OCR RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2610/1

Differential Equations (Mechanics 4)

Friday 14 JUNE 2002 Morning 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The approximate allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- Take $g = 9.8 \text{ m s}^{-2}$ unless otherwise instructed.
- The total number of marks for this paper is 60.

This question paper consists of 5 printed pages and 3 blank pages.

[Turn over

1 A car of mass 800 kg travels along a straight horizontal road at $10\,\mathrm{m\,s^{-1}}$. As it passes a point O, the car accelerates by increasing the power from the engine to 40 kW so that the driving force is $\frac{40\,000}{v}$ newtons, where $v\,\mathrm{m\,s^{-1}}$ is the speed of the car. The only other horizontal force on the car is a resistance to motion of magnitude $16v$ newtons. The time elapsed after passing O is t seconds and the displacement from O is x metres.

(i) Show that the motion of the car may be modelled by the differential equation

$$50v\frac{\mathrm{d}v}{\mathrm{d}t} = 2500 - v^2,$$

and state the initial conditions. [3]

(ii) Solve the equation to find the particular solution for v in terms of t. Sketch a graph of the solution. [7]

(iii) Formulate a differential equation relating v and x. Solve your equation to show that

$$x = 1250\ln\frac{50+v}{50-v} - 50v + A,$$

where A is a constant. [You are given that $\frac{v^2}{2500-v^2} = \frac{25}{50+v} + \frac{25}{50-v} - 1$.] [6]

(iv) Calculate the displacement after 10 seconds, giving your answer correct to the nearest metre. [4]

2 One end of a light spring is initially fixed at a point O and a particle of mass 2 kg is attached at the other end of the spring which hangs vertically beneath O. The tension in the spring is $50(x + 0.392)$ newtons, where x metres is the displacement of the particle below the equilibrium position.

The particle is held at a point 0.2 m below the equilibrium position and when $t = 0$ it is released from rest.

(i) Show that the motion of the particle may be modelled by the simple harmonic motion equation

$$\ddot{x} + 25x = 0. \qquad [3]$$

(ii) Write down the general solution of this equation. Find the particular solution in this case. [4]

The upper end of the spring is made to oscillate about O. The motion of the particle is now modelled by the differential equation

$$\ddot{x} + 25x = 0.5\sin 5t.$$

When $t = 0$, $x = 0$ and the particle is at rest.

(iii) Solve the differential equation to find x in terms of t. Describe briefly the motion of the particle. [10]

The particle is now submerged in liquid in order to damp the oscillations. The motion of the particle is now modelled by the differential equation

$$\ddot{x} + k\dot{x} + 25x = 0.5\sin 5t,$$

where k is a constant.

(iv) Explain why k must be positive. For what range of values of k will the system be under-damped? [3]

[Turn over

3 The differential equations

$$\frac{dx}{dt} = 2x - 10y, \qquad (1)$$

$$\frac{dy}{dt} = 5x - 12y, \qquad (2)$$

are to be solved simultaneously.

(i) Eliminate y from the equations to show that

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 26x = 0.$$ [5]

(ii) Find the general solution for x in terms of t. Hence use this solution and equation (1) to find the general solution for y in terms of t. [7]

The differential equations

$$\frac{dx}{dt} = 2x - 10y + 26, \qquad (3)$$

$$\frac{dy}{dt} = 5x - 12y + 13, \qquad (4)$$

are to be solved simultaneously.

(iii) Verify that the constant solution $x = 7$, $y = 4$ is a particular solution of equations (3) and (4). [2]

Let the solutions found in part **(ii)** be $x = \mathrm{f}(t)$, $y = \mathrm{g}(t)$.

(iv) Verify that $x = \mathrm{f}(t) + 7$, $y = \mathrm{g}(t) + 4$ satisfy equations (3) and (4). [6]

4 The differential equation

$$x\frac{dy}{dx} = y + 2\sqrt{x(y-x)}, \qquad x > 0,$$

is to be investigated numerically using Euler's method. The algorithm is given by

$$x_{r+1} = x_r + h,$$
$$y_{r+1} = y_r + hy'_r,$$

where y'_r is the value of $\frac{dy}{dx}$ at (x_r, y_r).

x_r	y_r	y'_r	y_{r+1}
1	2	4	2.4
1.1	2.4	4.3560	2.8356
1.2	2.8356	4.6980	3.3054
1.3	3.3054	5.0267	3.8081
1.4	3.8081	5.3431	
1.5			
1.6			
1.7		6.2269	6.1241
1.8	6.1241	6.5021	6.7743
1.9	6.7743	6.7688	7.4512
2.0	7.4512	7.0275	8.1539

The table shows the solution calculated from the initial conditions $x = 1$, $y = 2$.

