OCR
RECOGNISING ACHIEVEMENT

General Certificate of Education
Advanced Supplementary (AS) and Advanced Level
former Oxford and Cambridge modular syllabus

MEI STRUCTURED MATHEMATICS 5521/1

Numerical Analysis

Friday 19 JANUARY 2001 Morning 1 hour 20 minutes

Additional materials:
Answer paper
Graph paper
Students' Handbook

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

Write your name, Centre number and candidate number in the spaces provided on the answer paper/answer booklet.

Answer any **three** questions.

Write your answers on the separate answer paper provided.

If you use more than one sheet of paper, fasten the sheets together.

INFORMATION FOR CANDIDATES

The approximate allocation of marks is given in brackets [ ] at the end of each question or part question.

You advised that an answer may receive no marks unless sufficient detail of the working is shown on the answer paper to indicate that a correct method is being used.

This question paper consists of 3 printed pages and 1 blank page.

HN

[Turn over

1 The equation

$$e^{-x}(1+x) = 0.05$$

is to be solved numerically.

(i) Show, graphically or otherwise, that the equation has exactly two roots. Give approximate values for these roots. [5]

(ii) Show that the equation can be rearranged as $x = a\mathrm{e}^x + b$ where a and b are constants to be determined. Use the iterative formula based on this rearrangement to find one of the roots correct to 4 significant figures.

Show numerically that this iterative formula will not converge to the second root, and confirm this by considering the derivative of the right-hand side. [9]

(iii) Find a different iterative formula which does converge to the second root. Determine the second root correct to 4 significant figures. [6]

2 A function $f(x)$ depends upon a variable x which is never negative. Two values of $f(x)$ are known: $f(0) = 7.1$ and $f(0.5) = 2.9$.

(i) Estimate the values of

(*A*) $\int_0^{0.5} f(x)\,dx$,

(*B*) $f'(0)$. [4]

(ii) Suppose that $f(x)$ is now known at a third point, $x = c$. What value of c would be the most useful for improving the estimate of $\int_0^{0.5} f(x)\,dx$? Explain your answer.

Discuss whether the same value of c would be the most useful for improving the estimate of $f'(0)$. [4]

In fact, $c = 1$ and $f(1) = 4.4$.

(iii) Find the quadratic interpolating polynomial which passes through all three data points, expressing your answer as simply as possible. [6]

(iv) Hence obtain further estimates of

(*A*) $\int_0^{0.5} f(x)\,dx$,

(*B*) $f'(0)$. [4]

(v) Are the estimates in part (iv) likely to be better or worse than those in part (i)? Explain your answer briefly. [2]

3 In this question you are asked to obtain a Taylor approximation to $y = \arctan x$. You are given that

$$\frac{dy}{dx} = \frac{1}{1+x^2}.$$

(i) Find $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.

Hence obtain the third order Taylor polynomial for $y = \arctan x$ about $x = 0$. [6]

You are now given that

$$\frac{d^4y}{dx^4} = \frac{24x(1-x^2)}{(1+x^2)^4},$$

and that this function increases for $0 < x < 0.3$.

(ii) Write down an expression for the error term associated with the Taylor polynomial in part (i). Explain clearly the meaning of the symbol(s) used in this expression. [5]

(iii) Find the error bound when this Taylor polynomial is used to estimate $\arctan\frac{1}{5}$. Show that the corresponding error bound for the estimate of $\arctan\frac{1}{239}$ is negligible. [5]

(iv) Given that

$$\pi = 16\arctan\tfrac{1}{5} - 4\arctan\tfrac{1}{239},$$

obtain an approximation for π together with its error bound.

Show that the actual error is substantially smaller in magnitude than this error bound. [4]

4 The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = 1 + \frac{1}{x+y}$$

with $y = 1$ when $x = 1$. The differential equation is to be solved numerically to estimate the value of y when $x = 0.8$.

(i) Obtain two preliminary estimates by using Euler's method with step lengths $h = -0.2$ and $h = -0.1$. [5]

(ii) Obtain further estimates by using the modified Euler method with $h = -0.2$ and $h = -0.1$. [8]

