OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2612

Mechanics 6

Thursday 23 MAY 2002 Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The approximate allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- Take $g = 9.8$ m s^{-2} unless otherwise instructed.
- The total number of marks for this paper is 60.

This question paper consists of 4 printed pages.

[Turn over

1 *Option 1: Rotation of a rigid body*

A uniform cylinder of mass m and radius a is spinning with angular speed ω about its horizontal central axis. When it is placed gently on a rough plane inclined at an angle α to the horizontal, it immediately begins to slip and simultaneously begins to move up the plane. At time t later, the speed of the centre of mass of the cylinder parallel to the surface is $\dot{x}$ and the angular speed is $\dot{\theta}$. The coefficient of friction between the cylinder and the plane is μ.

(i) Write down the equation of motion for the linear motion of the centre of mass of the cylinder and the rotation equation about the central axis. [6]

(ii) Show that the distance moved by the cylinder while still slipping up the plane at time t is given by

$$x = \frac{gt^2}{2}(\mu\cos\alpha - \sin\alpha),$$

and find an expression for the angular speed at time t. [7]

(iii) Show that the cylinder ceases to slip after a time

$$\frac{a\omega}{3\mu g\cos\alpha - g\sin\alpha}.$$ [4]

In the special case when $\alpha = 0$ and the plane is horizontal, the distance moved while slipping is $L = \frac{a^2\omega^2}{18\mu g}$, the loss of energy is $\frac{ma^2\omega^2}{6}$ and the frictional force is μmg. [*You are not required to establish these results.*]

(iv) Verify that the loss of energy is greater than μmgL. Give an explanation of this result in terms of the mechanics of the situation. [3]

2 *Option 2: Vectors*

A system of forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ acts on a rigid body and the sum of the moments of the forces about the origin O is $\mathbf{M}$.

(i) Show that, if the system of forces reduces to a single force, $\mathbf{F}$, then $\mathbf{F}.\mathbf{M} = 0$. [4]

In a particular case, $\mathbf{F}_1 = (2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$, $\mathbf{F}_2 = (-\mathbf{i} + A\mathbf{j} - \mathbf{k})$ and $\mathbf{F}_3 = (2\mathbf{i} + \mathbf{k})$ where A is a constant. The three forces act at the points with position vectors $(\mathbf{i} + \mathbf{j})$, $(\mathbf{i} + \mathbf{j} + \mathbf{k})$ and $(\mathbf{j} + \mathbf{k})$ respectively relative to O.

(ii) Find the resultant force $\mathbf{F}$ in terms of A, and show that the sum of the moments of the forces about the origin is $(3 - A)\mathbf{i} - \mathbf{j} + (A - 1)\mathbf{k}$. [6]

(iii) Find the value of A for which the three forces reduce to the single force $\mathbf{F}$ and find the line of action of $\mathbf{F}$ in this case. [10]

3 *Option 3: Stability and oscillations*

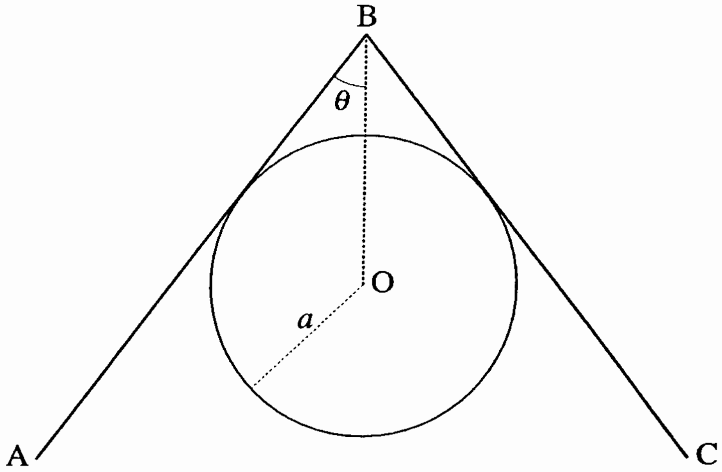

Fig. 3

Two light rods AB and BC, each of length l, are smoothly jointed at B and are placed on a smooth fixed cylinder, as shown in Fig. 3.