(i) Calculate the values for the missing entries in the table. [7]

(ii) Sketch the graph of y against x for $1 \leqslant x \leqslant 2$. [2]

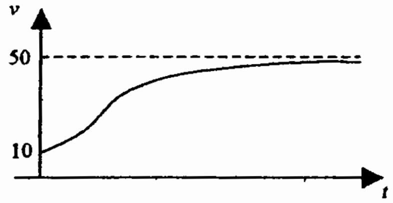
(iii) Explain why the graph suggests that $y'' > 0$. Are the estimates for y over-estimates or under-estimates? Explain your answer. [4]

The algorithm may be used to generate solutions to this differential equation from different starting points.

(iv) Why will the algorithm fail if at any stage $y_r < x_r$? [2]

(v) Given that $h > 0$, show that if $y_r > x_r > 0$ then $y'_r > 1$, and hence that $y_{r+1} > x_{r+1} > 0$. [5]

Mark Scheme

1(i)	$800\frac{dv}{dt} = \frac{40000}{v} - 16v$	M1	N2L (3 terms)	
	$\Rightarrow 50v\frac{dv}{dt} = 2500 - v^2$	A1		
	$t = 0,\ v = 10$	B1		
				3
(ii)	$\int \frac{50v}{2500-v^2}\,dv = \int dt$	M1	separating variables	
	$-25\ln\lvert 2500 - v^2\rvert = t + c$	M1 A1	integrating	
	$2500 - v^2 = Ae^{-t/25}$			
	$v = \sqrt{2500 - Ae^{-t/25}}$	M1	making v subject	
	$t = 0, v = 10 \Rightarrow$	M1	condition on v	
	$A = 2400$ so $v = \sqrt{2500 - 2400e^{-t/25}}$	A1		
		B1	labelled sketch with graph increasing to horizontal asymptote	
				7
(iii)	$\frac{dv}{dt} = v\frac{dv}{dx}$	B1	$v\mathrm{d}v/\mathrm{d}x$ seen	
	$50v^2\frac{dv}{dx} = 2500 - v^2$	M1	substitute into their equation	
	$50\int \frac{v^2}{2500-v^2}\,dv = \int dx$	M1	separating variables	
	$50\int\left(\frac{25}{50+v} + \frac{25}{50-v} - 1\right)dv = x$	B1	RHS	
	$1250\ln\lvert 50+v\rvert - 1250\ln\lvert 50-v\rvert - 50v + A = x$	M1	integrating	
	$x = 1250\ln\left(\frac{50+v}{50-v}\right) - 50v + A$	E1		
				6
(iv)	$x = 0, v = 10 \Rightarrow$	M1	condition on x	
	$A = -6.83139$	A1		
	$t = 10 \Rightarrow v = 29.8535$	M1	using $t = 10$	
	$\Rightarrow x = 222$	A1		
				4

2(i)	$2\ddot{x} = 2g - 50(x + 0.392)$	M1	N2L (3 terms)	
		A1		
	$\Rightarrow \ddot{x} + 25x = 0$	E1		
				3
(ii)	$x = A\sin 5t + B\cos 5t$	B1	or equivalent	
	$t = 0, x = 0.2 \Rightarrow B = 0$	M1	condition on x	
	$\dot{x} = 5A\cos 5t - 5B\sin 5t$			
	$t = 0, \dot{x} = 0 \Rightarrow A = 0$	M1	condition on $\dot{x}$	
	$x = 0.2\cos 5t$	A1	cao	
				4
(iii)	CF $x = C\sin 5t + D\cos 5t$	B1		
	PI $x = at\sin 5t + bt\cos 5t$	B1		
	$\ddot{x} = a(10\cos 5t - 25t\sin 5t) + b(-25t\cos t - 10\sin 5t)$	M1	differentiate twice (product rule) and substitute	
	in DE $\Rightarrow 10a\cos 5t - 10b\sin 5t = 0.5\sin 5t$	dM1	compare coefficients	
	$\Rightarrow a = 0, b = -0.05$	A1		
	$x = C\sin 5t + D\cos 5t - 0.05t\cos 5t$	F1	their CF + PI	
	$t = 0, x = 0 \Rightarrow D = 0$	M1	condition on x	
	$\dot{x} = 5C\cos 5t - 5D\sin 5t - 0.05\cos 5t + 0.25t\sin 5t$			
	$t = 0, \dot{x} = 0 \Rightarrow C = 0.01$	M1	condition on $\dot{x}$	
	$x = 0.01\sin 5t - 0.05t\cos 5t$	A1		
	oscillations unbounded (resonance)	B1		
				10
(iv)	$k > 0$ as resistance opposes motion	B1		
	underdamped $\Rightarrow k^2 - 4 \times 25 < 0$	M1	use of discriminant	
	$(0 <)\ k < 10$	A1		
				3

