RECOGNISING ACHIEVEMENT

Oxford, Cambridge and RSA Examinations

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2604

Pure Mathematics 4

Thursday **10 JANUARY 2002** Morning 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The approximate allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

This question paper consists of 3 printed pages and 1 blank page.

© OCR 2002 Registered Charity 1066969

1 A curve has equation $y = \dfrac{8(x-4)(x-14)}{4-x^2}$.

(i) Write down the equations of the three asymptotes. [3]

(ii) Show that $\dfrac{dy}{dx} = -\dfrac{48(3x^2 - 20x + 12)}{(4-x^2)^2}$.

Hence find the coordinates of the stationary points. [6]

(iii) Sketch the curve. [4]

(iv) On a separate diagram, sketch the curve with equation $y^2 = \dfrac{8(x-4)(x-14)}{4-x^2}$.

Give the equations of the asymptotes and the coordinates of the stationary points on this curve. [7]

2 **(a)** Find the sum of the series

$$1\times 9 \ + \ 2\times 33 \ + \ 3\times 73 \ + \ldots + \ n(8n^2+1),$$

giving your answer in a fully factorised form. [5]

(b) Solve the inequality $\dfrac{x+10}{x(x-4)} < 3$. [6]

(c) Prove by induction that $\displaystyle\sum_{r=1}^{n} \frac{r!(r-1)}{(r+1)(r+3)} = \frac{(n+1)!}{(n+2)(n+3)} - \frac{1}{6}$. [9]

3 In this question, α is the complex number $-1 + 3\mathrm{j}$.

(i) Find α^2 and α^3. [3]

It is given that λ and μ are real numbers such that $\lambda\alpha^3 + 8\alpha^2 + 34\alpha + \mu = 0$.

(ii) Show that $\lambda = 3$, and find the value of μ. [4]

(iii) Solve the equation $\lambda z^3 + 8z^2 + 34z + \mu = 0$, where λ and μ are as in part **(ii)**.

Find the modulus and argument of each root, and illustrate the three roots on an Argand diagram. [11]

(iv) On your diagram, draw the locus of points representing complex numbers z which satisfy

$$|z| = |z-\alpha|.$$ [2]

4 **(i)** Find the equation of the line of intersection of the two planes

$$5x - 2y + z = 28 \quad \text{and} \quad 3x - 4y - 5z = 0. \qquad [5]$$

(ii) Find **AB**, where $\mathbf{A} = \begin{pmatrix} 5 & -2 & k \\ 3 & -4 & -5 \\ -2 & 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 3k+8 & 4k+10 \\ -2 & 2k+20 & 3k+25 \\ 1 & -11 & -14 \end{pmatrix}$.

Hence write down the inverse matrix $\mathbf{A}^{-1}$, stating a necessary condition on k for this inverse to exist. [6]

(iii) Using the results in parts **(i)** and **(ii)**, or otherwise, solve the equation

$$\begin{pmatrix} 5 & -2 & k \\ 3 & -4 & -5 \\ -2 & 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 28 \\ 0 \\ m \end{pmatrix}$$

in each of the cases

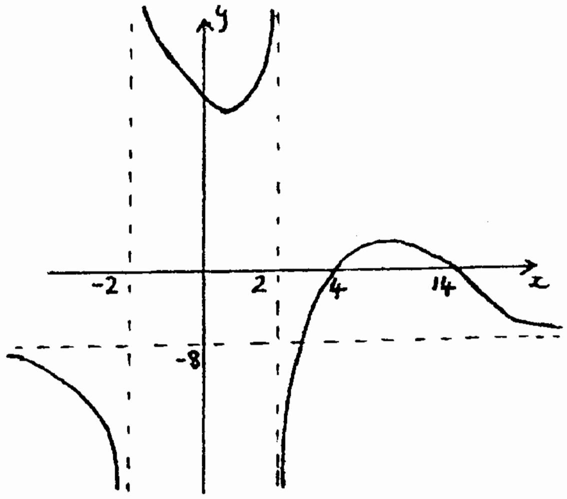
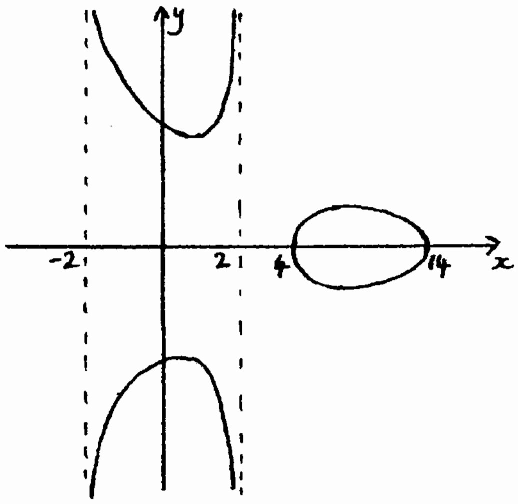
(A) $k = 8$, giving x, y and z in terms of m,

