OCR
RECOGNISING ACHIEVEMENT

General Certificate of Education
Advanced Supplementary (AS) and Advanced Level
former Oxford and Cambridge modular syllabus

MEI STRUCTURED MATHEMATICS 5505

Pure Mathematics 5

Thursday **11 JANUARY 2001** Morning 1 hour 20 minutes

Additional materials:
Answer paper
Graph paper
Students' Handbook

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

Write your name, Centre number and candidate number in the spaces provided on the answer paper/answer booklet.

Answer any **three** questions.

Write your answers on the separate answer paper provided.

If you use more than one sheet of paper, fasten the sheets together.

INFORMATION FOR CANDIDATES

The approximate allocation of marks is given in brackets [ ] at the end of each question or part question.

You are advised that an answer may receive no marks unless sufficient detail of the working is shown on the answer paper to indicate that a correct method is being used.

This question paper consists of 3 printed pages and 1 blank page.

HN/2

[Turn over

1 (a) A curve has polar equation $r = a\left(1+\sqrt{2}\cos\theta\right)$ for $-\frac{3}{4}\pi \leqslant \theta \leqslant \frac{3}{4}\pi$, where a is a positive constant.

(i) Sketch the curve. [4]

(ii) Find the area of the region enclosed by the curve. [8]

(b) A conic has polar equation $\dfrac{5a}{r} = 3 + 2\cos\theta$.

(i) Find the eccentricity of the conic, and state what type of conic it is. [2]

(ii) Sketch the conic. Given that one focus of the conic is at the origin, find the polar coordinates of the other focus. [6]

2 (a) Find $\displaystyle\int_0^{\frac{3}{2}} \frac{1}{\sqrt{4x^2+9}}\,dx$, giving your answer in logarithmic form. [5]

(b) (i) Sketch the graph of $y = \arccos(2x)$. [3]

(ii) Differentiate $\arccos(2x)$ with respect to x. [2]

(iii) Use integration by parts to find $\int \arccos(2x)\,dx$. [4]

(iv) By first expanding $(1-4x^2)^{-\frac{1}{2}}$, find the series expansion of $\arccos(2x)$ as far as the term in x^5. [6]

3 (a) The cubic equation $x^3 + 3x^2 - 5 = 0$ has roots α, β and γ.

By making a suitable substitution, find a cubic equation with integer coefficients which has roots

$$\frac{2}{\alpha^2},\ \frac{2}{\beta^2} \text{ and } \frac{2}{\gamma^2}.$$ [6]

(b) When the polynomial $f(x) = 2x^8 + kx^7 + x^2 - 32$ is divided by $(x+2)$ the remainder is 100.

(i) Find the value of k, and evaluate $f(1)$. Show that $f'(1) = 39$. [5]

(ii) When $f(x)$ is divided by $(x-1)(x+2)$, the quotient is $g(x)$ and the remainder is $ax+b$, so that

$$f(x) = (x-1)(x+2)g(x) + ax + b.$$

Find a and b. [4]

(iii) Find the remainder when $f(x)$ is divided by $(x-1)^2$. [5]

4 (i) Express $e^{jk\theta}$ and $e^{-jk\theta}$ in the form $a + jb$, and show that

$$e^{2j\theta} - 1 = 2je^{j\theta}\sin\theta.$$ [4]

Series C and S are defined by

$$C = \cos\theta + \cos 3\theta + \cos 5\theta + \ldots + \cos(2n-1)\theta,$$

$$S = \sin\theta + \sin 3\theta + \sin 5\theta + \ldots + \sin(2n-1)\theta,$$

where n is a positive integer and $0 < \theta < \frac{\pi}{n}$.

(ii) Show that $C + jS$ is a geometric series, and write down the sum of this series. [4]

(iii) Show that $|C + jS| = \dfrac{\sin n\theta}{\sin\theta}$, and find $\arg(C + jS)$. [5]

(iv) Find C and S. [3]

The points A_0, A_1, A_2, A_3, A_4, A_5, A_6 in the Argand diagram correspond to complex numbers z_0, z_1, z_2, z_3, z_4, z_5, z_6 where $z_0 = 0$ and $z_1 = \cos\frac{1}{7}\pi + j\sin\frac{1}{7}\pi$. The points are the vertices of a regular heptagon with sides of length 1, as shown in Fig. 4.

