OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2603(A)

Pure Mathematics 3

Section A

Tuesday **10 JUNE 2003** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer **all** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

NOTE

- This paper will be followed by **Section B: Comprehension.**

This question paper consists of 4 printed pages.

[Turn over

1 (a) Find the first four terms of the binomial expansion of $(1-3x)^{-2}$. [4]

(b) Express $\dfrac{-3-x}{(x^2+1)(x-1)}$ in the form $\dfrac{Ax+B}{x^2+1}+\dfrac{C}{x-1}$, giving the values of A, B and C. [4]

(c) Find $\displaystyle\int x\cos 2x\,\mathrm{d}x$. [4]

(d) Given that $x^2+y^2=y$, show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2x}{1-2y}$. [4]

[Total 16]

2 Water drains from a container so that the height h metres of the water above the base after t seconds satisfies the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t}=-ah,$$

where a is a positive constant. Initially, the height of the water is 2 metres.

(i) Verify that $h=2\mathrm{e}^{-at}$ satisfies both the differential equation and the initial condition. [3]

(ii) It takes 10 seconds for the height to drop to 1 metre. Find a. [3]

In another container, the height H metres of the water varies according to the differential equation

$$\frac{\mathrm{d}H}{\mathrm{d}t}=-b\sqrt{H},$$

where b is a positive constant. The initial height of the water is 1 metre.

(iii) Using integration, show that $\sqrt{H}=1-\frac{1}{2}bt$. [4]

(iv) Given that it takes 10 seconds for the height to drop to a half of its initial value, find b.

Hence find how long it takes for the height to drop from 1 metre to zero. [5]

[Total 15]

3 **(a)** A curve has parametric equations

$$x = \cos\theta,\ y = \cos 2\theta, \qquad 0 \leqslant \theta \leqslant \pi.$$

(i) Show that $\frac{dy}{dx} = 4\cos\theta$. [4]

(ii) Find the cartesian equation of the curve. Sketch the curve. [3]

(b) Fig. 3 shows the curve with parametric equations

$$x = \cos\theta,\ y = \cos 3\theta, \qquad 0 \leqslant \theta \leqslant \pi.$$

The curve cuts the x-axis at the points A, O and B.

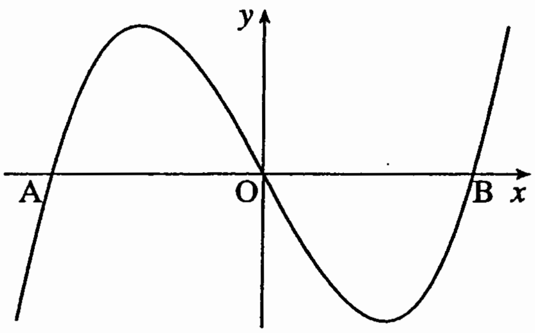

Fig. 3

(i) Find the coordinates of the points A and B. [4]

(ii) By first expanding $\cos(2\theta + \theta)$, show that

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

Hence write down the cartesian equation of the curve. [4]

[Total 15]

4 As part of a sculpture, an artist erects a flat triangular sheet ABC in his garden. The vertices are attached to vertical poles DA, EB and FC. The coordinate axes Ox and Oy are horizontal, and Oz is vertical. The coordinates of the triangle are A(2, 0, 2), B(−2, 0, 1) and C(0, 4, 3), with units in metres (see Fig. 4).