The radius of the cylinder is a and its axis is horizontal. The rods each carry a mass m at their free ends A and C, and B moves along the vertical line through O. In a general position, the angle OBA is equal to θ.

(i) Show that the potential energy V, relative to O, can be written

$$V = 2mg\left(\frac{a}{\sin\theta} - l\cos\theta\right).$$

Show that a position of equilibrium occurs where $a\cos\theta = l\sin^3\theta$.

Explain, graphically or otherwise, why this equation has only one solution for $0 < \theta < \frac{1}{2}\pi$. Show further that the position of equilibrium is stable. [8]

(ii) Show that if the rods are in equilibrium with $\theta = \frac{1}{4}\pi$, then $l = 2a$. [2]

In the rest of the question $l = 2a$ and the rods move so that $\theta = \frac{1}{4}\pi + \phi$, where ϕ remains very small.

(iii) Show that $\sin\theta \approx \frac{1}{\sqrt{2}}(1 + \phi - \frac{1}{2}\phi^2)$ and $\cos\theta \approx \frac{1}{\sqrt{2}}(1 - \phi - \frac{1}{2}\phi^2)$.

Deduce that, near equilibrium, $V \approx 4mga\sqrt{2}\phi^2$. [7]

(iv) You are given that, near equilibrium, the kinetic energy $T \approx 2ma^2\dot{\phi}^2$.

Show that the system moves in approximate simple harmonic motion and find the period of oscillation. [3]

4 *Option 4: Variable mass*

A particle falls under gravity from rest in a stationary cloud. It picks up mass as it falls through the cloud. The rate at which the mass of the particle increases is equal to mkv where m and v are the mass and speed at time t and k is a constant.

(i) Write down the equation of motion for the particle. [2]

(ii) Write down an expression for $\frac{dm}{dt}$. Using this and the equation of motion, show that

$$v\frac{dv}{dx} + kv^2 = g,$$

where x is the distance fallen in time t. [4]

(iii) Solve the equation in part **(ii)** to show that

$$v^2 = \frac{g}{k}(1 - e^{-2kx}).$$ [9]

(iv) Explain what happens to the speed and mass as time increases. [5]

Mark Scheme

Question 1

(i)	$\mu R - mg\sin\alpha = m\ddot{x}$	M1A1	
	$\frac{1}{2}ma^2\ddot{\theta} = -Fa = -\mu Ra$	M1A1	
	$mg\cos\theta = R$	M1A1	[6]
(ii)	$m\ddot{x} = \mu mg\cos\alpha - mg\sin\alpha$	M1	
	$\dot{x} = gt(\mu\cos\alpha - \sin\alpha)$	M1A1	
	$x = \dfrac{gt^2}{2}(\mu\cos\alpha - \sin\alpha)$	A1	
	$\ddot{\theta} = \dfrac{-2\mu g\cos\alpha}{a}$	M1	
	$\dot{\theta} = \omega - \dfrac{2\mu gt\cos\alpha}{a}$	M1A1	[7]
(iii)	$\dot{x} = a\dot{\theta}$ when rolling starts	M1	
	$gt(\mu\cos\alpha - \sin\alpha) = a\omega - 2\mu gt\cos\alpha$	M1A1	
	$t = \dfrac{a\omega}{3\mu g\cos\alpha - g\sin\alpha}$	A1	[4]
(iv)			
	$\mu mgL = \dfrac{ma^2\omega^2}{18} < \dfrac{ma^2\omega^2}{6}$ = loss of energy	B1	
	Work done is equal to product of frictional force and *relative* distance moved by surfaces in contact.	B1	
	i.e. difference between $a\theta$ and L.	B1	[3]