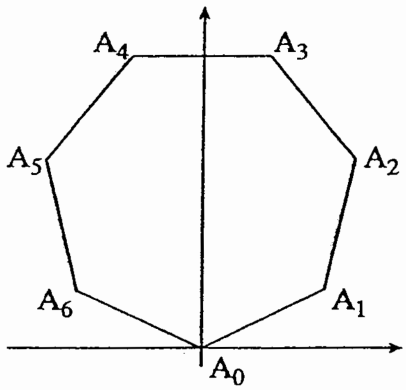


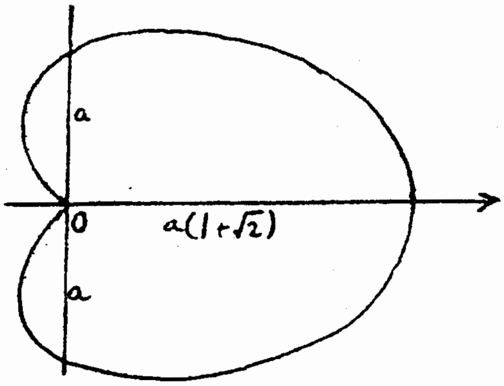
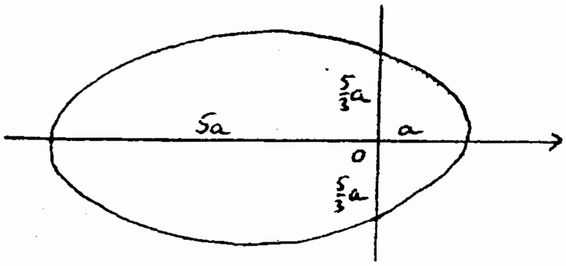
Fig. 4

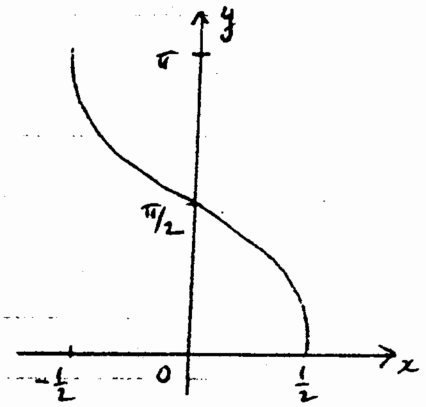
(v) Explain why $z_n = e^{\frac{1}{7}j\pi} + e^{\frac{3}{7}j\pi} + \ldots + e^{\frac{1}{7}(2n-1)j\pi}$ for n = 1, 2, 3, 4, 5, 6.

Hence, or otherwise, show that $\arg(z_n) = \frac{1}{7}n\pi$ for n = 1, 2, 3, 4, 5, 6. [4]

Mark Scheme

MARKING SCHEME

1 (a)(i)		B1 B1 B1 B1 **4**	Decreasing r from $\theta = 0$ to $\theta = \frac{3}{4}\pi$ Decreasing r from $\theta = 0$ to $\theta = -\frac{3}{4}\pi$ Cusp at O (and no loop inside) Indication of intercepts a and $a(1+\sqrt{2})$
(ii)	Area $= \int_{-\frac{3}{4}\pi}^{\frac{3}{4}\pi} \frac{1}{2}a^2(1+\sqrt{2}\cos\theta)^2\,d\theta$ $= \frac{1}{2}a^2\int_{-\frac{3}{4}\pi}^{\frac{3}{4}\pi}(1+2\sqrt{2}\cos\theta+1+\cos 2\theta)\,d\theta$ $= \frac{1}{2}a^2\left[2\theta+2\sqrt{2}\sin\theta+\frac{1}{2}\sin 2\theta\right]_{-\frac{3}{4}\pi}^{\frac{3}{4}\pi}$ $= \frac{1}{2}a^2\left(\frac{3}{2}\pi+2-\frac{1}{2}\right)\times 2$ $= \frac{3}{2}(\pi+1)a^2$	M1 A1 B1 B1 B1B1 ft M1 A1 **8**	For $\int(1+\sqrt{2}\cos\theta)^2\,d\theta$ Expansion of $(1+\sqrt{2}\cos\theta)^2$ For $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ Integration of $(a+b\cos\theta)$ and $\cos 2\theta$ Evaluation of $\sin\frac{3}{4}\pi$ and $\sin\frac{3}{2}\pi$
(b)(i)	$e = \frac{2}{3}$ The conic is an ellipse	B1 B1 **2**	
(ii)	 Ends of major axis are at distances a, $5a$ from O Other focus is at distance a from LH end, i.e. at $(4a, \pi)$	B1 B1 B1 B1 M1 A1 **6**	For any ellipse For O in approx correct position Indication of $\frac{5}{3}a$