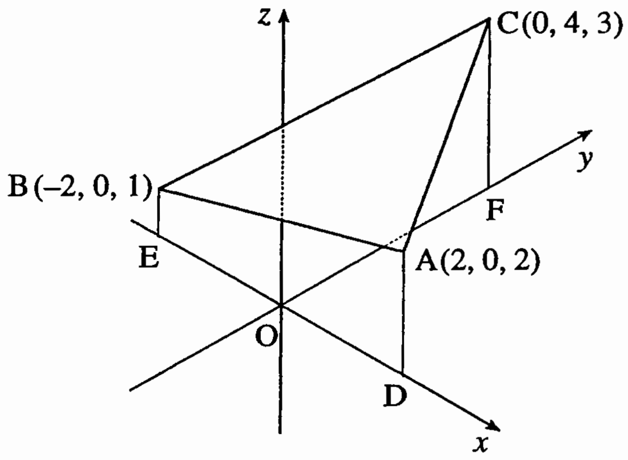

Fig. 4

(i) Find the length of the side AC. [2]

(ii) Find the scalar product $\overrightarrow{AB}.\overrightarrow{AC}$, and the angle BAC. [4]

(iii) Show that $\begin{pmatrix} 2 \\ 3 \\ -8 \end{pmatrix}$ is perpendicular to the lines AB and AC.

Hence find the cartesian equation of the plane ABC. [5]

(iv) The artist decides to erect another vertical pole GH based at the point G(1, 1, 0). Calculate the height of the pole if H is to lie in the plane ABC. [3]

[Total 14]

Mark Scheme

SECTION A

1(a) $(1-3x)^{-2} =$ $1+(-2)(-3x)+\frac{(-2)(-3)}{1\times 2}(-3x)^2+\frac{(-2)(-3)(-4)}{1\times 2\times 3}(-3x)^3+\ldots$ $= 1 + 6x + 27x^2 + 108x^3 + \ldots$	M1 B1 B1 B1 [4]	Four correct binomial coefficients as shown (allow one slip) s.o.i. $+ 6x$ $+ 27x^2$(Allow this B1 only, from the use of $(3x)^2$ instead of $(-3x)^2$) $+ 108x^3$
(b) $\frac{-3-x}{(x^2+1)(x-1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x-1}$. $\Rightarrow -3-x=(Ax+B)(x-1)+C(x^2+1)$ $x=1 \Rightarrow -4=2C \Rightarrow C=-2$ coeff^t of x^2: $0=A+C \Rightarrow A=2$ constants: $-3=-B+C \Rightarrow B=1$	M1 A1 A2,1 or 0 [4]	Equating the numerators s.o.i. If the brackets round Ax+B are omitted allow M1 only if the equations involving A and B are correct. For any one correct equation s.o.i. Deduct 1 for each incorrect value of A,B or C
(c) $\int x\cos 2x\,dx = \int x\frac{d}{dx}(\frac{1}{2}\sin 2x)dx$ $=\frac{1}{2}x\sin 2x-\int\frac{1}{2}\sin 2x\,dx$ $=\frac{1}{2}x\sin 2x+\frac{1}{4}\cos 2x+c$ ~~Other correct forms are possible e.g. ¼ x sin 2x + ¼ cos²2x + c etc.~~	M1 A1 A1f.t. A1www [4]	Using the method of integration by parts with $u=x$ and $dv/dx=\cos 2x$ leading to 2 terms. $\frac{1}{2}x\sin 2x$ $-\int\frac{1}{2}\sin 2x\,dx$ f.t. their $v=\pm\,1/2$ or $\pm 2\sin 2x$ Condone the omission of c
(d) $x^2+y^2=y$ $2x + 2y\,dy/dx = dy/dx$ or $x^2=t=y-y^2$ $dt/dx = 2x$ $dt/dy = 1-2y$ $\frac{dy}{dx}=\frac{2x}{1-2y}$ Or M1 for an attempt to express x or y explicitly in terms of the other and to differentiate explicitly. A1 for the correct explicit equation and A1 for the correct differentiation. E1 for a correct and complete manipulation to obtain the result. Or Done by integration. M1 for separating the variables of the result . A1 for the correct integration condoning the omission of the arbitrary constant. E2 for including a constant and explaining that it can be given the value 0.	M1 A1 A1 E1 [4] Total 16	For an attempt at implicit differentiation wrt x or y or t Correct differentiation of x terms Ditto y terms Brackets must be seen where they are necessary Correct and complete manipulation and result