3(i)	$\frac{d^2x}{dt^2} = 2\frac{dx}{dt} - 10\frac{dy}{dt}$	M1 A1	differentiate	
	$= 2\frac{dx}{dt} - 10(5x - 12y)$	M1	substitute d*y*/d*t*	
	$= 2\frac{dx}{dt} - 50x + 120(\frac{1}{10})(2x - \frac{dx}{dt})$	M1	substitute *y*	
	$\Rightarrow \frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 26x = 0$	E1		5
(ii)	$\alpha^2 + 10\alpha + 26 = 0$	M1	auxiliary equation	
	$\alpha = -5 \pm \mathrm{i}$	A1		
	$x = \mathrm{e}^{-5t}(A\sin t + B\cos t)$	F1	CF for complex roots	
	$y = \frac{1}{10}(2x - \frac{dx}{dt})$	M1	make *y* subject	
	$= \frac{1}{5}\mathrm{e}^{-5t}(A\sin t + B\cos t)$	M1	substitute *x*	
	$-\frac{1}{10}(-5\mathrm{e}^{-5t}(A\sin t + B\cos t) + \mathrm{e}^{-5t}(A\cos t - B\sin t))$	M1	substitute d*x*/d*t*	
	$y = \frac{1}{10}\mathrm{e}^{-5t}((7A + B)\sin t + (7B - A)\cos t)$	A1		7
(iii)	constant solutions $\Rightarrow \frac{dx}{dt} = \frac{dy}{dt} = 0$	B1		
	$2.7 - 10.4 + 26 = 0$			
	$5.7 - 12.4 + 13 = 0$	B1		2
(iv)	**either** $x - 7$ and $y - 4$	M1	considering these	
	are general solutions of (1) and (2)	M1	recognise as general solutions	
	$\frac{d}{dt}(x - 7) = 2(x - 7) - 10(y - 4)$	M1		
	$\Rightarrow \frac{dx}{dt} = 2x - 10y + 26$	E1		
	$\frac{d}{dt}(y - 4) = 5(x - 7) - 12(y - 4)$	M1		
	$\Rightarrow \frac{dy}{dt} = 5x - 12y + 13$	E1		
	or substitute RHS with	M1	for each equation	
	$x = 7 + \mathrm{e}^{-5t}(A\sin t + B\cos t)$			
	$y = 4 + \frac{1}{10}\mathrm{e}^{-5t}((7A + B)\sin t + (7B - A)\cos t)$			
	substitute LHS with derivative	M1		
	verify equality	E1		6

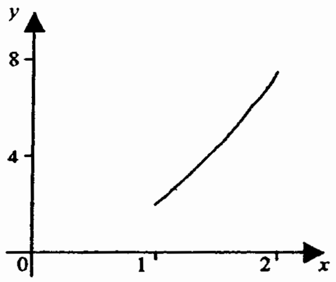
4(i)

x_r	y_r	y_r'	y_{r+1}
1.4			4.3424
1.5	4.3424	5.6480	4.9072
1.6	4.9072	5.9424	5.5014
1.7	5.5014		

M1 use of algorithm
A1 $y(1.5)$
M1 $y'(1.5)$
M1
A1 $y(1.6)$
M1
A1 $y(1.7)$

[7]

(ii)



B1 positive gradient
B1 correct sketch including labels

[2]

(iii) gradient increasing
$\Rightarrow y'' > 0$
so each step underestimates the increase in y as it is based on the minimum gradient for the interval

M1 reasoning
E1 conclusion
M1 reasoning or suitable sketch
A1 conclusion

[4]

(iv) if $y < x$ then y' is complex
so algorithm does not generate real numbers

M1
A1

[2]

(v)

$y_r > x_r > 0 \Rightarrow y_r - x_r > 0$ B1

$\Rightarrow \frac{2\sqrt{x_r(y_r - x_r)}}{x_r} > 0$ M1

$\Rightarrow y_r' > \frac{y_r}{x_r} > 1$ A1

$\Rightarrow y_{r+1} = y_r + hy_r' > x_r + h = x_{r+1} > 0$ M1

A1

[5]

Examiner’s Report

2610 Differential Equations (Mechanics 4)

General Comments

The paper seemed accessible to candidates with most finding three questions at which they could make a reasonable attempt. No question stood out as being more unpopular than the others, but there were a noticeable number of candidates who abandoned their attempt at question 1 in order to tackle the other three questions.

