OCR
RECOGNISING ACHIEVEMENT

General Certificate of Education
Advanced Supplementary (AS) and Advanced Level
former Oxford and Cambridge modular syllabus

MEI STRUCTURED MATHEMATICS 5510/1

Differential Equations (Mechanics 4)

Friday 19 JANUARY 2001 Morning 1 hour 20 minutes

Additional materials:
Answer paper
Graph paper
Students' Handbook

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

Write your name, Centre number and candidate number in the spaces provided on the answer paper/ answer booklet.

Answer any **three** questions.

Write your answers on the separate answer paper provided.

If you use more than one sheet of paper, fasten the sheets together.

INFORMATION FOR CANDIDATES

The approximate allocation of marks is given in brackets [ ] at the end of each question or part question.

You are advised that an answer may receive no marks unless sufficient detail of the working is shown on the answer paper to indicate that a correct method is being used.

Take $g = 9.8$ m s^{-2} unless otherwise instructed.

This question paper consists of 3 printed pages and 1 blank page.

[Turn over

1 (a) A solution is sought to the differential equation

$$\frac{dy}{dx} + \frac{y}{x} = \sin x \qquad (x > 0).$$

(i) Find the general solution for y in terms of x. [8]

(ii) Given that $y = \frac{1}{\sqrt{2}}$ when $x = \frac{1}{4}\pi$, find y when $x = \pi$. [4]

(b) A numerical solution is sought to the differential equation

$$\frac{dy}{dx} + \frac{y}{x} = \tan x$$

using Euler's method. The algorithm is given by

$$x_{n+1} = x_n + h, \qquad y_{n+1} = y_n + hf(x_n, y_n),$$

where h is the step length and $f(x, y) = \frac{dy}{dx}$.

(i) Given that $y = 1$ when $x = 1$, use a step length of 0.2 to estimate the value of y when $x = 1.4$. [6]

(ii) Explain why it would be inappropriate to continue this process for one further step. [2]

2 A solution is sought to the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + ky = 12e^{-x},$$

where k is a constant.

Consider the case $k = -2$.

(i) Find the general solution. [8]

You are given that y tends to zero as x tends to infinity and that $\frac{dy}{dx} = 2$ when $x = 0$.

(ii) Find the particular solution. [4]

(iii) Find the finite value of x for which y is zero. [2]

Consider now the case $k = \frac{1}{4}$. You are again given that y tends to zero as x tends to infinity and that $\frac{dy}{dx} = 2$ when $x = 0$.

(iv) Show that, in this case, the given conditions are not sufficient to determine the particular solution. [6]

3 A ball of mass 0.1 kg is projected vertically upwards from ground level with an initial speed of 49 m s^{-1}. When the ball is in flight, the forces acting on it are its weight and air resistance which is assumed to have magnitude $0.02v$ newtons, where v m s^{-1} is the velocity of the ball. The time after projection is denoted by t seconds.

(i) Show that, for the upward motion, $\frac{dv}{dt} = -0.2(49 + v)$. Hence find an expression for v in terms of t. [9]

The ball reaches a maximum height of H metres when $t = T_1$.

(ii) Calculate T_1 and show that $H = 245(1 - \ln 2)$. [6]

(iii) Explain briefly why the differential equation in part (i) and its solution apply unchanged to the downward motion as well as the upward motion. [3]

The ball reaches the ground again when $t = T_2$.

(iv) Show that $\frac{1}{10}T_2 + e^{-0.2T_2} = 1$. [2]

4 Two reservoirs, A and B, are connected by a river. Water flows from A to B at a rate proportional to the volume of water in A. After heavy rainfall, water also flows into each reservoir from the surrounding land at an exponentially decaying rate. The equations modelling the situation are:

$$\frac{dx}{dt} = 10e^{-6t} - x, \qquad (1)$$

$$\frac{dy}{dt} = 8e^{-6t} + x, \qquad (2)$$

where x km^3 is the volume of water in reservoir A, y km^3 is the volume of water in reservoir B and t is the time in days.

When $t = 0$, $x = 5$ and $y = 4$.

(i) Solve equation (1) to show that $x = 7e^{-t} - 2e^{-6t}$. [8]

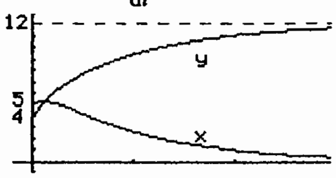
(ii) Hence solve equation (2) to find y in terms of t. [4]

(iii) Determine whether x is increasing or decreasing initially. Sketch the graphs of x and y against t. [4]

(iv) Find the maximum rate of decrease of the volume of water in reservoir A. [4]