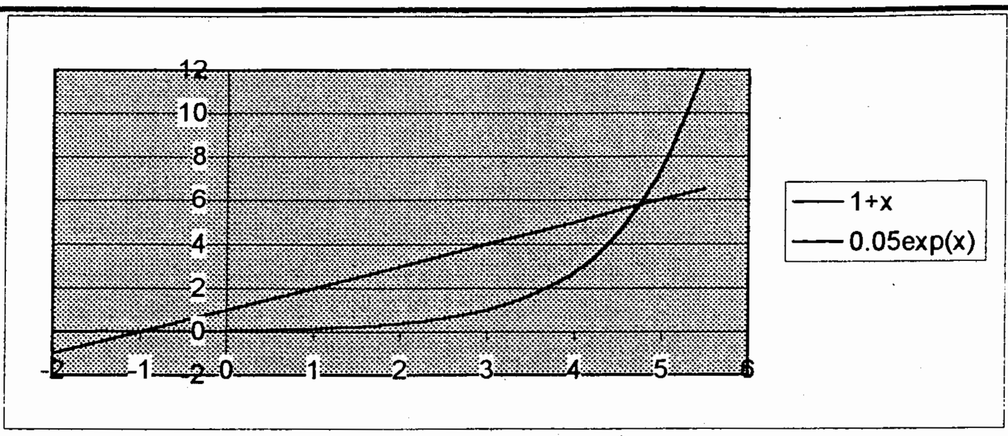
(iii) Given that the errors in the modified Euler method are approximately proportional to h^2, obtain a better estimate of the required value, giving your answer to an appropriate number of significant figures. [4]

(iv) Explain briefly, without doing any further calculations, how you would estimate the value of x when $y = 0.8$. [3]

Mark Scheme

Numerical Analysis (5521) January 2001 Mark scheme

Q1(i)



G3

Only two roots: in (-1,0) and (4,5). Solutions at approximately -1 and 5 — A1A1

(Other graphs and non-graphical approaches possible.) — **5**

(ii) re-arranging: $1 + x = 0.05\,e^x$, $x = 0.05\,e^x - 1$. So a = 0.05 and b = -1. — M1A1

iterating, eg:

x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
-1	-0.98161	-0.98126	-0.98126	-0.98126	-0.98126	-0.98126	-0.98126	M1A1
5	6.420658	29.72036	4.04E+11					A1

Convergence to negative root (-0.9813 to 4 dp) but not to the positive root. — A1

Gradient of RHS is $0.05\,e^x$ which has value approx 4.500857 at the positive root — A1A1

Since this is greater than 1 the iteration will not converge. Confirmed numerically above. — E1

9

(iii) The inverse iteration will converge:

$e^x = 20(x + 1)$, $x = \ln(20(x + 1))$ (Other functions possible) — B3

iterating:

x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
5	4.787492	4.751431	4.745181	4.744094	4.743904	4.743871	4.743866	M1A1

The solution is 4.744 to 4 sf — A1

TOTAL 20 — **6**

Q2(i) Integral = 0.5 (7.1 + 2.9) / 2 = 0.25 — M1A1

f ' (0) = (2.9 - 7.1) / (0.5 - 0) = -8.4 — M1A1

4

(ii) Best value of c for the integral is 0.25 in order to apply Simpson's rule. — E1E1

Best value of c for f ' (0) is close to, but not too close to, zero. (c = 0.1 or 0.01 is — E1

reasonable; c = 0.000001 is not.) — E1

4

(iii)

$$p(x) = 7.1\,\frac{(x - 0.5)(x - 1)}{(0 - 0.5)(0 - 1)} + 2.9\,\frac{x\,(x - 1)}{0.5\,(0.5 - 1)} + 4.4\,\frac{x\,(x - 0.5)}{1\,(1 - 0.5)}$$ — M1, A1,1,1

$$= 14.2\,(x^2 - 1.5\,x + 0.5) - 11.6\,(x^2 - x) + 8.8\,(x^2 - 0.5x)$$ — A1

$$= 11.4\,x^2 - 14.1\,x + 7.1$$ — A1

6

(iv) Integral (p(x), 0, 0.5) = $11.4\,(0.5)^3/3 - 14.1\,(0.5)^2/2 + 7.1\,(0.5) = 2.2625$ — M1A1

p ' (x) = 22.8 x - 14.1 so p ' (0) = - 14.1 — M1A1

4

(v) Can't be certain, but because the interpolating polynomial takes some account of — E1

the curvature the estimates may well be better. — E1

TOTAL 20 — **2**

Numerical Analysis (5521) January 2001 Mark scheme

Q3(i)	$y'' = -2x / (1+x^2)^2 \qquad y''' = -2 / (1+x^2)^2 + 8x^2 / (1+x^2)^3$	A1A2
	$y(0) = 0 \quad y'(0) = 1 \quad y''(0) = 0 \quad y'''(0) = -2$	M1A1
	Hence $y = x - x^3/3 + \dots$	A1
		6
(ii)	error term is $\dfrac{x^4}{4!} f^{iv}(\xi) = \dfrac{x^4}{4!} \dfrac{24\xi(1-\xi^2)}{(1+\xi^2)^4} = \dfrac{x^4\,\xi(1-\xi^2)}{(1+\xi^2)^4}$	M1 A1A1
	where x is the value at which the function is evaluated,	E1
	and ξ is an unknown point in the interval (0,x).	E1
		5
(iii)	Since y^{iv} is increasing, error bound at x = 1/5 is when ξ = 1/5	M1A1
	Error bound is 0.0002626	A1
	Similarly, error bound at x = 1/239 is at ξ = 1/239	M1
	Error bound there is about 10^{-12}: negligible.	A1
(iv)	Error bound for approximation to 16 arctan(1/5) - 4 arctan(1/239)	M1
	is 16 x 0.0002626 = 0.0042	A1
	Actual error is 3.1405969 - pi = - 0.001 (substantially smaller than 0.0042)	A1A1
	TOTAL 20	4

