OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2611

Mechanics 5

Thursday **23 MAY 2002** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The approximate allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- Take $g = 9.8 \text{ m s}^{-2}$ unless otherwise instructed.
- The total number of marks for this paper is 60.

This question paper consists of 3 printed pages and 1 blank page.

[Turn over

1 *Variable Forces*

A particle is constrained to move from the origin O in a two dimensional coordinate system to the point P(4, 3) along various paths under the influence of the force $\mathbf{F} = xy\mathbf{i} + x^2y\mathbf{j}$, in appropriate units.

(i) Write down an expression for the work done by the force on the particle, expressing your answer as the sum of two scalar integrals. [3]

(ii) Find the work done by the force as the particle follows the path described in each of the folowing three cases:

(*A*) The straight line OP; [5]

(*B*) A straight line from O to R(5, 0) followed by the straight line RP; [6]

(*C*) The straight line OR followed by the smaller arc of the circle $x^2 + y^2 = 25$ through the points R and P. [6]

2 *Relative Motion*

An aircraft flies with a constant speed relative to the air (air speed) of $200\,\mathrm{km\,h^{-1}}$. On a particular day, the wind blows at $30\,\mathrm{km\,h^{-1}}$ from the north-east. The aircraft takes off at noon and flies from airport P to another airport Q which is 300 km due north of P.

(i) Draw a relative velocity diagram for the aircraft and hence find the speed of the aircraft relative to the ground (i.e. its ground speed). [7]

A second identical aircraft also takes off at noon and flies at the same air speed from airport Q to another airport R due west of Q. The wind still blows at $30\,\mathrm{km\,h^{-1}}$ from the north-east.

(ii) Draw a relative velocity diagram for the second aircraft and hence find the ground speed in this case. [6]

(iii) Find the shortest distance between the two aircraft and the time taken to reach this position. [7]

3 *Motion described in Polar Coordinates*

A plane polar coordinate system (r,θ) has unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ in the radial and transverse directions respectively and O is the origin.

(i) Show that $\dfrac{\mathrm{d}}{\mathrm{d}t}(\hat{\mathbf{r}}) = \dot{\theta}\,\hat{\boldsymbol{\theta}}$ and $\dfrac{\mathrm{d}}{\mathrm{d}t}(\hat{\boldsymbol{\theta}}) = -\dot{\theta}\,\hat{\mathbf{r}}$. [6]

The position vector of a particle in plane polar coordinates is $\mathbf{r} = r\hat{\mathbf{r}}$, where r is a function of θ, and the only force acting on the particle is directed towards O.

(ii) Show that the velocity vector $\dot{\mathbf{r}}$ is given by

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}},$$

and find an expression for the acceleration vector $\ddot{\mathbf{r}}$. Hence deduce that $r^2\dot{\theta} = k$, where k is a constant. [7]

(iii) By writing $r = \dfrac{1}{u}$, where u is a function of θ, show that

$$\dot{\mathbf{r}} = -k\frac{\mathrm{d}u}{\mathrm{d}\theta}\hat{\mathbf{r}} + ku\hat{\boldsymbol{\theta}}.$$

Deduce that, when the speed of the particle is a constant, either $\dfrac{\mathrm{d}u}{\mathrm{d}\theta} = 0$ or $\dfrac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = 0.$

Give a geometrical interpretation of the path described by the equation $\dfrac{\mathrm{d}u}{\mathrm{d}\theta} = 0.$ [7]

4 *Rotation of a rigid body*

(You are given that the moment of inertia about a diameter of a hollow sphere of mass m and radius b is $\frac{2}{3}mb^2$.)

A solid sphere of radius a and mass M has density ρ which varies with radius r according to the formula $\rho = \rho_0(1 - \frac{r^3}{a^3})$, where ρ_0 is a constant.

(i) Show that $M = \frac{2}{3}\pi \rho_0 a^3$. [6]

(ii) Find the moment of inertia of the sphere about a diameter in terms of M and a. [6]

A uniform hoop has radius a and mass m.

(iii) Assuming that the moment of inertia of this hoop about the axis perpendicular to its plane through its centre is ma^2, deduce its moment of inertia about any diameter.