2 (i) $h = 2e^{-at}$ $\Rightarrow dh/dt = -2a\, e^{-at}$ $= -a\, h$ When $t = 0$, $h = 2\, e^0 = 2$ as required	M1 E1 E1 [3]	At least $2\, e^{-at}$ correct plus something else Independent of M1.
Or $dh/dt = -ah \Rightarrow \int \frac{dh}{h} = \int -a\,dt$ $\Rightarrow \ln h = -at + c$ When $t = 0$, $h = 2 \Rightarrow \ln 2 = c$ $\Rightarrow \ln h = -at + \ln 2 \Rightarrow h = 2e^{-at}$	M1 E1 E1 [3]	Separating the variables and integrating. At least ln h and t. Condone the omission of c Finding c. Dependent on M1
(ii) $1 = 2\, e^{-10a}$ $\Rightarrow e^{-10a} = ½$ $\Rightarrow -10a = \ln(1/2)$ $\Rightarrow a = -\ln(1/2)/10 = 0.069(3\ldots)$	M1 DM1 A1 [3]	Substituting h=1 and t=10 taking lns or 0.07 or better.
(iii) $\frac{dH}{dt} = -b\sqrt{H}$ $\Rightarrow \int \frac{dH}{\sqrt{H}} = \int -b\,dt$ $\Rightarrow 2H^{1/2} = -bt + k$ When $t = 0$, $H = 1 \Rightarrow 2 = k$ $\Rightarrow 2\, H^{1/2} = -bt + 2$ $\Rightarrow \sqrt{H} = 1 - ½\, bt$ * A similar scheme can be applied to the integration of dt/dH	M1 A1 B1 ft E1www. [4]	Separating variables Condone the absence of k Evaluating their k. f.t. their solution; or substituting correct limits into their definite integrals.
(iv) When $t = 10$, $H = ½$ $\Rightarrow \sqrt{1/2} = 1 - 5b$ $\Rightarrow 5b = 1 - 1/\sqrt{2}$ $\Rightarrow b = (1 - 1/\sqrt{2})/5 = 0.05857\ldots$ $H = 0$ when $½\, bt = 1$ $\Rightarrow t = 2/b$ $= 34(.142\ldots)$ s	M1 DM1 A1 M1 A1 [5] Total 15	Substituting values in * Solving for b Either exact or 0.06 or better Solving $1-1/2$ bt =0 Answers rounding to 34 secs.

3(a)(i) $\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}$ $=\frac{-2\sin 2\theta}{-\sin\theta}$ $=\frac{4\sin\theta\cos\theta}{\sin\theta}$ $= 4\cos\theta$ * **or** Finding the Cartesian equation of the curve as in part (ii) (which may be given M1 A1 if it is referred to in part (ii)) and differentiating to give $dy/dx = 4x$ M1 A1 $= 4\cos\theta$ M1 E1	M1 A1 M1 E1www [4]	$\frac{their\,dy/d\theta}{their\,dx/d\theta}$ Allow $\frac{2\sin 2\theta}{\sin\theta}$ without other working but give final E0 $\sin 2\theta = 2\sin\theta\cos\theta$ used
(ii) $y = \cos 2\theta = 2\cos^2\theta - 1$ $= 2x^2 - 1$ **or** $dy/dx = 4\cos\theta = 4x \Rightarrow y = 2x^2 + c$ $x = 0$ $\theta = \pi/2$ $\cos 2\theta = -1 = y = c$	M1 A1 B1 [3] M1 A1	$\cos 2\theta = 2\cos^2\theta - 1$ used. Or any equivalent equation. Parabola with vertex at (0, -1) which must be indicated in some way c must be seen
(b) (i) $\cos 3\theta = 0$ $\Rightarrow 3\theta = \pi/2, 3\pi/2, 5\pi/2$ $\Rightarrow \theta = \pi/6, \pi/2, 5\pi/6$ $\Rightarrow x = \cos\theta = \sqrt{3}/2, 0, -\sqrt{3}/2$ So A is $(-\sqrt{3}/2, 0)$, B is $(\sqrt{3}/2, 0)$ For the correct coordinates as shown opposite given without any working, ie by the use of a graphical calculator, give B2, B2 for A is (-0.87, 0) and B is (0.87, 0)	M1 A1 A1www A1 www	s.o.i. θ = either $\pi/6$ or $5\pi/6$ accept 30° or 150° A is $(-\sqrt{3}/2, 0)$ Accept $x = -\sqrt{3}/2$ but not $(0, -\sqrt{3}/2)$ Accept –0.87. B is $(\sqrt{3}/2.\ 0)$ Accept as for A
(ii) $\cos(2\theta+\theta) = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$ $= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta$ $= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$ $= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta$ $= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$ $= 4\cos^3\theta - 3\cos\theta$ So Cartesian equation is $y = 4x^3 - 3x$.	M1 M1 E1 B1 [4] Total 15	Compound angle formula correct Correct use of the double angle formulae using $\sin^2\theta + \cos^2\theta = 1$ and simplifying.

