OCR RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2604

Pure Mathematics 4

Monday **13 JANUARY 2003** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

This question paper consists of 3 printed pages and 1 blank page.

[Turn over

1 A curve has equation $y = \dfrac{(2x+1)(x-11)}{(x+2)(x-5)}$.

(i) Write down the equations of the three asymptotes. [2]

(ii) Show that $\dfrac{dy}{dx} = \dfrac{3(5x^2 - 6x + 59)}{(x^2 - 3x - 10)^2}$. Hence show that there are no stationary points. [5]

(iii) Sketch the curve. [4]

(iv) On a separate diagram, sketch the curve with equation $y = \left|\dfrac{(2x+1)(x-11)}{(x+2)(x-5)}\right|$. [3]

(v) Solve the inequality $\left|\dfrac{(2x+1)(x-11)}{(x+2)(x-5)}\right| < 2$. [6]

2 **(a)** Find $\sum_{r=1}^{n} r(3r+2)$, giving your answer in a fully factorised form. [6]

(b) Show that $\dfrac{r+2}{r(r+1)} - \dfrac{r+3}{(r+1)(r+2)} = \dfrac{r+4}{r(r+1)(r+2)}$.

Hence or otherwise find the sum of the first n terms of the series

$$\frac{5}{1\times2\times3} + \frac{6}{2\times3\times4} + \frac{7}{3\times4\times5} + \frac{8}{4\times5\times6} + \dots .$$ [6]

(c) Prove by induction that $\sum_{r=1}^{n} \dfrac{2^r(r^2-2)}{(r+1)^2(r+2)^2} = \dfrac{2^{n+1}}{(n+2)^2} - \dfrac{1}{2}$. [8]

3 **(a)** Two complex numbers are $\alpha = -2 - 2\text{j}$, $\beta = \sqrt{2}(\cos\frac{3}{5}\pi + \text{j}\sin\frac{3}{5}\pi)$.

Find the modulus and argument of each of the following complex numbers, giving the arguments in radians between $-\pi$ and π:

$$\alpha,\quad \beta,\quad \alpha\beta,\quad \frac{\alpha}{\beta}.$$

Illustrate these four complex numbers on an Argand diagram. [10]

(b) The complex numbers w and z satisfy the equations

$$\frac{w+\text{j}}{1+z} = 6 - \text{j} \qquad \text{and} \qquad \frac{(1+\text{j})w+\text{j}}{1+\text{j}+z} = 3 + 4\text{j}.$$

(i) Show that

$$w = (6-\text{j})z + 6 - 2\text{j}$$

and

$$(1+\text{j})w = (3+4\text{j})z - 1 + 6\text{j}.$$ [3]

(ii) Hence find w and z. [7]

4 **(a)** **(i)** Find the equation of the line of intersection of the two planes

$$3x + 4y + 2z = 32$$

and

$$8x + 9y + 5z = 82.$$ [5]

(ii) Hence or otherwise solve the equation

$$\begin{pmatrix} 3 & 4 & 2 \\ 8 & 9 & 5 \\ k & 5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 32 \\ 82 \\ 14 \end{pmatrix},$$

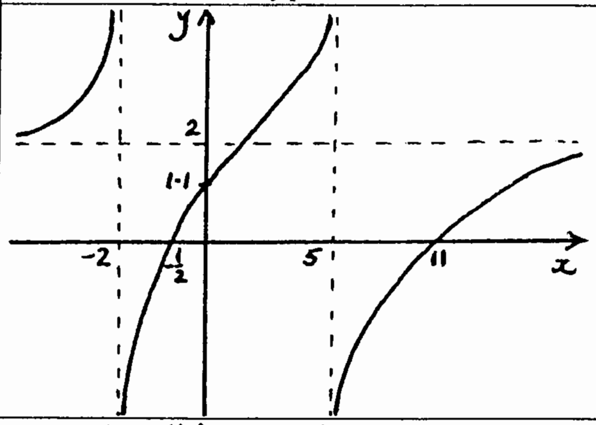
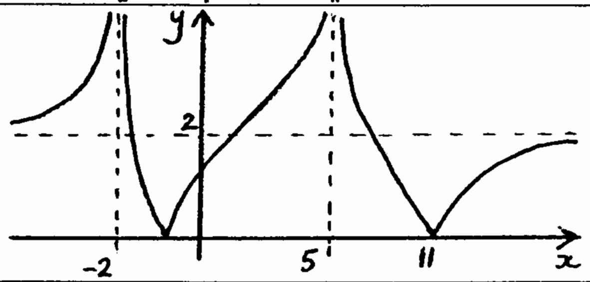
where $k \neq 0$, giving x, y and z in terms of k. [6]

