OCR
RECOGNISING ACHIEVEMENT

Oxford, Cambridge and RSA Examinations

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2623/1

Numerical Methods

Monday **14 JANUARY 2002** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer **all** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The approximate allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

This question paper consists of 3 printed pages and 1 blank page.

[Turn over

1 **(i)** The iterative formula $x_{r+1} = 0.8\,(1 - x_r^3)$ is used with starting values of

(*A*) $x_0 = 1.3$, (*B*) $x_0 = 0.6$.

Describe in each case how the sequence of iterates behaves. [5]

(ii) Show graphically that the equation $x = 0.8(1 - x^3)$ has only one real root, α. Use the Newton-Raphson method to determine α, correct to 5 significant figures. [6]

(iii) Evaluate $f'(\alpha)$, where $f(x) = 0.8(1 - x^3)$. Explain how this value relates to the behaviour of the iteration in part **(i)**(*B*). [4]

2 A student finds estimates of the roots α, β of the quadratic equation $ax^2 + bx + c = 0$ using the following algorithm.

Step 1 calculate $d = \sqrt{b^2 - 4ac}$ correct to 4 significant figures,

Step 2 calculate $\alpha = \dfrac{-b - d}{2a}$,

Step 3 calculate $\beta = -\left(\dfrac{b}{a} + \alpha\right)$.

In a particular case, $a = 1$, $b = -50$, $c = 1$.

(i) Obtain the values of d, α, β, as found by the student.

Solve the equation correct to 6 decimal places and hence determine the relative errors in the values of α and β found by the student. [6]

A second student uses the following algorithm.

Step 1 calculate $d = \sqrt{b^2 - 4ac}$ correct to 4 significant figures,

Step 2 calculate $\beta = \dfrac{-b + d}{2a}$,

Step 3 calculate $\alpha = \dfrac{c}{a\beta}$.

(ii) Obtain the values of α, β found by the second student. [3]

(iii) Identify and explain *two* features of the second student's algorithm which makes it better than the first. [4]

The second student now adds a fourth step to the algorithm as follows.

Step 4 calculate $\beta^* = -\left(\dfrac{b}{a} + \alpha\right)$.

(iv) Comment on the accuracy of β^* as an estimate of one of the roots of the equation. [2]

3 The table below shows some values, correct to 4 decimal places, of a function f(x).

x	0	1	2	3
f(x)	1.5557	1.0642	1.0154	1.3054

(i) Use a difference table to show that f(x) cannot be approximated well by a quadratic. [4]

(ii) Use Newton's interpolation formula to find, in the form $a + bx + cx^2 + dx^3$, the cubic function which passes through the data points. Hence estimate f(1.5) and f′(1.5). [8]

(iii) It is known that f(x) is never less than 1 for $0 \leqslant x \leqslant 3$. Discuss briefly the usefulness or otherwise of the approximating cubic found in part **(ii)**. [3]

4 [Note: in this question, S_1 denotes the value obtained from a single application of Simpson's rule. A single application of Simpson's rule is often referred to as using two "strips". Similarly S_2 and S_4 denote the values obtained when two and four applications of Simpson's rule are used to cover the range of integration.]

In this question you are asked to find estimates of the value of the integral

$$I = \int_1^2 \frac{\ln x}{x} \mathrm{d}x.$$

A standard notation is used; so, for example, T_1, M_1 and S_1 denote the values obtained from single applications of the trapezium rule, the mid point rule and Simpson's rule.

(i) Find the values of T_1 and M_1, and hence obtain the value of S_1, giving your answers to 6 decimal places. [6]

(ii) Find similarly the values of T_2, M_2, S_2, T_4, M_4, S_4. [6]

By considering the differences $S_2 - S_1$ and $S_4 - S_2$, determine the most accurate value you can of I. Give your answer to an appropriate number of significant figures. [3]

