OCR

RECOGNISING ACHIEVEMENT

General Certificate of Education
Advanced Supplementary (AS) and Advanced Level
former Oxford and Cambridge modular syllabus

MEI STRUCTURED MATHEMATICS **5512**

Mechanics 6

Thursday **7 JUNE 2001** Afternoon 1 hour 20 minutes

Additional materials:
Answer paper
Graph paper
Students' Handbook

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

Write your name, Centre number and candidate number in the spaces provided on the answer paper/ answer booklet.

Answer any **three** questions.

Write your answers on the separate answer paper provided.

If you use more than one sheet of paper, fasten the sheets together.

INFORMATION FOR CANDIDATES

The approximate allocation of marks is given in brackets [ ] at the end of each question or part question.

You are advised that an answer may receive no marks unless sufficient detail of the working is shown on the answer paper to indicate that a correct method is being used.

Take $g = 9.8 \text{ m s}^{-2}$ unless otherwise instructed.

This question paper consists of 5 printed pages and 3 blank pages.

[Turn over

Option 1: Rotation of a rigid body

1

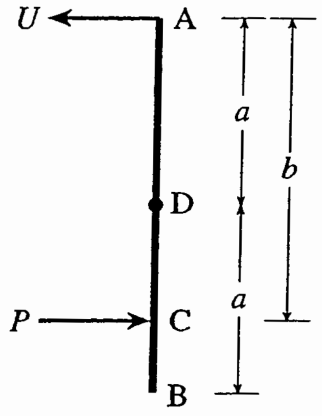

Fig. 1

Fig. 1 shows a simple model for the design of a cricket bat. A uniform rod AB of mass M and length $2a$ can rotate freely in a vertical plane about a fixed horizontal axis through the end A. The rod is hanging vertically at rest when it is acted on by a horizontal impulse P at the point C where AC = b. The impulsive reaction at A is denoted by U and the initial angular speed of the rod is ω.

(i) Write down two impulse equations for the initial motion of the rod. Hence show that $U = 0$ when $b = \frac{4}{3}a$. (The position of C for which $U = 0$ is known as the *centre of percussion*.) [7]

In a particular case, a batsman requires the centre of percussion to be $\frac{7}{5}a$ below the point A. To achieve this, a particle of mass m is added at the end B.

(ii) Find m in terms of M. [6]

When the rod is hanging vertically with this particle added, a horizontal impulse R acts at the centre D of the rod.

(iii) Find the impulsive reaction at A in terms of R.

With the same impulse R acting at D, show that a second particle of arbitrary mass can be added to the rod without changing the impulsive reaction at A. Find the positions at which this second particle may be attached. [7]

Option 2: Vectors

2

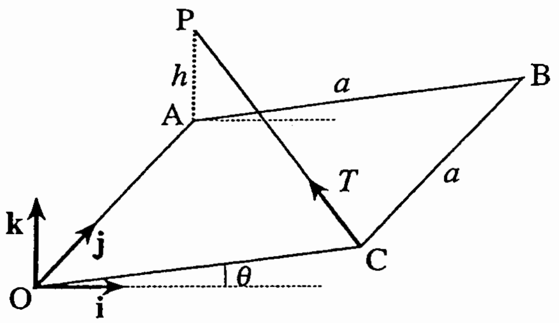

Fig. 2

A heavy concrete slab being slowly lowered into a horizontal position is shown in Fig. 2. The slab is modelled as a uniform square lamina of side a and mass m. A rope CP, attached to the slab, can only be pulled from P and the slab rotates about the horizontal line OA. P is at a height h vertically above A and the diagram shows the slab being held at an angle θ to the horizontal. Unit vectors **i**, **j** and **k** are chosen with **i** and **j** horizontal, with **j** along OA, and **k** vertically upwards.

