OCR
RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2612

Mechanics 6

Thursday **19 JUNE 2003** Morning 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- Take $g = 9.8$ m s^{-2} unless otherwise instructed.
- The total number of marks for this paper is 60.

This question paper consists of 5 printed pages and 3 blank pages.

1 *Option 1: Rotation of a rigid body*

A uniform pulley of mass m and radius r hangs freely from a light string attached to its centre O and to a fixed point P. Another light string is attached to the rim of the pulley and then wound round it with its free end attached to a fixed point Q, as shown in Fig. 1.

The two strings shown are vertical.

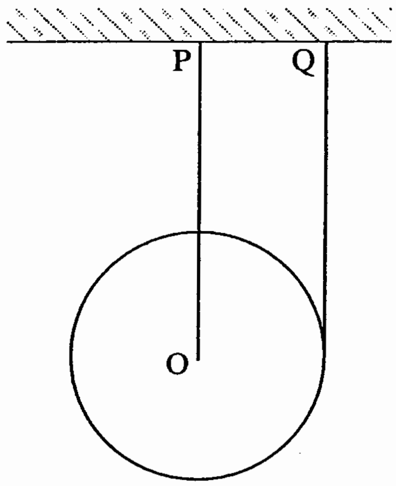

Fig. 1

(i) Write down the tensions in the two strings. [2]

The string through P then snaps and the pulley falls, with the other string unwinding.

(ii) Explain why, in the subsequent motion, the string through Q remains vertical. [2]

(iii) Write down the equation of motion for the centre of mass and a rotation equation for the pulley at time t when it has turned through an angle θ.

Hence show that the angular momentum of the pulley when it has made one complete revolution is

$$m\sqrt{\tfrac{2}{3}\pi g r^3}.$$ [12]

(iv) The remaining string now breaks. Explain why the angular momentum of the pulley at time t is proportional to t before the string breaks but remains constant afterwards. Describe briefly the behaviour of the linear momentum of the pulley before and after the string breaks. [4]

2 *Option 2: Vectors*

The forces $(\mathbf{j} - \mathbf{k})$ and $(-\mathbf{i} + 2\mathbf{k})$ act through the points with position vectors $(\mathbf{i} + 5\mathbf{j})$ and $(c\mathbf{j} + 4\mathbf{k})$ respectively, where c is a constant, relative to an origin O.

(i) Find the resultant force and show that the sum of the moments of the forces about O is

$$(2c - 5)\mathbf{i} - 3\mathbf{j} + (c + 1)\mathbf{k}.$$

Hence find the value of c for which the forces reduce to a single force. [7]

With this value of c, the forces act on a stationary sphere whose centre is fixed at the origin but which can turn freely about any diameter. The moment of inertia about any diameter is I.

(ii) Write down the equation of the line about which the sphere rotates. Find the angular velocity vector and the angular momentum of the sphere at time t. [4]

In another case, a third force $(\mathbf{i} + B\mathbf{j} - 2\mathbf{k})$ acting through the point with position vector $(A\mathbf{i} - \mathbf{j} + \mathbf{k})$ is added, where A and B are constants and c has the same value as in part **(i)**.

When the set of three forces is applied to the same stationary sphere, the sphere rotates about the line $\mathbf{r} = \lambda(6\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ and after a time T has elapsed, the sphere has kinetic energy

$$\frac{41T^2}{2I}.$$

(iii) Find the values of A and B. [9]

[Turn over

3 *Option 3: Stability and oscillations*

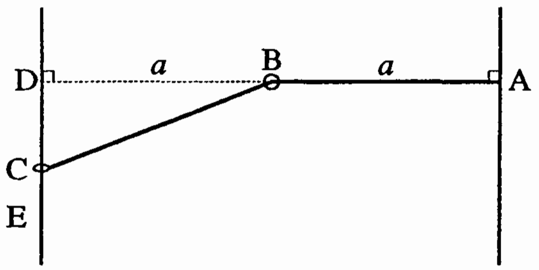

Fig. 3

In Fig. 3, ABC is an elastic string of natural length a and stiffness k. At the end C is a bead of mass m which can slide freely along the vertical wire DE. At B, the string passes through a fixed smooth ring and the end A is fixed. ABD is horizontal and AB = BD = a.

(i) Taking DC as a distance x, find the potential energy of the system. Show that there is one position of stable equilibrium. [7]

(ii) Find the period of small oscillations about this position of equilibrium. [6]

When the elastic string is shortened (for example by cutting a piece off one end and refastening), the position of equilibrium is moved. Suppose the string is shortened by d, where $d = a\varepsilon$ and ε is very small.