Comments on Individual Questions

Q.1 Although there was a small number of candidates who felt the need to be creative with their algebra to reach the stated result, most candidates had no problem in achieving it. Some candidates omitted to give the initial conditions as requested. A number of candidates rather than by separating variables, approached the solution by integrating factor, which was unsuccessful. Many of these candidates abandoned this question at this point in favour of the other three. The differential equation for x was sometimes poorly derived with some candidates not seeming to be aware of the standard expression for acceleration as $v\mathrm{d}v/\mathrm{d}x$. Solving the differential equation was often done well although some made errors which they then managed to erroneously manipulate into the given answer. A few candidates turned their integral into the given answer directly without any working shown. Candidates must be reminded of the need to show sufficient working, especially for given answers. The calculation of A and hence x was often done well but some candidates ran into problems at this stage and some left A in their answer.

(i) $t=0, v=10$, (ii) $v=\sqrt{2500-2400\mathrm{e}^{-\frac{t}{25}}}$, (iv) 1195.

Q.2 Most candidates were able to derive the differential equation, but a few made sign errors, some omitted the weight and some did not realise the need to start from Newton's second law. Solving the equation was usually done well. When solving the forced harmonic motion equation, many used the wrong form of the particular integral. However, it was encouraging to see some candidates realise this and have a second attempt with the correct form. Algebraic and numeric slips were common but there were also many completely correct solutions seen. The most common major error, apart from the wrong particular integral, was to use the particular solution from the previous part, rather than the general solution for the complementary function. The last part of the question was not answered well. Reasons for k being positive were often vague or omitted. Many did not know the correct condition for the discriminant to give underdamping.

(ii) 0.2cos5t, (iii) $x=0.01\sin 5t-0.05t\cos 5t$ oscillations unbounded, (iv) $k<10$.

Q.3 The derivation of the differential equation was often done well. Candidates seem to be definitely improving in their ability at this particular technique. Solving was often done well also. Candidates were for the first time directed how to find y from x and this resulted in much more success than in past exams. Candidates should be familiar with this technique without prompting. The remaining parts of the question were not so well done. Candidates often demonstrated the given constant solutions satisfied equations (3) and (4), although most with less clarity than would have been liked. Verifying the given general solutions to the equations often caused problems. Some candidates opted for direct verification, and although some were successful, the resulting algebraic manipulations defeated many, in particular clearly showing that the two sides of the equation were the same. A minority of candidates made some connection with the earlier parts of the question, but concise solutions based on the known solutions to (1) and (2) were very rare.

(ii) $\mathrm{e}^{-5t}(A\sin t+B\cos t)$, $\frac{\mathrm{e}^{-5t}}{10}((B+7A)\sin t+(7B-A)\cos t)$.

Q.4 Full marks were often awarded for the numeric solution. Sketches of the solution were sometimes good but sometimes sloppy, even suggesting a minimum point at $x = 1$. Candidates were often unclear what led to the conclusion that the second derivative was positive with the absence of a turning point being the favourite wrong answer. Although the majority realised that the solution by Euler's method would give underestimates, many were unable to give a clear explanation why, with reasons often totally unrelated to the shape of the curve. Candidates who used a sketch here were usually successful in giving a convincing explanation. When explaining why the algorithm would fail for $y < x$, most candidates spotted the negative but described the problem as getting a 'negative square root' rather than the 'square root of a negative' - there is a difference! In the final part of the question few gave fully correct answers and even those who did often relied heavily on words rather than using inequalities efficiently.

(i)

x_n	y_n	y_n'	y_{n+1}
1.4			4.3424
1.5	4.3424	5.6480	4.9072
1.6	4.9072	5.9424	5.5014
1.7	5.5014		

(iii) underestimates, (iv) $\frac{\mathrm{d}y}{\mathrm{d}x}$ is complex.