2 (a)	$\int_0^{\frac{3}{2}} \frac{1}{\sqrt{4x^2+9}}\,dx = \left[\tfrac{1}{2}\operatorname{arsinh}\frac{2x}{3}\right]_0^{\frac{3}{2}}$ $= \tfrac{1}{2}\operatorname{arsinh} 1$ $= \tfrac{1}{2}\ln(1+\sqrt{2})$	M1 A1 A1 M1 A1 5	For arsinh *or* any sinh substitution For $\operatorname{arsinh}\frac{2x}{3}$ *or* $2x = 3\sinh\theta$ For factor $\tfrac{1}{2}$ *or* $\int \tfrac{1}{2}\,d\theta$ Use of $\ln(x+\sqrt{x^2+1})$
	OR $\int_0^{\frac{3}{2}} \frac{1}{\sqrt{4x^2+9}}\,dx = \left[\tfrac{1}{2}\ln(2x+\sqrt{4x^2+9})\right]_0^{\frac{3}{2}}$ M2 A1 A1 $= \tfrac{1}{2}\ln(3+\sqrt{18}) - \tfrac{1}{2}\ln 3$ $= \tfrac{1}{2}\ln(1+\sqrt{2})$ A1		Integral of form $\ln(x+\sqrt{x^2+1})$ $\ln(2x+\sqrt{4x^2+9})$ *or* $\ln(x+\sqrt{x^2+\frac{9}{4}})$ For factor $\tfrac{1}{2}$
(b)(i)		B1 B1 B1 3	Correct shape Indication of $(0, \tfrac{1}{2}\pi)$ Indication of domain $-\tfrac{1}{2} \le x \le \tfrac{1}{2}$ *Max 2 if any 'extra' included*
(ii)	$\frac{d}{dx}(\arccos 2x) = \frac{-2}{\sqrt{1-4x^2}}$	M1 A1 2	For $\frac{k}{\sqrt{1-(2x)^2}}$
(iii)	$\int \arccos 2x\,dx = x\arccos 2x + \int \frac{2x}{\sqrt{1-4x^2}}\,dx$ $= x\arccos 2x - \tfrac{1}{2}\sqrt{1-4x^2} + C$	M1A1 ft M1 A1 cao 4	For $\int \frac{x}{\sqrt{1-ax^2}}\,dx = k\sqrt{1-ax^2}$
(iv)	$(1-4x^2)^{-\frac{1}{2}} = 1 + 2x^2 + 6x^4 + \ldots$ $\arccos 2x = -2\int (1-4x^2)^{-\frac{1}{2}}\,dx$ $= D - 2\left(x + \tfrac{2}{3}x^3 + \tfrac{6}{5}x^5 + \ldots\right)$ $\arccos 0 = \tfrac{1}{2}\pi \Rightarrow D = \tfrac{1}{2}\pi$ $\arccos 2x = \tfrac{1}{2}\pi - 2x - \tfrac{4}{3}x^3 - \tfrac{12}{5}x^5 - \ldots$	M1A1 M1 A1 ft M1 A1 cao 6	Integrating series *Constant not required*