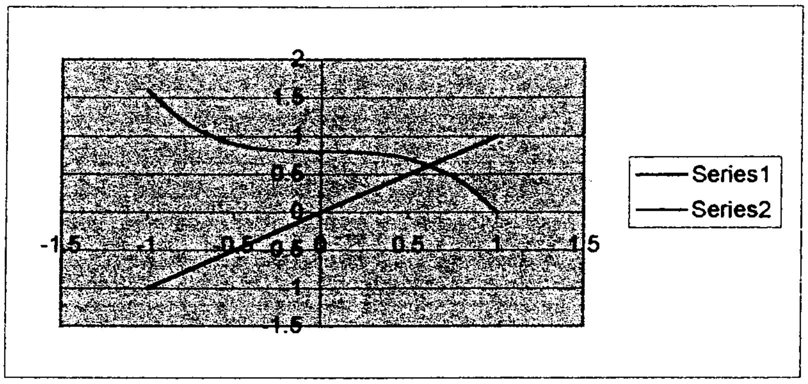
Mark Scheme

1 (i)

	r	0	1	2	3	4	5	6	
(A):	x_r	1.3	-0.9576	1.502494	-1.91349	6.404892	-209.396	7345109	**[M1A1]**
(B):	x_r	0.6	0.6272	0.602618	0.624928	0.604755	0.623059	0.606501	**[A1]**

(A) diverges **[E1]** (B) converges but slowly **[E1]** ***[subtotal 5]***

(ii)



[G2]

Newton-Raphson: $x_{r+1} = x_r - (x_r - 0.8(1-x_r^3))/(1+2.4x_r^2)$ **[M1A1]**

r	0	1	2	3	4	5	6	
x_r	0.6	0.614592	0.61443	*0.61443*	*0.61443*	*0.61443*	*0.61443*	**[A1A1]**

[subtotal 6]

(iii) $f'(x) = -2.4x^2$ **[A1]** $f'(0.61443) = -0.90606$ **[A1]**

the negative sign indicates oscillation **[E1]**

the magnitude, just less than 1, indicates slow convergence **[E1]**

[subtotal 4]

[TOTAL 15]

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2 (i) d = 49.96 $\alpha = 0.02$ $\beta = 49.98$ **[M1A1A1]**

roots to 6 d.p.: 0.020008 49.979992 **[A1]**

relative errors: 0.04% (-)0.000016% **[A1A1]**

[subtotal 6]

(ii) $\beta = 49.98$ $\alpha =$ 0.020008 (accept 0.02001) **[M1A1A1]**

[subtotal 3]

(iii) The second algorithm finds the larger root first, thus avoiding large relative errors. **[E1E1]**

It then uses division rather than subtraction for the second root, preserving accuracy. **[E1E1]**

[subtotal 4]

(iv) $\beta^* = 49.979992$ **[A1]**

This is more accurate than β. **[B1]**

[subtotal 2]

[TOTAL 15]

3 (i)

x	f(x)	Δf(x)	$\Delta^2 f(x)$	$\Delta^3 f(x)$	*(3rd diff*	
0	1.5557	-0.4915	0.4427	-0.1039	*not reqd.)*	**[M1A2]**
1	1.0642	-0.0488	0.3388			
2	1.0154	0.29				
3	1.3054					

The second differences are substantially different. **[E1]**

[subtotal 4]

(ii) Interpolating cubic: P3(x) = 1.5557 - 0.4915x + 0.4427x(x-1)/2 - 0.1039x(x-1)(x-2)/6 **[M1A1A1]**

= 1.5557 - 0.7475x + 0.2733x^2 - 0.0173x^3 **[A1A1]**

P3(1.5) = 0.9910 **[A1]**

P3'(x) = -0.7475x + 0.5466x - 0.0519x^2 **[A1]**

P3'(1.5) = -0.04438 **[A1]**

[subtotal 8]

(iii) The interpolating cubic clearly cannot be exactly right **[E1]**, but nevertheless it could be a good approximation **[E1]**. We need more data to be sure one way or the other **[E1]**.
*(Up to **[E3]** awarded for any sensible discussion.)* ***[subtotal 3]***

[TOTAL 15]

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4 (i)

x	1	1.5	2
ln(x)/x	0	0.27031	0.346574

T1 = (f(1) + f(2))/2 = 0.173287 **[M1A1]**
M1 = f(1.5) = 0.270310 **[M1A1]**
S1 = (T1+2*M1)/3= 0.237969 **[M1A1]**

[subtotal 6]