Mark Scheme

1.(a)(i) $I = \exp(\int \frac{1}{x}dx) = x$	M1 A1	
$x\frac{dy}{dx} + y = x\sin x$	M1	
$xy = \int x\sin x dx$	M1	
$= -x\cos x + \int \cos x dx$	M1 A1	
$= -x\cos x + \sin x + A$	A1	
$y = -\cos x + \frac{\sin x}{x} + \frac{A}{x}$	A1	
(ii) $\frac{1}{\sqrt{2}} = -\cos\frac{\pi}{4} + \frac{4}{\pi}\sin\frac{\pi}{4} + \frac{4A}{\pi}$	M1	condition
$A = \frac{\pi-2}{2\sqrt{2}}$	M1	solve (follow equation of similar nature)
$x = \pi, y = 1 + \frac{\pi-2}{2\sqrt{2}\,\pi} \approx 1.128$	M1	substitute
	A1 cao	
(b)(i) $\frac{dy}{dx} = \tan x - \frac{y}{x}$		
$y'(1) = 0.55741$	M1	
$y(1.2) = 1 + 0.2 \times 0.55741 = 1.11148$	M1 A1	
$y'(1.2) = 1.64592$	M1	
$y(1.4) = 1.11148 + 0.2 \times 1.64592 = 1.44066$	M1 A1	
(ii) $1.6 > \frac{\pi}{2}$ so when $x = \frac{\pi}{2}$, $\frac{dy}{dx}$ undefined	M1	
cannot continue solution across discontinuity	A1	
2. (i) $\lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda = 1$ or -2	M1 A1	
CF $y = Ae^{x} + Be^{-2x}$	F1	
PI $y = ae^{-x}$	B1	
$y' = -ae^{-x}, y'' = ae^{-x}$		
$(a + -a - 2a)e^{-x} = 12e^{-x}$	M1	differentiate and substitute
$a = -6$	M1 A1	equate coefficients
$y = Ae^{x} + Be^{-2x} - 6e^{-x}$	F1	
(ii) $y \to 0 \Rightarrow A = 0$	M1 F1	
$y = Be^{-2x} - 6e^{-x}$		
$\frac{dy}{dx} = -2Be^{-2x} + 6e^{-x}$		
$2 = -2B + 6 \Rightarrow B = 2$	M1	differentiate and sub. values
$y = 2e^{-2x} - 6e^{-x}$	A1 cao	
(iii) $0 = 2e^{-2x} + 6e^{-x}$		
$e^{-x} = 3$	M1	attempt to solve
$x = -\ln 3$	F1	follow equation of similar nature
(iv) $\lambda^2 + \lambda + \frac{1}{4} = 0 \Rightarrow \lambda = -\frac{1}{2}$ (repeated)	B1	
CF $y = (A + Bx)e^{-\frac{x}{2}}$	M1	
PI $y = ae^{-x}$	M1	
$y = ae^{-x} + (A + Bx)e^{-\frac{x}{2}}$	A1 cao	
All terms tend to zero as x tends to infinity		
so condition provides no information.	M1	
Condition on y' insufficient to determine		
two arbitrary constants	A1	

	Working	Mark	Comment
3.(i)	$0.1\frac{dv}{dt} = -0.1g - 0.02v$	M1	N2L with all terms
	$\frac{dv}{dt} = -0.2(49+v)$	E1	
	$\int \frac{dv}{49+v} = \int -0.2dt$	M1	
	$\ln\lvert 49+v\rvert = -0.2t + c$	M1 A1	with constant
	$v = Ae^{-0.2t} - 49$	M1	v in terms of t
	$v = 49, t = 0 \Rightarrow A = 98$	M1 A1	
	$v = 49(2e^{-0.2t} - 1)$	A1	
(ii)	$v = 0 \Rightarrow e^{-0.2T_1} = \frac{1}{2}$	M1	set $v = 0$ and attempt to solve
	$\Rightarrow T_1 = 5\ln 2$	F1	follow equation of similar form
	$x = \int v dt = 49(-10e^{-0.2t} - t + B)$	M1	
	$t = 0, x = 0 \Rightarrow B = 10$	M1	
	$H = 49(-10(\frac{1}{2}) - 5\ln 2 + 10)$	M1	substitute constant and value for t
	$H = 245(1 - \ln 2)$	E1	
(iii)	Down $\Rightarrow v < 0 \Rightarrow -0.02v > 0$ i.e. opposing motion	B1	
	so DE applies throughout motion	M1	
	solution applies as same DE same initial conditions	A1	some comment required
(iv)	$0 = 49(-10e^{-0.2T_2} - T_2 + 10)$	M1	equate to zero
	$\Rightarrow \frac{T_2}{10} + e^{-0.2T_2} = 1$	E1	
4.(i)	$\frac{dx}{dt} + x = 10e^{-6t}$	M1	rearranging
	CF $a + 1 = 0$ [or $I = e^{\int 1dt}$]	M1	
	$x = Ae^{-t}$ [$= e^{t}$]	A1	
	PI ae^{-6t} [$\frac{d}{dt}(xe^{t}) = 10e^{-5t}$]	M1	
	$(-6a + a)e^{-6t} = 10e^{-6t}$ [$xe^{t} = -2e^{-5t} + A$]	M1	
	$a = -2$ [$x = -2e^{-6t} + Ae^{-t}$]	A1	
	$t = 0, x = 5 \Rightarrow 5 = -2 + A \Rightarrow A = 7$	M1	condition on x
	$x = 7e^{-t} - 2e^{-6t}$	E1	
(ii)	$\frac{dy}{dt} = 8e^{-6t} + 7e^{-t} - 2e^{-6t} = 6e^{-6t} + 7e^{-t}$	M1	sub. for x and integrate
	$y = -e^{-6t} - 7e^{-t} + B$	A1	
	$t = 0, y = 4 \Rightarrow 4 = -1 - 7 + B \Rightarrow B = 12$	M1	condition on y
	$y = 12 - e^{-6t} - 7e^{-t}$	A1	
(iii)	$t = 0 \Rightarrow \frac{dx}{dt} = 5 > 0$ so rising	B1	
		B1	x decreasing after initial rise (from 5)
		B1	y increasing (from 4)
		B1	y asymptote
(iv)	$\frac{dx}{dt} = -7e^{-t} + 12e^{-6t}$		
	$\frac{d^2x}{dt^2} = 7e^{-t} - 72e^{-6t}$	M1	
	$\frac{d^2x}{dt^2} = 0 \Rightarrow e^{5t} = \frac{72}{7}$	M1	setting second derivative to zero
	$t = 0.466$	M1	solving
	$\frac{dx}{dt} = -3.66$ km^3 per day	A1	