Q4(i)	One step: $y_1 = y_0 + h(1 + 1/(x_0 + y_0))$ with $x_0 = 1$, $y_0 = 1$, $h = -0.2$	M1
	$= 0.7$	A1
	Two steps: $y_1 = 1 - 0.1\,(1 + 1/(1+1)) = 0.85$	A1
	$y_2 = 0.85 - 0.1\,(1 + 1/(0.9 + 0.85)) = 0.6928571$	M1A1
		5
(ii)	One step: $y_1 = y_0 + 0.5\,(k_1 + k_2)$ $\quad k_1 = -0.2\,(1 + 1/(1+1)) = -0.3$	M1
	$k_2 = -0.2\,(1 + 1/(0.8 + 0.7)) = -0.333\dots$	M1
	$= 0.68333\dots$	A1
	Two steps: $k_1 = -0.1\,(1 + 1/(1+1)) = -0.15$	
	$k_2 = -0.1\,(1 + 1/(0.9 + 0.85)) = -0.157143$	M1
	$y_1 = 0.8464286$	A1
	$k_1 = -0.1\,(1 + 1/(0.9 + 0.846\dots) = -0.15726$	M1
	$k_2 = -0.1\,(1 + 1/(0.8 + 0.68916\dots)) = -0.167\dots$	M1
	$y_2 = 0.684223$	A1
		8
(iii)	Extrapolation: 0.684223 - 0.683333 = 0.000890	M1A1
	Since errors decrease by a factor of 4, best estimate is	M1
	0.684223 + 0.000890 (1/4 + 1/16 + ...) = 0.684520	A1
		4
(iv)	To estimate x when y = 0.8 *either* solve this equation with greater accuracy and interpolate *or* (better) solve the equation $dx/dy = (1 + 1/(x+y))^{-1}$.	B3
	TOTAL 20	3

Examiner’s Report

MEI Structured Mathematics
5521 Numerical Analysis
Examiner's report

The entry for this paper was small, but most of the candidates were well prepared. Indeed some showed a grasp of numerical mathematics which was very impressive.

Question 1 (Numerical solution of an equation)
This question was on a very familiar topic, but it was not popular. Perhaps that was because candidates do not care for the negative exponential function. In part (i), the graphical approach used generally involved sketching $y = e^{-x}(1 + x)$ and $y = 0.05$. There were only a couple of explicit and clear demonstrations that there are only two roots. The numerical work in part (ii) was done well; the analysis, involving the derivative of the exponential function, was sometimes incorrect. Part (iii) was done well by almost everyone.
[Roots –0.9813, 4.744.]

Question 2 (Integration, differentiation and interpolation)
The numerical integral and derivative in part (i) caused no problems. In part (ii), almost everyone suggested $c = 0.25$ for the integral, but opinions were various on the best value for the derivative. A popular answer was $c = -0.5$ so that the central difference formula could be used: a nice idea but specifically excluded by the statement that x is never negative. The Lagrange method in part (iii) was frequently well done, though some omitted the y values which form the coefficients, and others made errors in the algebra. Part (iv) caused no problems; erroneous polynomials were followed through here. In part (v) most candidates were properly cautious in suggesting that the interpolating polynomial is likely to give better estimates.
[(i) integral 2.5, derivative –8.4; (iii) integral 2.625, derivative –14.1.]

Question 3 (Taylor polynomial)
The differentiation in part (i) was well done, and so candidates had little difficulty obtaining the Taylor polynomial for arctan x. In part (ii) the error term did cause problems: some candidates were unclear about the distinction between x and ξ. Despite that, the error bounds (in which $x = \xi$) were calculated accurately by almost everyone. However, combining these error bounds in part (iv) was found more difficult.
[(iii) error bounds 0.00026 and 10^{-12}; (iv) error bound 0.0042, actual error –0.001.]

Question 4 (Differential equation)
This was a popular question with candidates taking the negative step length in their stride. Euler's method in part (i) was very straightforward; the modified Euler method in part (ii) caused some candidates to become muddled in method or to make arithmetical mistakes. The extrapolation in part (iii) was understood by almost everyone, but some applied the correction term with the wrong sign. Part (iv) was pleasingly done with almost all candidates able to suggest re-writing the equation in terms of $\mathrm{d}x/\mathrm{d}y$.
[(i) 0.7, 0.69286; (ii) 0.68333, 0.68422; (iii) 0.68452.]