(B) $k = 1$ and $m = 4$,

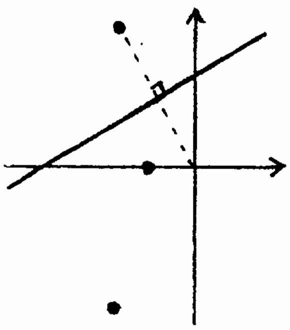
(C) $k = 1$ and $m = 2$. [9]

Mark Scheme

MARKING SCHEME

1 (i)	$x = 2,\ x = -2,\ y = -8$	B1B1B1 3	
(ii)	$\dfrac{dy}{dx} = \dfrac{8(4-x^2)(2x-18) - 8(x^2-18x+56)(-2x)}{(4-x^2)^2}$	M1 A1	Use of quotient rule (or equivalent) Any correct form
	$= \dfrac{-48(3x^2-20x+12)}{(4-x^2)^2}$	A1 (ag)	Correctly obtained
	$= 0$ when $x = \frac{2}{3},\ 6$	M1	Solving to obtain one value of x
	Stationary points are $\left(\frac{2}{3},\ 100\right)$ and $(6,\ 4)$	A2 cao 6	Give A1 for one stationary point, or for both values of x correct
(iii)		B1 B1 B1 B1 4	LH branch (negative gradient, below x-axis) Middle branch with minimum in correct (or consistent) place RH branch (cutting x-axis at 4 and 14) Fully correct shape and approaching asymptotes correctly
(iv)		B4 ft	Give B1 (ft) for every <u>two</u> of: • No graph in $x < -2$, graph in $x > -2$ • Graph in $4 < x < 14$ and no graph on either side • LH section above x-axis • LH section below x-axis • RH section above x-axis • RH section below x-axis • Vertical tangent at $x = 4$ • Vertical tangent at $x = 14$
	Asymptotes are $x = 2,\ x = -2$	B1 ft	B0 if any others given
	Stationary points $\left(\frac{2}{3},\ 10\right),\ \left(\frac{2}{3},\ -10\right)$ $(6, 2),\ (6, -2)$	B2 ft 7	Give B1 for any two correct

