OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS — 2623/1

Numerical Methods

Monday **20 JANUARY 2003** Morning 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer **all** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The approximate allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

This question paper consists of 3 printed pages and 1 blank page.

1 In the difference table below, the values of x are exact but the values of $f(x)$ may be subject to error.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	1.77				
		1.87			
1	3.64		0.26		
		2.13		0.23	
2	5.77		0.49		0.18
		2.62		0.41	
3	8.39		0.90		
		3.52			
4	11.91				

(i) Use Newton's forward difference method to obtain a sequence of four estimates, linear, quadratic, cubic and quartic, for $f(0.8)$.

Assuming that the values of $f(x)$ are exact, give an estimated value for $f(0.8)$ to the accuracy that appears justified. [9]

(ii) Now assume that the values of $f(x)$ are rounded to 2 decimal places. State the maximum possible error in each value.

Calculate, for the linear estimate, the maximum possible error due to rounding. Explain what this implies for the higher order estimates.

Explain briefly whether, in these circumstances, you would revise your final answer in part **(i)**. [6]

2 A function $f(x)$ has values as given in the following table.

x	0	0.2	0.4	0.6	0.8
$f(x)$	0	0.4199	0.6033	0.7598	0.9160

(i) Use Simpson's rule with $h = 0.4$ to find an estimate of $\int_0^{0.8} f(x)\,dx$.

Find a further estimate using Simpson's rule with $h = 0.2$.

Hence obtain the best estimate you can of the value of the integral. Give your answer to the number of significant figures you can justify. [8]

(ii) Use the central difference formula with two different values of h to find estimates of $f'(0.4)$.

Hence obtain the best estimate you can of the value of the derivative. Give your answer to the number of significant figures you can justify. [7]

3 **(a)** The equation

$$2x + \tan x - 2 = 0,$$

where x is in radians, has a root between 0 and $\frac{1}{2}\pi$.

Starting with $x_0 = 0.6$, use the iteration

$$x_{r+1} = 1 - \tfrac{1}{2}\tan x_r$$

to obtain x_1, x_2, x_3. Use these values to show that this fixed point iteration has first order convergence. (You are not required to find the root.) [6]

(b) The equation

$$x^3 - 9x^2 + 6 = 0$$

has a root, α, in the interval (0, 1).

Use the Newton-Raphson method, starting with $x_0 = 0.8$, to find α correct to 7 decimal places. (You are advised to work with as many decimal places as your calculator supports.)

Demonstrate, using x_0, x_1, x_2 and α, that this iteration has second order convergence. [9]

4 In certain computer applications, a rough approximation is required to $\sqrt{x}$ where $0.25 \leqslant x \leqslant 1$. A formula sometimes used is

$$\sqrt{x} \approx \tfrac{2}{3}x + 0.36. \qquad (*)$$

(i) Find the two values of x for which there is zero error in this approximation.
[Hint: form a quadratic equation in t, where $t = \sqrt{x}$.] [4]

(ii) Find the absolute and relative errors when the approximation is used for $x = 0.25$ and $x = 0.64$. [4]

If s is the approximation to $\sqrt{x}$ given by (*), then an improved approximation is given by

$$\frac{s^2 + x}{2s}.$$

(iii) Find the relative error in the improved approximation when $x = 0.25$. [3]

(iv) Suppose that s overestimates $\sqrt{x}$ with a relative error of 1%. Write down an equation for s in terms of $\sqrt{x}$. Hence show that $\dfrac{s^2 + x}{2s}$ is very nearly $1.000\,05\sqrt{x}$. State the relative error in the improved approximation. [4]

Mark Scheme

1	**(i)**	f(0.8) = 1.77 + 0.8 x 1.87	3.266	linear	**[M1A1]**
		f(0.8) = linear + 0.8 (-0.2) 0.26 / 2	3.2452	quadratic	**[M1A1]**
		f(0.8) = quadratic + 0.8 (-0.2) (-1.2) 0.23 / 6	3.25256	cubic	**[M1A1]**
		f(0.8) = cubic + 0.8 (-0.2) (-1.2) (-2.2) 0.18 / 24	3.249392	quartic	**[M1A1]**
			3.25 seems reliable		**[A1]**
					[subtotal 9]
	(ii)	mpe in each f value is 0.005			**[A1]**
		mpe in linear estimate is 0.005 (1 + 0.8 x 2) = 0.013			**[M1A1]**
		Higher order estimates will all have at least this much mpe			**[E1]**
		as they all contain the linear estimate			**[E1]**
		Hence 2nd dp is unreliable			**[E1]**
					[subtotal 6]
					[TOTAL 15]