(b) The matrix $\begin{pmatrix} -2 & 5 \\ -1 & 4 \end{pmatrix}$ defines a transformation in the x-y plane.

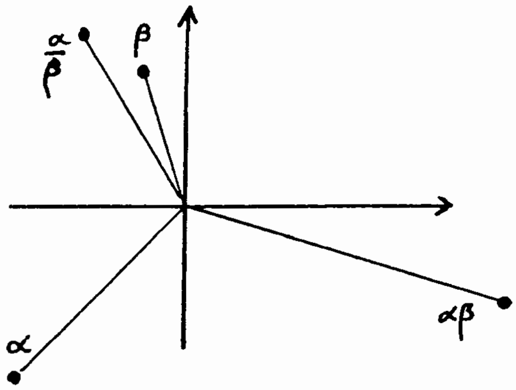
(i) Find the two values of m for which $y = mx$ is an invariant line of the transformation. [6]

(ii) Find the set of invariant points of the transformation. [3]

Mark Scheme

1 (i)	$x=-2,\ x=5$ $y=2$	B1 B1 2	
(ii)	$\frac{\mathrm{d}y}{\mathrm{d}x}$ $=\frac{(x^2-3x-10)(4x-21)-(2x^2-21x-11)(2x-3)}{(x^2-3x-10)^2}$ $=\frac{3(5x^2-6x+59)}{(x^2-3x-10)^2}$ Discriminant is $6^2-4\times5\times59=-1144<0$, So there are no stationary points.	M1 A1 A1 (ag) M1 A1 5	Use of quotient rule (or equivalent) Any correct form *Or equivalent* Correctly shown
(iii)		B1 B1 B1 B1 4	LH section: positive gradient, above x-axis Middle section: positive gradient, crossing axes at $x=-0.5$ and $y=1.1$ RH section: positive gradient, crossing x-axis at $x=11$ Fully correct shape and approaching asymptotes correctly
(iv)		B2 ft B1 ft 3	Negative sections reflected Give B1 if curve has just one negative section, or if one minor error made Two sharp points on x-axis *(No need for any numbers on axes)*
(v)	$\frac{(2x+1)(x-11)}{(x+2)(x-5)}=2$ when $x=0.6$ $\frac{(2x+1)(x-11)}{(x+2)(x-5)}=-2$ when $4x^2-27x-31=0$	M1A1 M1	If M0 then B2 for 0.6 (Accept $\frac{9}{15}$) If M0 then B1 for $-1,\ 7.75$
	OR $\frac{(2x+1)^2(x-11)^2}{(x+2)^2(x-5)^2}<4$ M1 $60x^3-441x^2-222x+279>0$ M1 Critical values 0.6 [$-1,\ 7.75$] A1		Cubic inequality For 0.6
	Considering intervals defined by critical values	M1	$x=-1,\ 0.6,\ 7.75$ (ft)
	OR $-2<\frac{(2x+1)(x-11)}{(x+2)(x-5)}<2$ M1 M1 A1 $x<-2,\ -1<x<5,\ x>7.75$ AND $-2<x<0.6,\ x>5$ M1		Method for solving *both* inequalities For 0.6 Combining the solutions
	$\left\|\frac{(2x+1)(x-11)}{(x+2)(x-5)}\right\|<2$ when $-1<x<0.6,\ x>7.75$	A2 cao 6	*Correct answer always earns 6 marks*