(i) Find the vector $\overrightarrow{CP}$ and find the length L of CP. Given that the tension in the rope is T, write down the vector representing the force in the rope. [6]

(ii) Show that the moment of this force about the point O is equal to

$$\frac{T\left(-a^2 \sin\theta\,\mathbf{i} - ah\cos\theta\,\mathbf{j} + a^2\cos\theta\,\mathbf{k}\right)}{\sqrt{2a^2 + h^2 - 2ah\sin\theta}}. \qquad [7]$$

(iii) Find the moment of the tension about the axis OA. Hence find an expression for T when the slab is at an acute angle θ to the horizontal and deduce that T is greatest when the slab is horizontal. [7]

[Turn over

Option 3: Stability and oscillations

3 A smooth circular hoop of radius a is fixed in a vertical plane. A small ring of mass m is threaded on the hoop and is joined to the highest point of the hoop by a light elastic spring of natural length d and stiffness k, where $ka^2 - mga > 0$. At time t, the angle between the spring and the downward vertical is θ where $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$.

(i) Show that the potential energy V, relative to the highest point of the hoop, can be written in the form

$$V = 2\left(ka^2 - mga\right)\cos^2\theta - 2kad\cos\theta + \tfrac{1}{2}kd^2.$$ [5]

(ii) Given that $d < 2\left(a - \frac{mg}{k}\right)$, show that there are three positions of equilibrium. [5]

Discuss the stability of each position. [4]

(iii) In the case when $d > 2\left(a - \frac{mg}{k}\right)$, show that there is just one equilibrium position and that this position is stable. [2]

(iv) Discuss briefly the case when $d = 2\left(a - \frac{mg}{k}\right)$. [4]

Option 4: Variable mass

4 At time t, a rocket moving in free space has total mass m and speed v. Fuel is burnt at a constant mass rate k and is ejected with speed c relative to the rocket. The initial total mass of the rocket is M, of which a mass αM is fuel. At time $t = 0$ the rocket is at rest.

(i) Find an expression for m in terms of M, k and t. Find also the time for which the fuel burns. [3]

(ii) Show that the equation of motion for the rocket can be written $m\dfrac{\mathrm{d}v}{\mathrm{d}t} = kc$. [3]

(iii) By solving the equation in part **(ii)**, show that

$$v = c\ln\left(\frac{M}{m}\right).$$

Hence find the final speed, V, of the rocket. [6]

In an early test flight of the rocket, a fault in the control system meant that the first 25% of the total mass of fuel was ejected at speed $\frac{1}{2}c$ relative to the rocket although the fuel was burnt at the correct rate. The remainder of the fuel was burnt at the correct rate and ejected at speed c relative to the rocket.

(iv) Show that the final speed, W, that was achieved in that case is given by

$$W = \tfrac{1}{2}c\ln\left(\frac{1-\frac{1}{4}\alpha}{(1-\alpha)^2}\right).$$ [8]

Mark Scheme

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Question 1

(i)	$P - U = Ma\omega$	M1A1	
	$Pb = \frac{4}{3}Ma^2\omega$	M1A1, B1(m of i)	
	$U = 0$ when $b = \frac{4a}{3}$	M1A1	[7]
(ii)	$P = Ma\omega + 2ma\omega$	M1 either equation	
	$P\left(\frac{7a}{5}\right) = \left(\frac{4}{3}Ma^2 + 4ma^2\right)\omega$	A1(1 eqn), B1(m of i)	
	$\frac{7a}{5} = \frac{\frac{4}{3}Ma^2 + 4ma^2}{Ma + 2ma}$	M1A1	
	$7M + 14m = \frac{20}{3}M + 20m$		
	$m = M/18$	A1	[6]
(iii)	$R - U = Ma\omega + 2ma\omega = 20ma\omega$		
	$Ra = \left(\frac{4}{3}Ma^2 + 4ma^2\right)\omega = 28ma^2\omega$	M1, either equation	
	$\frac{R-U}{R} = \frac{5}{7} \Rightarrow U = \frac{2R}{7}$	M1A1	
	Add mass λ at distance x from A	M1	
	$\frac{5}{7a} = \frac{20ma\omega + \lambda x\omega}{28ma^2\omega + \lambda x^2\omega} \Rightarrow 5\lambda x^2 = 7\lambda xa$	M1	
	Hence $x = 0$ or $7a/5$ with λ arbitrary.	B1,B1	[7]