2 (a)	$\sum_{r=1}^{n} r(8r^2+1) = \sum_{r=1}^{n}(8r^3+r)$	M1	
	$= 2n^2(n+1)^2 + \frac{1}{2}n(n+1)$	A1A1	
	$= \frac{1}{2}n(n+1)(4n^2+4n+1)$	M1	Two linear factors, e.g. $n(n+1)$
	$= \frac{1}{2}n(n+1)(2n+1)^2$	A1 5	
(b)	$3x^2(x-4)^2 - x(x-4)(x+10) > 0$	M1	Or $\frac{x+10-3x(x-4)}{x(x-4)} < 0$
	$x(x-4)(3x^2-13x-10) > 0$		Or writing $\frac{x+10}{x(x-4)} = 3$ as a quad eqn
	$x(x-4)(3x+2)(x-5) > 0$	A1A1	Factors $(3x+2)(x-5)$ or $x = -\frac{2}{3}, 5$ (Give A1 if both signs wrong) (If M0 then B1B1 for factors or values)
		M1	Considering intervals defined by four critical values $-\frac{2}{3}, 0, 4, 5$ (ft)
	$x < -\frac{2}{3},\quad 0 < x < 4,\quad x > 5$	A2 cao 6	*Always award 6 marks if correct* Give A1 for $\le$ etc
(c)	When $n=1$, LHS $= 0$		
	RHS $= \frac{2}{3\times 4} - \frac{1}{6} = 0$	B1	
	so it is true when $n=1$		
	Assume true for $n=k$, then		
	$\sum_{r=1}^{k+1} = \frac{(k+1)!}{(k+2)(k+3)} - \frac{1}{6} + \frac{(k+1)!(k)}{(k+2)(k+4)}$	M1A2	
	$= \frac{(k+1)!\{k+4+k(k+3)\}}{(k+2)(k+3)(k+4)} - \frac{1}{6}$	M1	Use of common denominator
	$= \frac{(k+1)!(k+2)^2}{(k+2)(k+3)(k+4)} - \frac{1}{6}$		*2nd, 3rd and 4th M1's are each dependent on the first M1*
	$= \frac{(k+1)!(k+2)}{(k+3)(k+4)} - \frac{1}{6}$	M1	Cancelling to denom $(k+3)(k+4)$
	$= \frac{(k+2)!}{(k+3)(k+4)} - \frac{1}{6}$	M1	Use of $(k+1)!(k+2) = (k+2)!$
		A1	Correctly obtained
	True for $n=k \Rightarrow$ True for $n=k+1$	A1	Stated or clearly implied
	(Hence true for all positive integers n)	9	*Dependent on previous 7 marks*

3 (i)	$\alpha^2 = -8 - 6j$ $\alpha^3 = (-8-6j)(-1+3j) = 8 + 6j - 24j + 18$ $= 26 - 18j$	B1 M1 A1 cao 3	 Multiplying out and using $j^2 = -1$
(ii)	$\lambda(26-18j) + 8(-8-6j) + 34(-1+3j) + \mu = 0$ Equating imaginary parts, $-18\lambda - 48 + 102 = 0$ $\lambda = 3$ Equating real parts, $26\lambda - 64 - 34 + \mu = 0$ $\mu = 20$	 M1 A1 (ag) M1 A1 cao 4	
(iii)	Two roots are $-1+3j,\ -1-3j$	B1	
	These are the roots of $z^2 + 2z + 10 = 0$ Cubic is $3z^3 + 8z^2 + 34z + 20 = 0$ $(z^2 + 2z + 10)(3z + 2) = 0$ Third root is $-\frac{2}{3}$	M1 A1 M1 A1 cao	For $(z+1-3j)(z+1+3j)$ or finding sum (-2) and product (10) *Always give 4 marks for* $-\frac{2}{3}$
	OR $(-1+3j)(-1-3j)\gamma = -\frac{20}{3}$ M1A1 $\gamma = -\frac{2}{3}$ M1A1		M1 for relating product of roots to the constant term in cubic
	OR $(-1+3j) + (-1-3j) + \gamma = -\frac{8}{3}$ M1A1 $\gamma = -\frac{2}{3}$ M1A1		M1 for relating sum of roots to the coefficient of z^2
	$-1+3j$ has modulus $\sqrt{10}$, argument 1.89 $-1-3j$ has modulus $\sqrt{10}$, argument -1.89 $-\frac{2}{3}$ has modulus $\frac{2}{3}$, argument π 	B1B1 B1 ft B1 ft B1 B1 ft 11	Accept 3.16; 108° Or $-108°$, 4.39, 252° ft for any real number Points $-1 \pm 3j$ Real root
(iv)	Line drawn on diagram (see above)	B2 ft 2	Perpendicular bisector of points $0, \alpha$