When the hoop is rotated about a diameter with angular speed ω, its kinetic energy is E. When the sphere is rotated about a diameter with angular velocity 2ω, its kinetic energy is $3E$. Find m in terms of M. [8]

Mark Scheme

Question 1

(i) WD = $\int_O^P \mathbf{F}.\mathbf{dr} = \int_0^4 xy\,\mathrm{d}x + \int_0^3 x^2 y\,\mathrm{d}y$	M1A1A1	[3]
(ii) (A) $y = \frac{3x}{4}, \quad dy = \frac{3}{4}\mathrm{d}x$	M1	
WD = $\int_0^4 \left[\frac{3x^2}{4} + x^2.\frac{3x}{4}.\frac{3}{4}\right]\mathrm{d}x = \int\left(\frac{3x^2}{4} + \frac{9}{16}x^3\right)\mathrm{d}x$	M1A1	
$= \left[\frac{x^3}{4} + \frac{9x^2}{64}\right] = 52$	A1	
Correct diagram	B1	[5]
(B) Along OR $y = 0$, hence WD = 0	B1	
Along RP, $y = 15 - 3x$, $\mathrm{d}y = -3\mathrm{d}x$	M1	
WD = $\int_5^4 \left[15x - 3x^2 - 3x^2(15 - 3x)\right]\mathrm{d}x$	M1A1	
$= -328 + 406.25 = 78.25$	A1	
Correct diagram	B1	[6]
(C) Along OR, WD = 0		
Along RP, $x^2 + y^2 = 25$ and $x\mathrm{d}x = -y\mathrm{d}y$	M1	
WD = $\int_5^4 (x(25 - x^2)^{\frac{1}{2}} - x^2.x)\mathrm{d}x$	M1A1A1	
$= [(-9 - 64) - (-156.25)]$		
$= 83.25$	A1	
Correct diagram	B1	[6]

Question 2

(i) ${}_wV_g$ = wind velocity relative to the ground = 30		
${}_aV_w$ = aircraft velocity relative to the wind = 200		
${}_aV_g = V_1$ = aircraft velocity relative to the ground		
$40000 = 900 + V_1^2 - 60V_1\cos(\frac{\pi}{4} + \frac{\pi}{2})$, by cos rule	M1A1	
solving quadratic	M1	
$V_1 = 177.66\,\text{km}\,\text{h}^{-1}$	A1	
correct diagram	B3	[7]
(ii) V_2 = new aircraft velocity relative to the ground		
$40000 = 900 + V_2^2 - 60V_2\cos\frac{\pi}{4}$, by cos rule	M1A1	
solving quadratic	M1	
$V_2 = 220.09\,\text{km}\,\text{h}^{-1}$	A1	
correct diagram	B2	[6]
(iii) d = shortest distance		
$\tan\theta = \dfrac{177.66}{220.09} \Rightarrow \theta = 38.9°$ or $0.679\,\text{rad}$	M1	
$d = 300\cos\theta = 233.44$ km	M1A1	
Time required $= \dfrac{300\sin\theta}{\sqrt{177.66^2 + 220.09^2}} = \dfrac{300\sin\theta}{282.85}$		
= 0.666 hours or 40 minutes	M1A1	
correct diagram	B2	[7]

Question 3

(i) $\hat{\mathbf{r}} = (\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}),\hat{\ } = (-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}),$	M1 either,A1 both	
$\frac{\mathrm{d}}{\mathrm{d}t}(\hat{\mathbf{r}}) = (-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}) = \dot{\theta}\,\hat{\ }$	M1A1	
$\frac{\mathrm{d}}{\mathrm{d}t}(\hat{\ }) = (-\cos\theta\,\mathbf{i} - \sin\theta\,\mathbf{j}) = -\dot{\theta}\hat{\mathbf{r}}$	M1A1	[6]
(ii) $\mathbf{r} = r\hat{\mathbf{r}},\therefore \dot{\mathbf{r}} = r\frac{\mathrm{d}}{\mathrm{d}t}(\hat{\mathbf{r}}) + \dot{r}\hat{\mathbf{r}} = r\dot{\theta}\,\hat{\ } + \dot{r}\hat{\mathbf{r}}$	M1A1, **ag**	
$\left.\begin{array}{l}\ddot{\mathbf{r}} = (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\ } + r\dot{\theta}(-\dot{\theta})\hat{\mathbf{r}} + \ddot{r}\hat{\mathbf{r}} + \dot{r}(\dot{\theta}\,\hat{\ }) \\ = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\ }\end{array}\right\}$	M1A1A1	
$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = 0$ hence $r^2\dot{\theta} = \text{constant} = k$	M1A1, **ag**	[7]
(iii) $r = \frac{1}{u}, u = u(\theta), \dot{\mathbf{r}} = \frac{1}{u}\dot{\theta}\,\hat{\ } + \left(\ \frac{1}{u^2}\right)u'\dot{\theta}\hat{\mathbf{r}}$	M1,	
$\therefore \dot{\mathbf{r}} = -ku'\hat{\mathbf{r}} + ku\hat{\boldsymbol{\theta}}$	A1, **ag**	
Speed is constant when $k^2(u'^2 + u^2) = \text{constant}$	M1	
$\frac{\mathrm{d}}{\mathrm{d}t}(u'^2 + u^2) = 2u'(u'' + u) = 0$	M1	
$\therefore$ either $\frac{\mathrm{d}u}{\mathrm{d}\theta} = 0$ or $\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = 0$	A1, A1,**ag**	
$\frac{\mathrm{d}u}{\mathrm{d}\theta} = 0 \Rightarrow u = \frac{1}{r} = \text{constant}$ ie a circle	B1	[7]