Question 2

(i) $\overrightarrow{CP} = \overrightarrow{CO} + \overrightarrow{OA} + \overrightarrow{AP}$	M1	
$\overrightarrow{CP} = -a\cos\theta\mathbf{i} + a\mathbf{j} + (h - a\sin\theta)\mathbf{k}$	A1	
$L = \sqrt{2a^2 + h^2 - 2ha\sin\theta}$	M1F1	
$\mathbf{F} = \text{force} = \dfrac{T\overrightarrow{CP}}{L}$	M1F1	[6]
(ii) $\overrightarrow{OC} = a\cos\theta\,\mathbf{i} + a\sin\theta\,\mathbf{k}$	B1	
$\mathbf{M} = \text{mt about O} = \overrightarrow{OC}\,\mathbf{x}\mathbf{F}$ (or $\overrightarrow{OP}\,\mathbf{xF}$)	M1F1	
$\mathbf{M} = \dfrac{T}{\sqrt{2a^2 + h^2 - 2ha\sin\theta}}\begin{pmatrix} a\cos\theta \\ 0 \\ a\sin\theta \end{pmatrix} \mathbf{x} \begin{pmatrix} -a\cos\theta \\ a \\ h - a\sin\theta \end{pmatrix}$	M1F1, (on OC and F), A1cao	
$\mathbf{M} = \dfrac{T}{\sqrt{2a^2 + h^2 - 2ha\sin\theta}}\begin{pmatrix} -a^2\sin\theta \\ -ha\cos\theta \\ a^2\cos\theta \end{pmatrix}$	A1, ag	[7]
(iii) mt about OA $= \mathbf{j.M}$	M1	
$= \dfrac{-Tha\cos\theta}{\sqrt{2a^2 + h^2 - 2ha\sin\theta}}$	F1	
mt of weight about OA $= \dfrac{mga\cos\theta}{2}$	M1A1 (any sign)	
$T = \dfrac{mg}{2h}\sqrt{2a^2 + h^2 - 2ha\sin\theta}$	M1A1	
max when $\theta = 0$	A1	[7]

Question 3

(i)	$V = \frac{1}{2}kx^2 - mg(d+x)\cos\theta$	M1A1A1	
	$V = \frac{1}{2}k(2a\cos\theta - d)^2 - 2mga\cos^2\theta$	M1	
	$V = 2(ka^2 - mga)\cos^2\theta - 2kad\cos\theta + \frac{1}{2}kd^2$	A1ag	[5]
(ii)	$V'(\theta) = -4(ka^2 - mga)\sin\theta\cos\theta + 2kad\sin\theta$	M1	
	$V'(\theta) = 0$ gives $\sin\theta = 0, \cos\theta = \dfrac{kd}{2(ka - mg)}$	M1A1A1	
	For 3 values $0, \pm\alpha, d < 2\left(a - \dfrac{mg}{k}\right)$	M1, clearly shown	
	$V''(0) < 0, \quad \therefore$ unstable	M1A1, any method	
	$V''(\pm\alpha) > 0, \quad \therefore$ stable	M1A1, any method	[9]
(iii)	$d > 2\left(a - \dfrac{mg}{k}\right) \Rightarrow \cos\theta > 1, \therefore$ only $\sin\theta = 0$ and hence $V''(0) > 0$ now	M1, A1 completion	[2]
(iv)	When $d = 2\left(a - \dfrac{mg}{k}\right), V'(\theta) = 0$ gives $\theta = 0$.	M1	
	But $V'(\theta) = -2(ka^2 - mga)\sin 2\theta + 2kad\sin\theta$	M1 examining behaviour near $\theta = 0$	
	$\Rightarrow V'(\theta) = kad(2\sin\theta - \sin 2\theta)$		
	$\Rightarrow V'''(\theta) = kad(-2\cos\theta + 8\cos 2\theta)$		
	$\Rightarrow V'''(\theta) \approx kad(6 - 15\theta^2)$	A1,any method	
	$V \approx \dfrac{kad}{4}\theta^4$, near θ = 0, ∴ stable.	A1 for reasonable deduction not nec. correct!.	[4]

(or from $V(\theta) = kad\cos^2\theta - 2kad\cos\theta + \frac{1}{2}kd^2$

$\Rightarrow V(\theta) = kad\left(\frac{1}{2}\cos 2\theta - 2\cos\theta\right) + \text{const}$)