4 (i)	When $x=0$, $-2y+z=28$, $-4y-5z=0$ $y=-10$, $z=8$	M1 A1	Finding one point One point correct
	Direction is given by $\begin{pmatrix}5\\-2\\1\end{pmatrix}\times\begin{pmatrix}3\\-4\\-5\end{pmatrix}$ $\left[=\begin{pmatrix}14\\28\\-14\end{pmatrix}\right]$	M1	*Or* finding a second point, e.g. $(5,0,3)$, $(8,6,0)$ and using it to find direction
	Direction is $\begin{pmatrix}1\\2\\-1\end{pmatrix}$	A1 cao	
	Equation of line is $\mathbf{r}=\begin{pmatrix}0\\-10\\8\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\-1\end{pmatrix}$	A1 ft 5	*Dependent on M1M1.* Accept any form A0 if '$\mathbf{r}=$' omitted
	OR Eliminating y, $7x+7z=56$ M1A1 M1A1 $x=\lambda$, $y=-10+2\lambda$, $z=8-\lambda$ A1 ft		Eliminating one variable Expressing (e.g.) y and z in terms of x
(ii)	$\mathbf{AB}=\begin{pmatrix}k-1&0&0\\0&k-1&0\\0&0&k-1\end{pmatrix}$	B2 B2	$(k-1)$'s on main diagonal (Give B1 for one correct) For zeros (Give B1 for two correct)
	$\mathbf{A}^{-1}=\dfrac{1}{k-1}\mathbf{B}$	B1	
	provided $k\neq 1$	B1 6	
(iii)	(*A*) $\begin{pmatrix}x\\y\\z\end{pmatrix}=\mathbf{A}^{-1}\begin{pmatrix}28\\0\\m\end{pmatrix}=\frac{1}{7}\begin{pmatrix}-1&32&42\\-2&36&49\\1&-11&-14\end{pmatrix}\begin{pmatrix}28\\0\\m\end{pmatrix}$	M1 M1	Finding $\mathbf{A}^{-1}$ when $k=8$ *Or* eliminating one variable in 2 ways Using $\mathbf{A}^{-1}$ correctly *Or* solving to obtain one of x, y, z *Dependent on previous M1*
	$=\frac{1}{7}\begin{pmatrix}-28+42m\\-56+49m\\28-14m\end{pmatrix}$	A2 cao	Give A1 for two correct
	$x=-4+6m$, $y=-8+7m$, $z=4-2m$		If M0 then B4 for all correct B2 for two correct B1 for one correct
	(*B*) When $k=1$ the first two equations have solution $x=\lambda$, $y=-10+2\lambda$, $z=8-\lambda$ Substituting into $-2x+3y+4z=4$, $-2\lambda+3(-10+2\lambda)+4(8-\lambda)=4$	M1	*Or* eliminating one variable in 2 ways *Can be earned in (C) if not in (B)*
	Inconsistent; there are no solutions	A1 ft A1 cao	*Or* two equations in two variables *cao* 'No solutions' correctly shown
	(*C*) Substituting into $-2x+3y+4z=2$, $-2\lambda+3(-10+2\lambda)+4(8-\lambda)=2$ Consistent; there are infinitely many solutions,	A1 ft	*Or* two equations in two variables *cao*
	$x=\lambda$, $y=-10+2\lambda$, $z=8-\lambda$	A1 cao 9	

Examiner’s Report

Pure Mathematics 4 (2604)

General Comments

This paper was found to be more straightforward than most of the previous ones. There were many excellent scripts, with about 30% of candidates scoring 50 marks or more (out of 60). There was nevertheless a wide range of performance, and about a quarter of the candidates scored less than 30. Most candidates appeared to have sufficient time to complete the paper; indeed a fair number attempted all four questions. Questions 1 and 3 were considerably more popular than questions 2 and 4. The work on vectors and matrices (Q.4) was of markedly lower quality than that seen on the other topics.

Comments on Individual Questions

Question 1 (Curve sketching)

This was the best answered question, with half the attempts scoring 17 marks or more (out of 20). In part (i) the vertical asymptotes were almost always found correctly, but some candidates were unable to find the horizontal asymptote. The differentiation in part (ii) and the graph in part (iii) were well understood. The square root graph in part (iv) was often correctly drawn, but the most common reason for dropping a mark in this question was failure to show clearly the infinite gradients at $x = 4$ and $x = 14$.