3 (a)	Put $y=\frac{2}{x^2}$, $x=\left(\frac{2}{y}\right)^{\frac{1}{2}}$, $\left(\frac{2}{y}\right)^{\frac{3}{2}}+3\left(\frac{2}{y}\right)-5=0$	M1A1	
	$\left(\frac{2}{y}\right)^3=\left(5-\frac{6}{y}\right)^2$	M1	
	$\frac{8}{y^3}=25-\frac{60}{y}+\frac{36}{y^2}$	A1A1	
	$25y^3-60y^2+36y-8=0$	A1 **6**	
	OR $\sum\alpha=-3$, $\sum\alpha\beta=0$, $\alpha\beta\gamma=5$		*Special ruling* *Maximum 5 if not done by substitution*
	$\sum\frac{2}{\alpha^2}=\frac{2\{(\sum\alpha\beta)^2-2(\alpha\beta\gamma)\sum\alpha\}}{(\alpha\beta\gamma)^2}=\frac{60}{25}$ M1A1		
	$\sum\frac{2}{\alpha^2}\frac{2}{\beta^2}=\frac{4\{(\sum\alpha)^2-2\sum\alpha\beta\}}{(\alpha\beta\gamma)^2}=\frac{36}{25}$ M1A1		
	Equation is $25y^3-60y^2+36y-8=0$ A1		
(b)(i)	$f(-2)=100 \Rightarrow 512-128k+4-32=100$	M1	
	$\Rightarrow k=3$	A1	
	$f(x)=2x^8+3x^7+x^2-32$, so $f(1)=-26$	A1	
	$f'(x)=16x^7+21x^6+2x$, so $f'(1)=39$	M1A1 (ag) **5**	
(ii)	Putting $x=1$, $-26=a+b$	B1 ft	
	Putting $x=-2$, $100=-2a+b$	B1	
	Hence $a=-42$, $b=16$	M1A1 cao **4**	
	OR By long division, quotient is $2x^6+x^5+3x^4-x^3+7x^2-9x+24$ A2; remainder is $-42x+16$ A1 (M1 for long division)		Complete long division leading to a linear remainder. Give A1 ft for $2x^6+(k-2)x^5+\ldots$
(iii)	Let $f(x)=(x-1)^2h(x)+px+q$		
	Putting $x=1$, $-26=p+q$	B1 ft	
	$f'(x)=(x-1)^2h'(x)+2(x-1)h(x)+p$	M1A1	
	Putting $x=1$, $39=p$	M1	
	Hence $q=-65$ and remainder is $39x-65$	A1 cao **5**	
	OR By long division, M2; quotient is $2x^6+7x^5+12x^4+17x^3+22x^2+27x+33$ A2; remainder is $39x-65$ A1		Complete long division leading to a linear remainder. Give A1 ft for $2x^6+(k+4)x^5+\ldots$

4 (i)	$e^{jk\theta} = \cos k\theta + j\sin k\theta$, $e^{-jk\theta} = \cos k\theta - j\sin k\theta$ $e^{2j\theta} - 1 = e^{j\theta}(e^{j\theta} - e^{-j\theta})$ $= e^{j\theta}\{(\cos\theta + j\sin\theta) - (\cos\theta - j\sin\theta)\}$ $= 2je^{j\theta}\sin\theta$	B1 M1 M1 A1 (ag) **4**	
	OR $e^{2j\theta} - 1 = \cos 2\theta - 1 + j\sin 2\theta$ $= -2\sin^2\theta + 2j\sin\theta\cos\theta$ M1 $= 2j\sin\theta(\cos\theta + j\sin\theta)$ M1 $= 2je^{j\theta}\sin\theta$ A1		
(ii)	$C + jS = e^{j\theta} + e^{3j\theta} + e^{5j\theta} + \ldots + e^{(2n-1)j\theta}$, a GP with common ratio $r = e^{2j\theta}$ $= \dfrac{e^{j\theta}(e^{2nj\theta} - 1)}{e^{2j\theta} - 1}$	M1 A1 M1A1 **4**	
(iii)	$C + jS = \dfrac{e^{j\theta}2je^{jn\theta}\sin n\theta}{2je^{j\theta}\sin\theta}$ $= \left(\dfrac{\sin n\theta}{\sin\theta}\right)e^{jn\theta}$ Hence $\lvert C + jS\rvert = \dfrac{\sin n\theta}{\sin\theta}$ and $\arg(C + jS) = n\theta$	M1M1 A1 A1 (ag) A1 **5**	
(iv)	$C = \dfrac{\sin n\theta\cos n\theta}{\sin\theta}$, $S = \dfrac{\sin^2 n\theta}{\sin\theta}$	M1A1A1 **3**	Accept any correct real form
(v)	$\overrightarrow{A_1A_2}$ has length 1, making angle $\dfrac{3\pi}{7}$ with the real axis, so $z_2 - z_1 = e^{j\frac{3\pi}{7}}$ Similarly $z_3 - z_2 = e^{j\frac{5\pi}{7}}$, etc so $z_n = e^{j\frac{\pi}{7}} + e^{j\frac{3\pi}{7}} + \ldots + e^{j\frac{(2n-1)\pi}{7}}$ $z_n = C + jS$ with $\theta = \dfrac{\pi}{7}$ Hence $\arg z_n = n\theta = \dfrac{n\pi}{7}$	B1 B1 M1 A1 **4**	

Examiner’s Report

Pure Mathematics 5 (5505)

General Comments

The marks on this paper were well spread out, with about one fifth of the candidates scoring 50 marks or more (out of 60) and about a quarter scoring less than 20. Questions 2 and 3 were considerably more popular than questions 1 and 4.