4 (i) AC $= \sqrt{\{2^2 + (-4)^2 + (-1)^2\}}$ $= \sqrt{21}$ or 4.58…	M1 A1 [2]	
(ii) $\vec{AB} = \begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix}$ $\vec{AC} = \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}$ $\vec{AB}.\vec{AC} = (-4)\times(-2) + 0\times 4 + (-1)\times 1$ ($= 7$ $\cos BAC = \dfrac{7}{\sqrt{17}\times\sqrt{21}} = 0.3704..$ $\Rightarrow BAC = 68.25°$	M1 A1 M1 A1 [4]	For vectors **AB** and **AC** , (accept **BA** ,**CA**. Condone one slip in each vector), and for evaluating the scalar product.soi 7 must be seen. Ft their vectors and their scalar product. Or 68.3° or 1.19 radians
(iii) $\begin{pmatrix} 2 \\ 3 \\ -8 \end{pmatrix}.\begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} = -8+0+8=0$ $\begin{pmatrix} 2 \\ 3 \\ -8 \end{pmatrix}.\begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix} = -4+12-8=0$ so perpendicular to AB and AC Equation of plane: $2x + 3y - 8z = c$ At A, $2\times2 + 3\times0 - 8\times2 = c$ $\Rightarrow$ $c = -12$ $\Rightarrow$ $2x + 3y - 8z = -12$	E1 E1 M1 DM1 A1 [5]	-8+8 must be seen -4+12-8 must be seen substituting the coordinates of A, B or C or using $\underline{a}.\underline{n}$ If a vector equation is used give M1 for a correct form, DM1 for eliminating the two parameters and A1 for the result
(iv) H is $(1, 1, h)$ where $2\times1 + 3\times1 - 8\times h = -12$ $\Rightarrow 8h = 17$ $\Rightarrow h = 2.125$ (m) <u>or</u> The equation of GH is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ Meets AB where $2(1) + 3(1) - 8(\lambda) = -12$ $\lambda = 17/8$ and so h=2.125(m)	M1 A1ft A1cao M1 A1 ft A1 cao [3] Total 14	Ft their equation from (iii) Ft their equation from (iii)

Examiner’s Report

2603 Pure Mathematics 3

General Comments

Candidates were given opportunities to perform well on this paper and many responded with high marks, a good number scoring in the range 70-75. A mark in this range was a notable achievement because there were some tricky points, which could catch out the strong as well as the weak. Some questions on the comprehension paper also gave able candidates as much trouble as the less able.

Generally there was a good spread of marks and only a small number of candidates scored fewer than 10 or 15.