2 (a)	$\sum_{r=1}^{n}(3r^2+2r)$	B1	
	$=\frac{1}{2}n(n+1)(2n+1)+n(n+1)$	M1A1A1 ft	Ft only for $\sum r^2$ and $\sum r$
	$=\frac{1}{2}n(n+1)(2n+1+2)$	M1	Common factor $n(n+1)$
	$=\frac{1}{2}n(n+1)(2n+3)$	A1 cao **6**	Accept $n(n+1)(n+\frac{3}{2})$ etc
(b)	$\text{LHS}=\dfrac{(r+2)(r+2)-r(r+3)}{r(r+1)(r+2)}$	M1	
	$=\dfrac{(r^2+4r+4)-(r^2+3r)}{r(r+1)(r+2)}$		
	$=\dfrac{r+4}{r(r+1)(r+2)}$	A1 (ag)	
	Sum is $\sum_{r=1}^{n}\left(\dfrac{r+2}{r(r+1)}-\dfrac{r+3}{(r+1)(r+2)}\right)$	M1	Using given result
	$=\left(\frac{3}{2}-\frac{4}{6}\right)+\left(\frac{4}{6}-\frac{5}{12}\right)+\left(\frac{5}{12}-\frac{6}{20}\right)+\ldots$	A1	Two correct terms
	$=\dfrac{3}{2}-\dfrac{n+3}{(n+1)(n+2)}\quad\left[=\dfrac{n(3n+7)}{2(n+1)(n+2)}\right]$	M1 A1 **6**	One fraction at start, one at end
	OR Sum is $\sum_{r=1}^{n}\left(\dfrac{2}{r}-\dfrac{3}{r+1}+\dfrac{1}{r+2}\right)$ M1		Using partial fractions
	$=\left(\frac{2}{1}-\frac{3}{2}+\frac{1}{3}\right)+\left(\frac{2}{2}-\frac{3}{3}+\frac{1}{4}\right)+\ldots$ A1		Two correct terms
	$=\dfrac{2}{1}-\dfrac{3}{2}+\dfrac{2}{2}+\dfrac{1}{n+1}-\dfrac{3}{n+1}+\dfrac{1}{n+2}$ M1		Three fractions at start, three at end
	$=\dfrac{3}{2}-\dfrac{2}{n+1}+\dfrac{1}{n+2}$ A1		
(c)	When $n=1$, $\text{LHS}=\dfrac{2(1-2)}{2^2\times3^2}=-\dfrac{1}{18}$		
	$\text{RHS}=\dfrac{2^2}{3^2}-\dfrac{1}{2}=-\dfrac{1}{18}$	B1	
	so it is true when $n=1$		
	Assume true for $n=k$, then		
	$\sum_{r=1}^{k+1}=\dfrac{2^{k+1}}{(k+2)^2}-\dfrac{1}{2}+\dfrac{2^{k+1}\{(k+1)^2-2\}}{(k+2)^2(k+3)^2}$	M1A1	
	$=\dfrac{2^{k+1}\left(k^2+6k+9+k^2+2k-1\right)}{(k+2)^2(k+3)^2}-\dfrac{1}{2}$	M1	Use of common denominator
		M1	Numerator as $2^{k+1}\times$ simple quadratic
	$=\dfrac{2^{k+1}2(k+2)^2}{(k+2)^2(k+3)^2}-\dfrac{1}{2}$	M1	Use of $2^{k+1}\times2=2^{k+2}$
	$=\dfrac{2^{k+2}}{(k+3)^2}-\dfrac{1}{2}$	A1	Correctly obtained
	True for $n=k\Rightarrow$ True for $n=k+1$ (Hence true for all positive integers n)	A1 **8**	Stated or clearly implied *Dependent on previous 6 marks*

