidates knew how to eliminate and solve, and the majority were able to maintain accuracy throughout. The consideration of the solutions for large t caused many problems. Although many were able to show that $\frac{y}{x} \to \frac{1}{2}$, few were able to identify the other possible values.

(ii) $x = Ae^{-t} + Be^{2t} + 2e^{-2t}$, $y = \frac{2}{3}Ae^{-t} + \frac{1}{2}Be^{2t} + \frac{3}{2}e^{-2t}$; (iv) $\frac{2}{3}, \frac{3}{4}$

Question 3 (Second order differential equation)

This question was also popular and well done. Most candidates knew how to solve the equation although minor errors often occurred. Some candidates ran into difficulties when they thought that their first choice of particular integral was in the complementary function and hence introduced a factor of t. Candidates often recognised which terms of their solution decayed for large t, but often gave unclear descriptions of the behaviour of the solution. Sketches often lacked the detail specifically asked for in the question. Many candidates recognised that the equation given in the last part would give different behaviour, but descriptions sometimes lacked detail, in particular the oscillatory nature.

(i) $y = \frac{1}{13}(3\sin t - 2\cos t) + e^{-3t}(A\cos t + B\sin t)$;

(ii) $y = \frac{1}{13}(3\sin t - 2\cos t) + \frac{1}{13}e^{-3t}(2\cos t + 3\sin t)$

Question 4 (Integrating factor method)

Candidates often began this question well. The technique was well known, but a common error was to forget to divide or multiply the right hand side of the equation when dividing or multiplying the left hand side. Consideration of the particular solutions was often done well, but use of the approximation for cos2x was prone to errors. In the last part of the question, the general solution was often correct, but candidates usually made errors when finding the solution with the required property.

(iii) $y = \frac{1 - \cos 2x}{2x^2}$; (iv) $y = -\frac{\cos 2x}{2x} + \frac{\sin 2x}{4x^2} + \frac{B}{x^2}$, $y = \frac{\sin 2x - 2x\cos x}{4x^2}$