Comments on Individual Questions

Question 1 (Polar coordinates)

Although this was the least popular question, it was also the best answered, with half the attempts scoring 16 marks or more (out of 20). In part (a)(i), the curve was usually drawn with the correct shape, but some candidates included the inner loop, and some gave no indication of the scale (or took a to be 1). In part (a)(ii), the method for finding the area was well understood and the integration of $(1+\sqrt{2}\cos\theta)^2$ was, on the whole, competently done; the correct answer was frequently obtained. Part (b), on the ellipse, was also well understood.

(a)(ii) $\frac{3}{2}(\pi+1)a^2$; (b)(i) $\frac{2}{3}$, ellipse; (ii) $(4a,\ \pi)$.

Question 2 (Inverse hyperbolic and circular functions)

Almost every candidate attempted this question and demonstrated competence in some of the techniques examined. However, there were many opportunities to make mistakes, and the average mark was about 11. In part (a), the form of the integral was usually correct, but many candidates omitted the factor $\frac{1}{2}$. In part (b)(i), most candidates lost marks for the sketch, usually by drawing a one-to-many relation, or showing the domain as $-1 \le x \le 1$ instead of $-\frac{1}{2} \le x \le \frac{1}{2}$, or giving the crossing point on the y-axis as $(0,\ \pi)$ instead of $(0,\ \frac{1}{2}\pi)$. In the differentiation in part (b)(ii), the factor 2 was often missing. In part (b)(iii), the integration by parts was well understood, but then the integration of $\dfrac{2x}{\sqrt{1-4x^2}}$ caused problems for many. In part (b)(iv), very many candidates omitted the constant when integrating the series.

(a) $\frac{1}{2}\ln(1+\sqrt{2})$;

(b)(ii) $\dfrac{-2}{\sqrt{1-4x^2}}$; (iii) $x\arccos 2x - \frac{1}{2}\sqrt{1-4x^2} + C$; (iv) $\frac{1}{2}\pi - 2x - \frac{4}{3}x^3 - \frac{12}{5}x^5 - \ldots$.

Question 3 (Algebra)

This question was quite well answered, with half the attempts scoring 14 marks or more. In part (a), the new cubic equation was often found correctly, although a fair number of candidates were unable to deal with the fractional powers resulting from the substitution. Some candidates considered sums of products of the roots (despite the instruction given in the question), and it was not possible to earn full marks if this method was used. In part (b), the remainder theorem was well understood, and parts (i) and (ii) were usually answered correctly. In part (iii), many candidates did not know how to find the remainder after division by $(x-1)^2$, and it was often omitted altogether. A fair number answered it efficiently and correctly by considering $f'(x)$, and some used long division (with occasional success).

(a) $25y^3 - 60y^2 + 36y - 8 = 0$; (b)(i) $k=3$, $f(1)=-26$; (ii) $a=-42$, $b=16$; (iii) $39x-65$.

Question 4 (Complex numbers)

This was the worst answered question, with half the attempts scoring 7 marks or less. Part (i) was often answered correctly, although a surprising number of candidates experienced difficulties, notably with signs in steps such as $-\sin\theta + \mathrm{j}\cos\theta = \mathrm{j}(\cos\theta + \mathrm{j}\sin\theta)$. In part (ii), most candidates obtained the geometric series, but several gave the sum to infinity instead of the first *n* terms. In parts (iii) and (iv), the methods for realising the denominator and taking real and imaginary parts were generally understood but rarely done accurately. Only a handful of candidates produced any work of value in part (v); many attempts stated that the vertices represented the seventh roots of unity.

(i) $\cos k\theta + \mathrm{j}\sin k\theta$, $\cos k\theta - \mathrm{j}\sin k\theta$; (ii) $\dfrac{\mathrm{e}^{\mathrm{j}\theta}(\mathrm{e}^{2n\mathrm{j}\theta} - 1)}{\mathrm{e}^{2\mathrm{j}\theta} - 1}$; (iii) $n\theta$;

(iv) $C = \dfrac{\sin n\theta \cos n\theta}{\sin\theta}$, $S = \dfrac{\sin^2 n\theta}{\sin\theta}$.