(ii)

x	1.25	1.75	1.125	1.375	1.625	1.875
ln(x)/x	0.178515	0.31978	0.104696	0.231603	0.298774	0.335258

T2 = (T1 + M1)/2 =	0.221798	**[A1]**	T4 = (T2 + M2)/2 =	0.235473	**[A1]**
M2=(f(1.25)+f(1.75))/2=	0.249148	**[A1]**	M4 = (...) =	0.242583	**[A1]**
S2 = (T2+2*M2)/3 =	0.240031	**[A1]**	S4 = (T4+2*M4)/3 =	0.240213	**[A1]**

[subtotal 6]

(iii) S2 - S1 = 0.002062
S4 - S2 = 0.000182
Differences reducing by a factor of about 11 (theoretically 16)
Suggests a limit of about 0.240213 + 0.000182 (1/11 + $(1/11)^2$ + ...) = 0.240231 **[M1A1]**
or 0.240213 + 0.000182 (1/16 + $(1/16)^2$ + ...) = 0.2402251
Certainly can't rely on 6th dp. Accept either 0.2402 or 0.24023. **[A1]**

[subtotal 3]

[TOTAL 15]

Examiner’s Report

Numerical Methods (2623)

General Comments

Overall, this paper attracted some very good attempts. There were a few candidates who were clearly out of their depth, but there were some who had an excellent grasp of the concepts involved. Routine numerical processes were carried out accurately for the most part. Analysis was more challenging.

Comments on Individual Questions

Q1 (solution of an equation)

In part (i), the fixed point iteration proved to be an easy starter with most candidates scoring full marks. It was intended that the first few terms of each iteration would be written down, but there was no penalty where candidates chose to describe the iterations without giving numerical values.

A number of candidates did not appreciate that in part (ii) it is necessary to rearrange the equation from $x = \mathrm{f}(x)$ into the form $x - \mathrm{f}(x) = 0$ in order to apply Newton-Raphson.

Further confusion arose in part (iii) where it is the un-rearranged f(x) which has to be differentiated. Only the very best candidates managed to keep a clear head throughout parts (i) and (ii).

[$\alpha = 0.61443$ to 5 s.f.]

Q2 (algorithm)

Parts (i) and (ii) were well done by all except those who ignored the information that d is calculated correct to 4 significant figures. When full accuracy in d is preserved the point of the question is lost.

In part (iii) it was rare to find two distinct features identified and explained. (The second algorithm finds the root of larger magnitude first, thereby avoiding large relative errors. It then uses division rather than subtraction to find the second root, thereby preserving accuracy.)

The comments offered in part (iv) were often unhelpfully vague. It was sufficient to find β^* and simply observe that it is more accurate that β.

[$\alpha = 0.020008$, $\beta = 49.979992$ to 6 decimal places]

Q3 (Newton's interpolation)

Part (i) was well done for the most part, although some candidates became very confused about the signs of the differences. As a consequence, these candidstes got the interpolating polynomial wrong.

The algebraic simplification in part (ii) defeated many. This led to large errors in the value of the function and the gradient at $x = 1.5$, thereby obscuring the point being made.

Part (iii) produced some excellent solutions pointing out that the function dips only slightly below it theoretical lower limit and does so near to its minimum point. Those who had part (ii) wrong, however, were frequently unable to comment sensibly here.

[$\mathrm{f}(1.5) = 0.9910$; $\mathrm{f}'(1.5) = -0.04438$]

Q4 (numerical integration)

Once again there were many candidates who ignored the conventional notation here, despite the very clear wording of the question. When it says, for example, that T_1 is the value obtained from a single application of the trapezium rule it is puzzling, to say the least, when candidates bisect the range of integration and apply the trapezium rule twice.

Most candidates were able to find numerical estimates in parts (i) and (ii) reasonably efficiently. Extrapolating the Simpson's rule estimates in part (iii) was more challenging. The final part, which involved justifying giving the answer to an appropriate degree of accuracy, was rarely tackled. Most candidates simply wrote down an answer to too many or too few significant figures with no attempt at justification.

[0173287, 0.270310, 0.237969, 0.221798, 0.249148, 0.240031, 0.235473, 0.242583, 0.240213, 0.24023]