Question 4

(i)	$\dfrac{dm}{dt} = -k, \quad m = M - kt$	M1A1	
	Burn out at $t = \dfrac{\alpha M}{k}$	B1	[3]
(ii)	Using conservation of momentum:		
	$(m+\delta \qquad m)(v+\delta v) - \delta \qquad m(v-c) = mv$	M1M1	
	$\Rightarrow m\dfrac{dv}{dt} = -c\dfrac{dm}{dt} = kc$	A1	[3]
(iii)	$\int dv = kc\int \dfrac{dt}{M-kt}$	M1A1	
	$v = A - c\ln(M-kt)$	A1	
	$a = c\ln M, v = c\ln\left(\dfrac{M}{M-kt}\right) = c\ln\left(\dfrac{M}{m}\right)$	A1, ag	
	When fuel gone, $m = (1-\alpha)M$	M1	
	$\therefore$ final $V = -c\ln(1-\alpha)$	A1	[6]
(iv)	end of phase 1: $V = \dfrac{-c}{2}\ln\left(\dfrac{M-\frac{\alpha M}{4}}{M}\right)$	M1A1A1	
	in second phase, $v = V - c\ln\left(\dfrac{\text{mass at } t}{\text{initial mass}}\right)$	M1	
	Final $v = V - c\ln\left(\dfrac{M-\alpha M}{M-\frac{\alpha M}{4}}\right)$	M1A1	
	Final $v = \dfrac{-c}{2}\ln\left(\dfrac{(1-\alpha)^2}{1-\frac{\alpha}{4}}\right) = \dfrac{c}{2}\ln\left(\dfrac{1-\frac{\alpha}{4}}{(1-\alpha)^2}\right)$	M1A1,ag	[8]

Examiner’s Report

Mechanics 6 (5512)

General Comments

The number of candidates for this paper was a disappointingly small 41, slightly higher than last year but still well down on previous years.

The candidates' response to the paper this year was also a little disappointing, with the mean mark marginally down. The candidates clearly understood the basic mechanics of the situations but often could not accurately complete the questions. Sometimes this was due to difficulty with the required algebraic manipulation (self-inflicted more often than not) and sometimes to the time spent sorting out their ideas. In the past the candidates were unable to do the last parts of the questions but this year this proved not to be the case

Comments on Individual Questions

Question 1 (Rotation of a rigid body)

The last time an impulse question was set, the candidates had some difficulty with the general principles. On this occasion that was not a problem. They understood what to do but sometimes got into difficulty when taking moments, particularly about the centre of the rod. Taking moments about a moving point requires great care and regrettably a few failed to get the correct expression in part (ii). Those who took moments about the fixed point fared much better. Apart from this difficulty, answers were quite good.

(i) $P - U = Ma\omega,\ Pb = \frac{4}{3} Ma^2\omega$ (ii) $m = \frac{1}{18} M$ (iii) $U = \frac{2}{7} R$, $x = 0$ or $7a/5$

Question 2 (Vectors)

For the candidates who were familiar with vectors this was reasonably well done. It was noticeable this year how few candidates confused vector and scalar notation. This was indeed welcome. Quite a few candidates completed the question on this occasion. The difficulty the majority had was in the algebraic manipulation, where little slips introduced a lot of complexity. The taking of moments about lines and points was not confused and, on the whole, this question was done well.

(i) $\overrightarrow{CP} = -a\cos\theta\mathbf{i} + a\mathbf{j} + (h - a\sin\theta)\mathbf{k}$, $L = \sqrt{2a^2 + h^2 - 2ha\sin\theta}$

(iii) moment of tension about OA $= \dfrac{-Tha\cos\theta}{\sqrt{2a^2 + h^2 - 2ha\sin\theta}}$, $T = \dfrac{mg}{2h}\sqrt{2a^2 + h^2 - 2ha\sin\theta}$

Question 3 (Stability and oscillations)

The question on stability and oscillations is always popular and this time was no exception. Candidates knew exactly what to do in the early parts of the question but, as before, their algebraic skills let them down somewhat. There was a regrettable tendency to anticipate the stability condition and adjust constants accordingly and arbitrarily. As a result of this some candidates, having found an error later in the question,