Examiner’s Report

Mechanics 4 (5510)

General Comments

There were many good scripts, with candidates demonstrating both a good understanding of the concepts and the ability to work accurately. Q.3 was the least popular question, with the other questions being chosen in approximately equal numbers.

Comments on Individual Questions

Question 1 (Integrating factor and numerical solution)

Many candidates used the correct method of solution for the first equation, and most of these were able to find the correct solution. However common errors were to fail to multiply the right hand side of the equation by the integrating factor and failing to divide the constant by x when rearranging the solution. The calculation of the constant and of the value of y in part (ii) was often done well.

The numerical solution was frequently correct, but also was often just a list of incorrect numbers with no evidence of method. Candidates must yet again be warned that such solutions are unlikely to gain any credit. A number of candidates recognized that the next step of the solution would go past a discontinuity, but not all clearly expressed the problem that this causes. Some candidates did not recognise the discontinuity and commented in general terms about loss of accuracy with Euler's method.

(a)(i) $y = -\cos x + \frac{\sin x}{x} + \frac{A}{x}$; (ii) 1.128; (b)(i) 1.4407.

Question 2 (Second order equation)

The first three parts of this question were frequently done very well. Some candidates were unable to use the information that the solution tended to zero, but most realised that the coefficient of the non-decaying term must therefore be zero. In the last part of the question, some candidates were unable to recall the correct form of the complementary function for a repeated root of the auxiliary equation. Most were successful, but many felt it was necessary to then find the general solution in full (which was usually correct but not necessary to answer the question). Most candidates who had got this far realised the problem with the given conditions, but often did not express themselves clearly enough.

(i) $y = Ae^{x} + Be^{-2x} - 6e^{-x}$, (ii) $y = 2e^{-2x} - 6e^{-x}$, (iii) $-\ln 3$.

Question 3 (Resisted motion under gravity: separable variables)

This question was not as popular as the others. Most candidates were able to set up the differential equation, although some attempts seemed to be led by the given answer rather than starting from a clear Newton's second law equation. The solution of the equation was often well done, although some did not recognise that it could be done using separation of variables and used other methods. A surprising number of candidates were unable to correctly deduce the expression for displacement - some forgot to include a constant when integrating, and some did not realise that they only had to integrate the expression for velocity. The explanation as to why the equation and solution were valid for the downward motion was often based on false or vague reasons and was rarely complete. Candidates who had previously found an expression for displacement were usually able to use it to establish the equation given in the last part of the question.

(i) $v = 49(2e^{-0.2t} - 1)$.

Question 4 (Water in reservoirs: simultaneous equations)

The solutions for x and y were often correct, although some candidates treated the equation for x as a separable variables equation and some even differentiated it to produce a second order equation. The sketch graphs were often well done and it was good to see most candidates making sure that the initial conditions were shown clearly on the graphs. Most candidates were able to use the gradient to correctly deduce that x was initially increasing, but it was surprising to see some candidates attempting to use values of x. Many candidates realised the need to find the second derivative and to set it to zero to find the maximum rate of decrease of x, but a few candidates attempted to use only the first derivative.

(ii) $y = 12 - e^{-6t} - 7e^{-t}$, (iv) 3.66 km³ per day.