[(i) $x = 2,\ x = -2,\ y = -8$; (ii) $\left(\frac{2}{3},\ 100\right)$ and $(6,\ 4)$;
(iv) $x = 2,\ x = -2$; $\left(\frac{2}{3},\ 10\right),\ \left(\frac{2}{3},\ -10\right),\ (6, 2),\ (6, -2)$]

Question 2 (Inequalities and series)

This question was fairly well answered, and the average mark was about 13. In part (a) the majority of candidates obtained the sum of the series as $8\sum r^3 + \sum r = 2n^2(n+1)^2 + \frac{1}{2}n(n+1)$, but surprisingly many were unable to simplify this efficiently, typically multiplying it out to givea quartic expression instead of factorising.

For the inequality in part (b), a great variety of methods were used, and there was considerable success. Many candidates multiplied by $x^2(x-4)^2$, but then multiplied everything out to obtain a quartic and were unable to proceed further. Some ignored the critical values 0 and 4 completely, but this was less prevalent than in similar questions in the past.

There was a lot of good work in part (c). The process of induction was well understood, and a fair proportion of candidates completed the algebraic steps successfully.

[(a) $\frac{1}{2}n(n+1)(2n+1)^2$; (b) $x < -\frac{2}{3},\quad 0 < x < 4,\quad x > 5$]

Question 3 (Complex numbers)

This question was answered well, with half the attempts scoring 16 marks or more. In part (i), α^2 and α^3 were usually calculated correctly. Part (ii) was also done correctly by most candidates, although some did not seem to know the technique of equating real and imaginary parts; such candidates just assumed the given result $\lambda = 3$ and were able to find μ. In part (iii), almost every candidate knew that there were two conjugate complex roots and one real root. However, finding the real root caused some difficulty; in particular the coefficient 3 of z^3 was often overlooked, leading to $z = -2$ instead of $z = -\frac{2}{3}$. Finding the modulus and argument, and drawing the Argand diagram, were done well, but the locus in part (iv) was very often drawn as a circle instead of a straight line.

[(i) $\alpha^2 = -8 - 6\text{j}$, $\alpha^3 = 26 - 18\text{j}$; (ii) $\mu = 20$;
(iii) $z = -1 \pm 3\text{j}$, $-\frac{2}{3}$; $|-1 \pm 3\text{j}| = \sqrt{10}$, $\arg(-1 \pm 3\text{j}) = \pm 1.89$, $\left|-\frac{2}{3}\right| = \frac{2}{3}$, $\arg(-\frac{2}{3}) = \pi$]

Question 4 (Vectors and matrices)

This was the least popular question, but even so it was attempted by about half the candidates. It was certainly the worst answered, with the majority of attempts scoring less than half marks. In part (i) most candidates knew a method for finding the line of intersection, and many completed it successfully. The equation of the line could be given in any form (e.g. vector, parametric or ratio form), but a proper equation was required; many candidates lost a mark for omitting '$\mathbf{r} =$' from the vector equation. In part (ii) the matrix product was often found correctly, but few candidates drew the correct conclusion about the inverse. Many stated that $\mathbf{A}^{-1} = \mathbf{B}$ when $k = 2$; this is a particular case of the general result required and it did not earn any marks. In part (iii) very few candidates made use of the earlier results. Elimination of variables was the most popular method; although this was quite often successful in part (*A*), the lack of a systematic approach usually prevented much progress in parts (*B*) and (*C*).

[(i) $\mathbf{r} = \begin{pmatrix} 0 \\ -10 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$; (ii) $\begin{pmatrix} k-1 & 0 & 0 \\ 0 & k-1 & 0 \\ 0 & 0 & k-1 \end{pmatrix}$, $\mathbf{A}^{-1} = \frac{1}{k-1}\mathbf{B}$, $k \neq 1$;
(iii)(*A*) $x = -4 + 6m$, $y = -8 + 7m$, $z = 4 - 2m$; (*B*) no solution;
(*C*) $x = \lambda$, $y = -10 + 2\lambda$, $z = 8 - \lambda$]