3 (a)	$\lvert\alpha\rvert=\sqrt{8}$, $\arg\alpha=-\frac{3}{4}\pi$ $\lvert\beta\rvert=\sqrt{2}$, $\arg\beta=\frac{3}{5}\pi$ $\lvert\alpha\beta\rvert=4$, $\arg(\alpha\beta)=-\frac{3}{20}\pi$ $\left\lvert\frac{\alpha}{\beta}\right\rvert=2$, $\arg\left(\frac{\alpha}{\beta}\right)=\frac{13}{20}\pi$	B1B1 B1 B1B1 ft B1B1 ft	*Accept decimals to at least 2 dp* *Penalise wrong form of arguments and/or insufficient accuracy once only*
		B1 B1 B1 **10**	α in 3rd quadrant, β in 2nd quadrant $\alpha\beta$ in 4th quadrant $\frac{\alpha}{\beta}$ in 2nd quadrant
(b)(i)	$\frac{w+j}{1+z}=6-j \Rightarrow w+j=6-j+(6-j)z$ $\Rightarrow w=(6-j)z+6-2j$	B1 (ag)	
	$\frac{(1+j)w+j}{1+j+z}=3+4j$ $\Rightarrow (1+j)w+j=(3+4j)(1+j)+(3+4j)z$ $\Rightarrow (1+j)w+j=3+7j-4+(3+4j)z$ $\Rightarrow (1+j)w=(3+4j)z-1+6j$	 M1 A1 (ag) **3**	 Multiplication using $j^2=-1$
(ii)	$(1+j)(6-j)z+(1+j)(6-2j)=(3+4j)z-1+6j$ $(7+5j)z+8+4j=(3+4j)z-1+6j$ $(4+j)z=-9+2j$	M1 M1	Eliminating w or z
	$z=\frac{-9+2j}{4+j}=\frac{(-9+2j)(4-j)}{(4+j)(4-j)}$ $=\frac{-34+17j}{17}$ $=-2+j$	M1 A1	Use of complex conjugate
	$w=(6-j)(-2+j)+6-2j$ $=-11+8j+6-2j$ $=-5+6j$	M1 M1 A1 **7**	Substitution Necessary manipulation
	OR Let $w=a+bj$, $z=c+dj$ $a=6c+d+6$, $b=-c+6d-2$ $a-b=3c-4d-1$, $a+b=4c+3d+6$ M1 Solving, $a=-5$, $b=6$ M2 $w=-5+6j$ A1 $c=-2$, $d=1$ M2 $z=-2+j$ A1		 Four equations Obtaining two of a, b, c, d Obtaining the other two

4(a)(i)	When $x=0$, $4y+2z=32$, $9y+5z=82$ $y=-2$, $z=20$	M1 A1	Finding one point One point correct
	Direction is given by $\begin{pmatrix}3\\4\\2\end{pmatrix}\times\begin{pmatrix}8\\9\\5\end{pmatrix}$	M1	*Or* finding a second point, e.g. $(4,0,10)$, $(8,2,0)$ and using it to find direction
	Direction is $\begin{pmatrix}2\\1\\-5\end{pmatrix}$	A1 cao	
	Equation of line is $\mathbf{r}=\begin{pmatrix}0\\-2\\20\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-5\end{pmatrix}$	A1 ft 5	*Dependent on M1M1.* Accept any form A0 if '$\mathbf{r}=$' omitted
	OR Eliminating z, $x-2y=4$ M1A1 M1A1 $x=2\lambda$, $y=-2+\lambda$, $z=20-5\lambda$ A1 ft		Eliminating one variable Expressing (e.g.) y and z in terms of x
(ii)	Substituting into $kx+5y+z=14$, $k(2\lambda)+5(-2+\lambda)+(20-5\lambda)=14$	M1 A1 ft	*Or* eliminating one variable in 2 ways *Or* 2 correct equations (in 2 variables)
	$\lambda=\dfrac{2}{k}$	M1 A1 cao	Obtaining λ *or* one of x, y, z *Or* one variable correct
	$x=\dfrac{4}{k}$, $y=-2+\dfrac{2}{k}$, $z=20-\dfrac{10}{k}$	M1 A1 cao 6	Obtaining at least two of x, y, z
(b)(i)	$\begin{pmatrix}-2&5\\-1&4\end{pmatrix}\begin{pmatrix}x\\mx\end{pmatrix}$	M1	
	$=\begin{pmatrix}-2x+5mx\\-x+4mx\end{pmatrix}$	A1	
	$y=mx$ is invariant if $-x+4mx=m(-2x+5mx)$	M1A1	
	$5m^2-6m+1=0$	M1	Quadratic in m
	$m=1,\ \dfrac{1}{5}$	A1 cao 6	
(ii)	For invariant points, $\begin{pmatrix}-2&5\\-1&4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$		
	$-2x+5y=x$ $-x+4y=y$	M1	
	$x=0$, $y=0$ only	A2 3	