Question 2

(i) sum of moments about O of forces = **M**	M1	
moment of resultant F about O $= \mathbf{r}\times\mathbf{F} = \mathbf{M}$	A1	
$\mathbf{F}.\mathbf{M} = \mathbf{F}.(\mathbf{r}\times\mathbf{F}) = 0$	M1A1	[4]
(ii) $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \begin{pmatrix} 3 \\ 2+A \\ 3 \end{pmatrix}$	M1A1	
$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}$	M1A1	
$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ A \\ -1 \end{pmatrix} = \begin{pmatrix} -1-A \\ 0 \\ A+1 \end{pmatrix}$	A1	
$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$		
sum $= \mathbf{M} = \begin{pmatrix} 3-A \\ -1 \\ A-1 \end{pmatrix}$	A1	[6]
(iii) $\mathbf{F}.\mathbf{M} = 0 \Rightarrow 3(3-A) + (-1)(2+A) + 3(A-1) = 0$	M1A1	
$\Rightarrow \quad A = 4$	A1	
line of action $r = \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix}$	M1A1	
$\begin{pmatrix} 1 \\ a \\ b \end{pmatrix} \times \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$	M1A1	
$3a - 6b = -1$ $3b - 3 = -1$ $6 - 3a = 3$	M1	
$a = 1,\ b = 2/3$	A1	
$\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ \frac{2}{3} \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix}$	A1	[10]

Question 3

(i)	PE = mg(OB-ABcosθ) rel to O for each mass	M1	
	$V = 2mg\left(\frac{a}{\sin\theta} - l\cos\theta\right)$	A1	
	$V' = 2mg\left(-\frac{a\cos\theta}{\sin^2\theta} + l\sin\theta\right)$	M1	
	$= 0$ when $l\sin^3\theta = a\cos\theta$	A1	
	Compare curves y = cosθ and $y = \frac{l}{a}\sin^3\theta$ to show one solution only	B1	
	$V'' = 2mg\left(l\cos\theta + a\left(\frac{\sin^3\theta + 2\sin\theta\cos^2\theta}{\sin^4\theta}\right)\right)$	M1	
	> 0 since all terms are > 0 in $0 < \theta < \frac{\pi}{2}$	M1	
	hence stable equilibrium	A1	[8]
(ii)	when $\theta = \frac{\pi}{4}, V' = 0 \therefore \frac{a}{\sqrt{2}} = \frac{l}{2\sqrt{2}} \Rightarrow l = 2a$	M1A1	[2]
(iii)	$\sin\theta = \sin(\frac{\pi}{4} + \phi) = \frac{1}{\sqrt{2}}(\cos\phi + \sin\phi)$ $\approx \frac{1}{\sqrt{2}}(1 + \phi - \frac{\phi^2}{2})$, near $\phi = 0$	M1A1	
	$\therefore \frac{1}{\sin\theta} \approx \sqrt{2}(1 + \phi - \frac{\phi^2}{2})^{-1} \approx \sqrt{2}(1 - \phi + \frac{\phi^2}{2} + \phi^2)$ $\approx \sqrt{2}(1 - \phi + \frac{3\phi^2}{2})$, near $\phi = 0$	A1	
	$\cos\theta = \frac{1}{\sqrt{2}}(\cos\phi - \sin\phi)$ $\approx \frac{1}{\sqrt{2}}(1 - \phi - \frac{\phi^2}{2})$, near $\phi = 0$	M1A1	
	Hence, $V \approx 2mga\sqrt{2}(1 - \phi + \frac{3\phi^2}{2} - 1 + \phi + \frac{\phi^2}{2})$ $\approx 2mga\sqrt{2}(2\phi^2) \approx 4mga\sqrt{2}\phi^2$	M1A1	[7]
(iv)	$T+V$ = constant and hence $2ma^2\dot{\phi}^2 + 4mga\sqrt{2}\phi^2 \approx$ constant $\therefore \ddot{\phi} + \frac{2g}{a}\sqrt{2}\phi \approx 0$	M1A1	
	i.e.SHM of period $2\pi\sqrt{\frac{a}{2g\sqrt{2}}}$	A1	[3]