(iii) Show that the stiffness in the string is now approximately $k(1 + \varepsilon)$. [3]

In the special case for which $mg = ka$, you are **given** that the new position of equilibrium is at $x = x_0$, where

$$(x_0^2 + a^2)^{\frac{1}{2}} = \frac{dx_0}{a - d - x_0}.$$

(iv) Assuming further that $x_0 = a + \beta\varepsilon$, show that the position of equilibrium moves a distance of approximately $a\varepsilon\left(1 + \frac{1}{\sqrt{2}}\right)$. [4]

4 *Option 4: Variable mass*

As a mound of mud slides from rest down a slope inclined at an angle α to the horizontal, it picks up more mud that is initially at rest. When it has slid a distance x, its total mass M is given by $M = m(1 + kx)$ where k and m are constants. The sliding resistance to movement is modelled as a frictional force with coefficient of friction μ.

(i) Show that the speed v at time t satisfies the differential equation

$$(1+kx)v\frac{\mathrm{d}v}{\mathrm{d}x}+kv^2=(1+kx)(\sin\alpha-\mu\cos\alpha)g.$$ [8]

(ii) By differentiating $(1 + kx)^2v^2$ with respect to x, show that the solution of the differential equation is $v^2 = A(1 + kx) + B(1 + kx)^{-2}$, where $A = \frac{2g}{3k}(\sin\alpha - \mu\cos\alpha)$ and B is a constant to be determined. [8]

(iii) Show that when x becomes very large, the acceleration of the mud tends to a constant value. [4]

Mark Scheme

Question 1

(i)		
Tension in OP = mg, tension in other = 0	B1, B1	[2]
(ii)		
Forces on centre of mass are vertical	B1, B1	[2]
(iii)		
$mg - T = m\ddot{x},\ \ Tr = I\ddot{\theta} = \frac{mr^2}{2}\ddot{\theta},\ \ x = r\theta;$	M1A1, M1A1, B1	
$g = \frac{3r\ddot{\theta}}{2},\ \ \frac{3r\dot{\theta}\ddot{\theta}}{2} = g\dot{\theta},\ \ \dot{\theta}^2 = \frac{4g\theta}{3r}$	M1A1, M1, A1	
When $\theta = 2\pi$, $\dot{\theta} = \sqrt{\frac{8g\pi}{3r}}$	M1	
Angular momentum $= \frac{mr^2}{2}\dot{\theta} = m\sqrt{\frac{2\pi g r^3}{3}}$ at $\theta = 2\pi$	M1A1, **ag**	[12]
(iv)		
Ang mom before $I\dot{\theta} = \frac{2gt}{3r}$, hence prop to t	B1	
No tangential force on pulley after, hence constant	B1	
Linear mom before $= \dot{x} = r\dot{\theta} \Rightarrow \dot{x} = \frac{2gt}{3}$	B1	
After, $\dot{x} = \sqrt{\frac{8g\pi r}{3}} + gt$ (in freefall)	B1	[4]

Question 2

(i)

$\sum \mathbf{r}\mathbf{x}\mathbf{F} = \begin{pmatrix}1\\5\\0\end{pmatrix}\mathbf{x}\begin{pmatrix}0\\1\\-1\end{pmatrix}+\begin{pmatrix}0\\c\\4\end{pmatrix}\mathbf{x}\begin{pmatrix}-1\\0\\2\end{pmatrix}$	M1	
$=\begin{pmatrix}-5\\1\\1\end{pmatrix}+\begin{pmatrix}2c\\-4\\c\end{pmatrix}=\begin{pmatrix}2c-5\\-3\\c+1\end{pmatrix}=\mathbf{M}_1$	A1, A1	
$\sum \mathbf{F} = \begin{pmatrix}-1\\1\\1\end{pmatrix}=\mathbf{R}$	B1	
Single force if $\mathbf{R}.\mathbf{M}_1 = 0, \Rightarrow c = 3$	M1, M1A1	[7]
(ii)		
Direction of rotation $\mathbf{r} = \lambda(\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})$ when c =3	F1	
$I\dfrac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$	M1	
$\mathbf{w} = (\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})t / I$	F1	
$I\mathbf{w} = (\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})t$ =angular momentum	F1	[4]
(iii)		
New sum of moments $\mathbf{M}_2 = \begin{pmatrix}3-B\\2A-2\\AB+5\end{pmatrix}$	M1A1	
Equating $\begin{pmatrix}3-B\\2A-2\\AB+5\end{pmatrix}=\dfrac{\lambda T}{I}\begin{pmatrix}6\\2\\-1\end{pmatrix}$	M1F1	
Using KE $=\dfrac{1}{2}I\Omega^2$	M1	
Equating $\dfrac{41T^2}{2I}=\dfrac{41\lambda^2T^2}{2I}$	M1A1	
$\Rightarrow \lambda^2 = 1 \Rightarrow \lambda = 1$	M1	
Hence $B = -3$ and $A = 2$	A1 (both)	[9]

Question 3

(i)

$V = -mgx + \frac{1}{2}k\left(\sqrt{x^2+a^2}+a-a\right)^2$	M1A1	
$= -mgx + \frac{1}{2}k\left(x^2+a^2\right)$	A1	
$V' = 0 = -mg + kx \quad \Rightarrow \quad x = \frac{mg}{k}$	M1A1	
$V'' = k > 0 \quad \therefore$ stable	M1, B1	[7]