2	**(i)**	S(0.4) = (0.4 / 3) (0 + 4 x 0.6033 + 0.9160) =		0.443893	**[M1A1]**
		S(0.2) = (0.2 / 3) (0 + 4 x 0.4199 + 2 x 0.6033 + 4 x 0.7598 + 0.9160) =		0.456093	**[M1A1]**
		difference 0.0122			
		extrapolating (any valid method)	$0.456093 + 0.0122\,(1/16 + (1/16)^2 + \ldots)$ =	0.456907	**[M1A1]**
			3rd dp unreliable so	0.46	**[E1A1]**
					[subtotal 8]
	(ii)	h = 0.4	f ' (0.4) = (0.9160 - 0) / 0.8 =	1.145	**[M1A1]**
		h = 0.2	f ' (0.4) = (0.7598 - 0.4199) / 0.4 =	0.84975	**[A1]**
		difference -0.29525			
		extrapolating (any valid method)	$0.84975 - 0.29525\,(1/4 + (1/4)^2 + \ldots)$ =	0.751333	**[M1A1]**
		Can't be sure of 0.7 (or 0.8): have to settle for an estimate of		1	**[E1A1]**
					[subtotal 7]
					[TOTAL 15]

3 (a)

x_0	0.6	Δ	ratio	
x_1	0.657932	0.057932		**values [A1]**
x_2	0.613602	-0.04433	-0.76521	**diffs [M1A1]**
x_3	0.647853	0.034251	-0.77264	**ratio [M1A1]**

differences reducing by (nearly) constant factor so first order **[E1]**

[subtotal 6]

(b) NR: $x_{r+1} = x_r - (x_r^3 - 9x_r^2 + 6) / (3x_r^2 - 18x)$

		$x_i - \alpha$	$(x_i - \alpha)^2/(x_{i+1} - \alpha)$		
					NR formula [M1A1]
x_0	0.8	-0.05846			**root [M1A1]**
x_1	0.86025641	0.001792	1.907809	these values	**diffs [M1A1]**
x_2	0.85846631	1.55E-06	2.065791	approx equal	**ratio [M1A1]**
x_3	0.85846476				**explanation [E1]**
x_4	0.85846476	$= \alpha$			

[subtotal 9]

[TOTAL 15]

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4 (i) $2t^2 - 3t + 1.08 = 0$ where $t = \text{sqrt}(x)$ **[M1A1]**
$t = 0.6, 0.9$ **[A1]**
$x = 0.36, 0.81$ **[A1]**

[subtotal 4]

(ii)

x = 0.25	approx: 0.5267	error: 0.0267	rel error: 0.053	**[M1A1]**
x = 0.64	approx: 0.7867	error: -0.0133	rel error: -0.017	**[A1]**
		consistent use of signs		**[A1]**

[subtotal 4]

(iii) x = 0.25 gives s = 0.5267 which leads to the improved estimate 0.500675 **[M1A1]**
with relative error 0.00135 **[A1]**

[subtotal 3]

(iv) s = 1.01 sqrt(x) **[B1]**
improved estimate $=((1.01\ \text{sqrt}(x))^2 + x) / (2.02\ \text{sqrt}(x))$ **[M1]**
= 2.0201 x / 2.02 sqrt(x) **[A1]**
approx 1.00005 sqrt(x)
I.e. the improved approximation has relative error 0.00005 **[A1]**

[subtotal 4]

[TOTAL 15]

Examiner’s Report

2623 Numerical Methods

General Comments

The performance of candidates on this paper was, in many cases, rather disappointing. There were very few able to demonstrate a thorough grasp of all of the standard techniques. Even the 'number-crunching', which is often very sound, was less good than usual in some questions. For whatever reason, it appears that candidates were just less well prepared than in the past.

Comments on Individual Questions

Question 1 (finite differences)

Most candidates were able to produce the required sequence of estimates in part (i), and f(0.8) was given as 3.25 in most cases. Part (ii), however, defeated the vast majority. The maximum possible error in the linear estimate can be calculated quite easily by considering its greatest and least values, but very few candidates managed to do that successfully. The point that errors in the linear estimate are carried forward into higher order estimates was understood by many, though often the explanation was vague as was the statement about revising the answer to part (i).

(i) 3.266, 3.2452, 3.25256, 3.249392, 3.25. (ii) 0.005, 0.013.

Question 2 (integration and differentiation)

In both parts of this question the numerical work was done reasonably well - though in part (i) quite a few candidates managed to get the composite Simpson rule formula wrong. In each part of the question required candidates to obtain the best estimate possible from the two initial estimates. As in many previous questions, the technique required is extrapolation from $h = 0.4$ and $h = 0.2$. Only a minority knew what to do.

(i) 0.443893, 0.456093, 0.456907 by extrapolation, 0.46.
(ii) 1.145, 0.84975, 0.75133 by extrapolation, 1.

Question 3 (solution of equations, order of convergence)

In part (a) the iterates were generally found successfully, but candidates were much less happy demonstrating first order convergence. What was required was to show that the ratio of successive differences between iterates is approximately constant. In part (b) the differences had to be taken from the root - which is why the root was asked for explicitly. Then it is a matter of examining the ratio of e_r^2 and e_{r+1}. Very few understood this.

(b) 0.858465.

Question 4 (approximating square roots)

The quadratic equation in part (i) was a difficulty for many. Those who solved correctly for t sometimes failed to transform back to x successfully. In part (ii) the magnitudes of the relative errors were often calculated accurately, but almost nobody preserved the signs. In part (iii) the relative error was generally found correctly. Part (iv) was algebraically challenging, but a pleasing number of candidates tackled it confidently and accurately.

(i) 0.36, 0.81. (ii) 0.053, –0.017. (iii) 0.00135.