The presentation of work was very varied; at the one extreme, immaculate scripts were a pleasure to mark, at the other, work was so muddled that it was difficult to decipher what the candidate was aiming to do. Poor notation was often a cause of this, brackets omitted, failure to be clear about which variable is being differentiated and with respect to which other variable, and attempts to integrate one variable with respect to a different variable. Perhaps the main cause of muddled work by many candidates, was the absence of any attempt to explain the logical progression of their work by the use of words or symbols; therefore, $\Rightarrow$, the correct use of the = sign, differentiating, etc. This was most evident in Section B, questions 3 and 4, and in Section A, question 1, part (d).

There was little evidence that candidates were short of time.

Comments on Individual Questions

Q.1

This question presented candidates with four standard procedures and generally they responded well. A very pleasing number scored full marks. Of course, there were errors from candidates who were clearly familiar with the methods involved.

(a) Most commonly, $-3x^2$ instead of $(-3x)^2$, or x instead of $3x$ etc., or perhaps just a careless sign error.

(b) Quite often the omission of brackets $Ax + B\ .\ (x - 1)$, although in many cases the subsequent working showed that the brackets had been implied. Errors sometimes occurred in the solution of correct equations in A, B and C.

(c) All too often, $\int \sin 2x dx = -1/2 \sin 2x$, or $2 \sin 2x$ or even $\frac{1}{2} \sin x$.

(d) Again, the omission of brackets, $1 - 2y . \frac{dy}{dx} = 2x$ was very common.

Of the four procedures in this question, implicit differentiation was the least familiar. This was a question where the notation of differentiation was crucial and where candidates who use the symbol $\frac{dy}{dx}$ as shorthand for 'differentiating' might not give convincing proofs of the stated result.

Q.2 Differential equations

(i) Many candidates attempted to solve the given differential equation rather than verify that $h = 2e^{-at}$ is a solution, as was suggested in the question. In the solution those candidates who omitted the arbitrary constant were restricted to one mark out of three because they were unable to use the initial condition. Those who included a constant often made the error $\ln h = -at + c \Rightarrow h = e^{-at} + e^c$. This was sometimes followed by a correct expression, $h = Ae^{-at}$, but without any justification. Those candidates who found the value of their constant as ln 2 were more likely to move correctly from $\ln h = -at + \ln 2$ to the required result.

Candidates who verified the result by differentiation were less likely to go wrong. Some poorer attempts at this question were a confused mixture of both methods. It is possible that some of these attempts failed because the candidates did not grasp that 'initially' implies $t = 0$.

(ii) This part was generally well done even by weaker candidates, most candidates making the correct substitutions and finding the value of a. The most likely error was $1 = 2\,e^{-10a} \Rightarrow \ln 1 = -10a \ln 2e$.

(iii) Only the most able candidates were able to solve this differential equation and they often produced immaculate solutions. For the rest there were many pitfalls. Some failed to separate the variables and attempted to integrate $\int -b\sqrt{H}\,dt$. Those who correctly obtained $\int \frac{dH}{\sqrt{H}} = \int -b\,dt$ often gave the LHS as $\ln\sqrt{H}$, or, if written as $\int H^{-\frac{1}{2}}dH$, integrated to $\frac{H^{\frac{1}{2}}}{2}$ or $\frac{H^{-\frac{3}{2}}}{-\frac{3}{2}}$. The RHS was sometimes given as $\frac{-b^2}{2}$ or $\frac{-b^2}{2}t$.

(iv) As in (ii) most candidates recovered to use the given result, substitute correctly and find the value of b, and very often go on to find the required time.

Q.3 Parametric equations

(a)(i) This part was generally very well done, candidates are familiar with the procedure and solutions were often correct.

(ii) Those candidates who thought to use the double angle formulae to express y in terms of $\cos\theta$ or $\sin\theta$ or both, usually managed to obtain the correct equation in some form, although just a few who chose the route $y = 1 - 2\sin^2\theta \Rightarrow \sin^2\theta = \frac{1-y}{2}$, sometimes then wrote $\left(\frac{1-y}{2}\right)^2 + x^2 = 1$. Candidates who obtained the equation usually made a correct sketch.