(ii)

Energy equation is		
$\frac{1}{2}m\dot{x}^2 - mgx + \frac{1}{2}k\left(x^2+a^2\right) =$ constant	M1A1A1	
Diff wrt t: $\ddot{x} + \frac{kx}{m} = g$, period $= 2\pi\sqrt{\frac{m}{k}}$	M1,F1,A1(cao)	[6]

(iii)

$k = \frac{\text{modulus}}{\text{nat length}} \Rightarrow E = ka = k_1(a - a\varepsilon)$	B1	
$\therefore k_1 = \frac{ka}{a(1-\varepsilon)} = \frac{k}{1-\varepsilon} \approx k(1+\varepsilon)$	M1E1	[3]

(iv)

$\left(x_0^2 + a^2\right)^{\frac{1}{2}} \approx \left(2a^2 + 2a\beta\varepsilon\right)^{\frac{1}{2}} \approx a\sqrt{2} + O(\varepsilon)$	M1, for sub $x_0 = a + \beta\varepsilon$	
$\frac{dx_0}{a-d-x_0} \approx \frac{a\varepsilon(a+\beta\varepsilon)}{a - a\varepsilon - a - \beta\varepsilon} \approx \frac{-a^2}{\beta+a} + O(\varepsilon)$	M1, for using $\varepsilon << 1$ in expansions	
$\therefore \beta + a = \frac{-a}{\sqrt{2}} \Rightarrow \beta = -a\left(1+\frac{1}{\sqrt{2}}\right)$	A1, both values	
Eqm position moves $\varepsilon a\left(1+\frac{1}{\sqrt{2}}\right)$	E1, **ag**	[4]

Question 4

(i)

$\frac{d}{dt}(Mv) = Mg\sin\alpha - \mu Mg\cos\alpha,\quad \frac{d}{dt} = v\frac{d}{dx}$	M1A1A1, B1	
$mv\frac{d}{dx}((1+kx)v) = Mg(\sin\alpha - \mu\cos\alpha)$	M1A1	
$(1+kx)v\frac{dv}{dx} + kv^2 = (1+kx)(\sin\alpha - \mu\cos\alpha)g$	M1A1,**ag**	[8]
(ii)		
$\frac{d}{dx}\left(v^2(1+kx)^2\right) = 2(1+kx)\left(v(1+kx)\frac{dv}{dx} + kv^2\right)$	M1A1	
$= 2(1+kx)(1+kx)(\sin\alpha - \mu\cos\alpha)g$	M1	
$= 2(1+kx)^2(\sin\alpha - \mu\cos\alpha)g$	A1	
$v^2(1+kx)^2 = \frac{2}{3k}(1+kx)^3(\sin\alpha - \mu\cos\alpha)g + c$	M1	
$c = -\frac{2}{3k}(\sin\alpha - \mu\cos\alpha)g$	M1A1	
$v^2 = A(1+kx) + \frac{B}{(1+kx)^2},\quad A = -c,\ B = c$	A1, **ag**	[8]
(iii)		
As $x \to \infty$, $v^2 \propto A(1+kx)$	M1A1	
Acceleration $= v\frac{dv}{dx} \propto \frac{kA}{2} =$ constant	M1A1, **ag**	[4]

Examiner's Report

2612 Mechanics 6

General Comments

There were only 24 candidates for this paper this year but the standard was very good. It is difficult to reach any general conclusions with so few candidates, but given the high sophistication required for this paper they did themselves justice and can feel proud of their achievements. Question 1 was the least popular and question 4 the best answered.

Comments on Individual Questions

Q.1 Rotation of a rigid body

The candidates answered the first two parts of this question on a falling pulley very well. Writing down the equation of motion was found distinctly harder as, indeed, was solving it. Usually in a problem like this, candidates prefer to use energy ideas (whether asked for or not). For some reason this time they didn't.

(i) mg, 0

Q.2 Vectors

In the last few years the calculation of vector products in this paper has been done very well and this year is no exception. Finding the moments of the forces in part (i) proved no problem and candidates were then able to relate the moments to the concepts of angular momentum and angular velocity. Introducing a further force and energy ideas caused some difficulties but many handled them quite well.

(i) $c = 3$ (ii) $\mathbf{r} = \lambda(\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})$ (iii) $A = 2, B = -3$.

Q.3 Stability and oscillations

Finding the equilibrium position in this stability question did not cause the candidates much trouble in parts (i) and (ii). Similarly, finding the new stiffness in the shortened string in part (iii) was well done. Candidates then found the approximations required in the last part rather more taxing but many worked their way through the problem successfully.

(ii) period $= 2\pi\sqrt{\frac{m}{k}}$

Q.4 Variable mass

This was easily the best answered question on the paper with many candidates deservedly scoring full marks. They were able to derive the equation of motion and then show clearly how to solve it. The only part that caused problems for a few was in the final part, where they had to show that the acceleration tended to a constant value.

(ii) $B = -A$.