Question 4

(i) $\delta M = 4\pi r^2 \rho \delta r$ for mass of shell thickness δr	M1A1	
$M = \int_0^a 4\pi r^2 \rho \mathrm{d}r = 4\pi\rho_0 \int_0^a r^2\left(1 - \frac{r^3}{a^3}\right)\mathrm{d}r$	M1A1 (ignore limits)	
$= 4\pi\rho_0\left(\frac{a^3}{3} - \frac{a^3}{6}\right) = \frac{2\pi\rho_0 a^3}{3}$	M1 (for integration), A1, **ag**	[6]
(ii) $\delta I = \frac{2}{3}.r^2 4\pi r^2 \rho_0 \delta r$, for a shell thickness δr	M1A1	
$I = \frac{8\pi}{3}\rho_0 \int_0^a \left(r^4 - \frac{r^7}{a^3}\right)\mathrm{d}r$	M1A1 (ignore limits)	
$= \frac{8\pi}{3}\rho_0\left(\frac{a^5}{5} - \frac{a^5}{8}\right) = \frac{3Ma^2}{10}$	M1A1	[6]
(iii) I_D =m of I of hoop about any diameter Hence m of I of hoop about normal = $2I_D = ma^2$, using perp axis theorem. Hence $I_D = ma^2/2$	M1A1	
For hoop, $\frac{1}{2}.\frac{ma^2}{2}.\omega^2 = E$	M1A1	
For sphere, $\frac{1}{2}.\frac{3Ma^2}{10}(2\omega)^2 = 3E$	M1A1	
Dividing, $\frac{3Ma^2}{10}.4\omega^2.\frac{2}{ma^2\omega^2} = \frac{3E}{E}$	M1	
$\frac{24M}{10m} = 3, \quad \therefore 4M = 5m$	A1	[8]

Examiner’s Report

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General Comments

There were about 68 candidates for this paper and I am delighted to say that the standard of the work presented this year was higher in my view than for some time. As always there were some candidates who had difficulty with some parts but usually more than made up for it elsewhere. The perennial problem with algebraic manipulation was still present but was not so crucial with this paper.

Comments on Individual Questions

Q.1 This question on integration round a non-linear path was very well done. The topic had not been specifically covered for a while but the candidates were well prepared for it. The marks scored were high reflecting the candidates' clear understanding of the method and technique required for the correct solution. The hardest part was found to be part (ii)(C) where the integration followed a circular arc. Here the problem lay in changing the variables because the change was no longer linear.

(ii) (A) 52 (B) 78.25 (C) 83.25.

Q.2 This question on relative velocity was found to be the hardest question on the paper. The problem as always in this topic is making sure the vector diagrams are correct. This proved to be as difficult as anything else on the paper, but many persevered, some through considerable variations until the correct ones were spotted. With correct diagrams, very few failed to complete the first two parts of the question, with the vector method proving more successful than the graphical one. The final part to find shortest separation between the aircraft was found to be quite a bit harder and the number of successful attempts was limited.

(i) 177.7 kmh^{-1}, (ii) 220.1 kmh^{-1}, (iii) 233.4 km, 40 minutes.

Q.3 There is always an element of surprise in examination papers where the candidates do something unexpected. Part (i) here proved to be such an example. In advance I expected this to be a welcome piece of bookwork to get the candidates started off with some confidence. I was quite taken aback by the high number of attempts which were inaccurate! By contrast the second part was found to be absolutely straightforward rather than being testing as anticipated. The third part was a mixture of the two with no particular pattern emerging. Instead of being just very good, the marks would have been excellent had the standard bookwork been more familiar.

(iii) a circle.

Q.4 This was extremely well done with one reservation. Calculating the total mass in part (i) was found to be quite straightforward. In calculating the moment of inertia in part (ii) however, many candidates used the moment of inertia for a *solid* sphere rather than the given result for a *hollow* sphere. The final part to find the kinetic energy posed no problems whatever except for the carrying forward of the wrong moment of inertia. The candidates knew the mathematics so little was lost.

(ii) $\frac{3Ma^2}{10}$, (iii) $\frac{ma^2}{2}$, $m=\frac{4M}{5}$.