Question 4

(i)	N2 gives $\frac{d}{dt}(mv) = mg$	M1A1	[2]
(ii)	$\frac{dm}{dt} = kmv$	B1	
	$mg = \frac{d}{dt}(mv) = m\frac{dv}{dt} + v\frac{dm}{dt},$	M1	
	$\frac{dv}{dt} = v\frac{dv}{dx} \Rightarrow v\frac{dv}{dx} + kv^2 = g$	B1, A1	[4]
(iii)	With $w = v^2$, $\frac{dw}{dt} + 2kw = 2g$	B1, M1A1	
	$\frac{d}{dx}\left(we^{2kx}\right) = 2ge^{2kx}$	M1A1	
	$we^{2kx} = A + \frac{g}{k}e^{2kx}$	A1	
	When $x = 0$, $w = v^2 = 0$, $\therefore A = -\frac{g}{k}$	M1A1	
	$w = v^2 = \frac{g}{k}\left(1 - e^{-2kx}\right)$	A1	[9]
(iv)	Emphasise that x and t increase together, either physically or mathematically.	B1	
	As $x \to \infty$, $e^{-2kx} \to 0$, $\therefore v \to \sqrt{\frac{g}{k}}$	B1	
	$\frac{1}{m}\frac{dm}{dt} = kv, \therefore \ln m = kx + \ln M$	M1	
	where $m = M$ at $x = 0$, $\therefore m = Me^{kx}$	A1	
	$\therefore m \to \infty$ as $x \to \infty$	A1	[5]

Examiner’s Report

2612 Mechanics 6

General Comments

The number of candidates for this paper this year was a disappointing 39. The small number however was more than made up for in quality as I feel this is the best group of candidates I have seen for this paper for a good number of years. There were very few really weak ones with most scoring very high marks indeed. Having well over a quarter of the entry score 55 or more out of 60 is testament alone to the high standards achieved. I congratulate the candidates and their teachers for a job well done.

It seems a little churlish to criticise on this occasion but even excellence can be improved!

Comments on Individual Questions

Q.1 Most candidates made an excellent start to this question on the slipping of a rough cylinder on an inclined slope. Sometimes they needed a couple of attempts to get going but once under way they recognised the topic and soon almost completed the question. The very last part caused some difficulty in that the candidates were unable to explain how energy was lost in the slipping phase and relate it to the distance moved.

(ii) $\omega - \frac{2\mu g t \cos\alpha}{a}$

Q.2 By far the most difficult part of this vector question for the candidates was the standard bookwork in part (i). They had the greatest difficulty in explaining clearly the condition for a system of forces to reduce to a single couple. They were very confident at applying the condition though and the vast majority went from this hesitant start to complete the rest of the question with some ease. The inevitable mistakes that were made revolved around the need to maintain accuracy especially in the finding of vector products.

(ii) $\mathbf{F} = 3\mathbf{I} + (2 + A)\mathbf{j} + 3\mathbf{k}$, (iii) $A = 4$, $r = (1 + \lambda)\mathbf{I} + (1 + 2\lambda)\mathbf{j} + \left(\frac{2}{3} + \lambda\right)\mathbf{k}$.

Q.3 The technique used by the candidates in dealing with this question was most impressive. As in question 2 there were those who had problems maintaining accuracy but on the whole this was a very well answered question by those who attempted it. The candidates understood the mechanics and applied the necessary tests with some confidence. Expanding $(\sin\theta)^{-1}$ for values of θ near to $\frac{\pi}{4}$ was somewhat hesitant but usually completed eventually.

(iv) $2\pi\sqrt{\frac{a}{2g\sqrt{2}}}$.

Q.4 This question on a particle picking up mass from a stationary cloud was completed quickly and easily by the majority of candidates. They understood the principles and applied them with confidence and clarity. The equation in part (ii) was most often solved using the method of separation of variables with the correct boundary conditions applied. Part (iv) required candidates to explain what happens as time increases. Many used the result in part (iii) for this either not noticing (or disregarding) the fact that the result in (iii) expresses variation in *distance* not time. This requires a crucial extra step in the explanation.