(b)(i) Most candidates started correctly with $\cos 3\theta = 0$. Those who solved this, most often gave only the solutions $3\theta = \pi/2$ and $3\pi/2$, thus omitting the solution $\theta = 5\pi/6$. This gave them the correct coordinates $(\sqrt{3}/2, 0)$ which they then identified as the point B. Many candidates then gave A correctly as the point $(-\sqrt{3}/2, 0)$ but failed in most cases to say why. Sometimes symmetry was mentioned but sometimes both $x = \pm\sqrt{3}/2$ were derived from $\theta = \pi/6$.

(ii) Again most candidates started correctly by expanding $\cos(2\theta + 0)$ as suggested in the question, but very many candidates were unable, successfully, to complete the question. Perhaps the most frequent breakdown came from dealing with $-\sin 2\theta \sin\theta = -2\sin\theta\cos\theta\sin\theta = -2\sin^2\theta\cos\theta$. A surprising number of candidates at this point used $2\sin^2\theta = 1 - \cos 2\theta$, thus returning to double angles, instead of using $\sin^2\theta + \cos^2\theta = 1$. If a Cartesian equation was given at the end of this question, it was usually the correct one.

Q.4 Vectors

This was a good question for very many candidates, they were familiar with the methods involved and carried them out accurately.

(i) Very well done by almost all candidates.

(ii) Most candidates obtained the correct angle but some did not state the scalar product clearly, as the question requested.

(iii) Almost all candidates knew that they had to show that $\begin{pmatrix}2\\3\\-8\end{pmatrix}.\begin{pmatrix}-4\\0\\-1\end{pmatrix}=0$ but a significant number did not state clearly that this scalar product was equal to $-8 + 0 + 8 = 0$, and similarly for $\begin{pmatrix}2\\3\\-8\end{pmatrix}.\mathbf{AC}=0$.

Those candidates approaching the Cartesian equation of the plane directly usually got the LHS correct but sometimes there was an error in sign on the RHS. Quite a large number of candidates still approach this question by starting with the vector equation of the plane and eliminating the parameters. Some did this successfully but others made errors with the algebra involved. In either event, of course, this procedure takes more time which could be better spent on another question.

(iv) The more able candidates did this question most successfully either by giving H the coordinates (1, 1, *h*) or writing down the equation of the line GH as $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}0\\0\\1\end{pmatrix}$ and substituting into the equation of the plane ABC, to find *h* or λ. The most common error was to use the equation of the perpendicular to the plane ABC through H instead of the vertical line through H. Some other candidates used the formula for the length of the perpendicular from a point to a plane, in effect, the same error.

Section B (Comprehension)

1&2 The first two questions presented candidates with no problem and were almost always done correctly.

3. Candidates who were confident with simple algebra were able to do this question successfully, but many got into a muddle with their attempt, and many solutions were difficult to follow through.

   Candidates who used the substitution $\frac{1}{2}(a + b + c) = 50$, in dealing with the entries for A, were able to say ‘and similarly for B and C’; but more candidates used the substitution $c = 100 - (a + b)$, given in the table, in which case a separate proof was needed for C where it was necessary to change the substitution to $a = 100 - (b + c)$.

   Many candidates in this, and the remaining questions, quoted numerical examples from the text to ‘justify’ the general cases, not understanding that the questions they were answering were justifying steps in the process of moving from particular cases to general cases.

4. Many candidates quoted the sentence from the text on line 165, ‘The figures in the last column in Table 12 are exactly 100 less than those in the last column of Table 11’, the statement that they were being asked to justify in the general case. It was most difficult to follow many of the algebraic proofs given because of the lack of clarity about the starting point and the steps taken.

5. Although the first part of this question was generally answered correctly very few candidates were able to go any further. They failed to realise that rewriting the last column, as had been done with Table 5, would enable them to make the comments needed. Just a few made the necessary change deliberately and a few more came across the form needed accidentally in making the entries required by part (i). Most other candidates simply repeated comments from the text.