Examiner’s Report

2604 Pure Mathematics 4

General Comments

This paper was found to be slightly more accessible than most of the previous ones. There was a wide range of ability, with about one third of candidates scoring 50 marks or more (out of 60), and about 20% scoring less than 30 marks. Many candidates used time consuming inefficient methods (notably in questions 1(v), 2(b), 3(b) and 4(a)(ii)) and then found themselves unable to complete the paper. Question 1 was attempted by almost every candidate, and question 4 was the least popular (attempted by about half the candidates).

Comments on Individual Questions

Question 1 (Curve sketching)

This was well answered, with half the attempts scoring 15 marks or more (out of 20), and about 20% scoring full marks. In part (i) the three asymptotes were almost always given correctly. In part (ii) the differentiation was usually done correctly and most candidates demonstrated that the quadratic equation had no real roots, although some simply stated that this was the case without any justification. In part (iii) the curve was sketched well; the main reasons for loss of marks were omitting the left-hand section or not giving the values where the curve crosses the axes. Sketching the modulus graph in part (iv) was very well understood. The inequality in part (v) was often quickly and accurately solved by finding the critical values and inspection of the graph. However, very many candidates attempted an algebraic solution without reference to the graph; this was usually a lengthy process and was rarely completed successfully. In particular, the 'squaring' method $|y| < 2 \Leftrightarrow y^2 < 4$ was often started but hardly ever proceeded as far as factorising the resulting cubic inequality.

(i) $x = -2,\ x = 5,\ y = 2$; (v) $-1 < x < 0.6,\ x > 7.75$.

Question 2 (Series)

This was the best answered question, with half the attempts scoring 17 marks or more, and about a third scoring full marks. In part (a) the use of the standard sums was well understood, but it was surprising how many candidates multiplied out the resulting expression instead of using the common factor $n(n+1)$. In part (b) the difference method was usually applied confidently and accurately, although many candidates used partial fractions both to prove the given result and to find the sum of the series. In part (c) the process of mathematical induction was very well understood and was frequently completed successfully.

(a) $\frac{1}{2}n(n+1)(2n+3)$; (b) $\dfrac{3}{2} - \dfrac{n+3}{(n+1)(n+2)}$.

Question 3 (Complex numbers)

In part (a) a large number of candidates were not able to find the moduli and arguments accurately. In part (b) there were a few attempts to equate real and imaginary parts treating w and z as if they were real numbers, but the great majority understood how to solve the equations. However, arithmetic slips were very common.

(a) $\alpha : \sqrt{8}, -\frac{3}{4}\pi,\ \beta : \sqrt{2}, \frac{3}{5}\pi,\ \alpha\beta : 4, -\frac{3}{20}\pi,\ \dfrac{\alpha}{\beta} : 2, \frac{13}{20}\pi$; (b)(ii) $w = -5 + 6\mathrm{j},\ z = -2 + \mathrm{j}$.

Question 4 (Vectors and matrices)

This was the worst answered question, with an average mark of about 11. In part (a)(i) most candidates understood how to find the equation of the line of intersection. In part (a)(ii) the 'expected' method was to substitute from the line of intersection into the third equation; candidates who did this were able to solve the equations quickly. However, most preferred to start afresh with three simultaneous equations and use elimination; very few then sustained the systematic approach necessary, and the constant k caused confusion, with many candidates producing a value for k. There were even a few attempts to find the inverse matrix. Part (b)(i) was very well understood, and the invariant lines were frequently found correctly. In part (b)(ii) the equations for invariant points were often written down correctly, but most candidates failed to make the correct conclusion from these equations.

(a)(i) $\mathbf{r} = \begin{pmatrix} 0 \\ -2 \\ 20 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$; (ii) $x = \frac{4}{k}$, $y = -2 + \frac{2}{k}$, $z = 20 - \frac{10}{k}$;

(b)(i) $1, \frac{1}{5}$; (ii) $(0, 0)$ only.