GCE Examinations

Pure Mathematics Module P6

Advanced Subsidiary / Advanced Level

Paper F

Time: 1 hour 30 minutes

Instructions and Information

Candidates may use any calculator except those with a facility for symbolic algebra and / or calculus.

Full marks may be obtained for answers to ALL questions.

Mathematical and statistical formulae and tables are available.

This paper has 7 questions.

Advice to Candidates

You must show sufficient working to make your methods clear to an examiner. Answers without working will gain no credit.

Written by Rosemary Smith & Shaun Armstrong

1. Prove by induction that, for all $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^{n} \ln\frac{r+1}{r} = \ln(n+1).$$ **(6 marks)**

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2. $$\mathbf{M} = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}.$$

(a) Find the eigenvalues of **M**. **(4 marks)**

(b) Find eigenvectors corresponding to each eigenvalue found in part *(a)*. **(4 marks)**

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3. A transformation T from the z-plane to the w-plane is defined by

$$w = \frac{z+2\text{i}}{z-\text{i}}, \quad z \neq \text{i},$$

where $z = x + \text{i}y$, $w = u + \text{i}v$ and x, y, u and v are real.

(a) Show that the circle $|z| = 1$ is mapped onto a straight line in the w-plane under T and find an equation of the line.

(5 marks)

The circle $|z - (a + \text{i}b)| = r$ in the z-plane is mapped under T onto the circle $|w| = 2$ in the w-plane, where a, b and r are real.

(b) Find the values of a, b and r. **(6 marks)**

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4. The points A, B and C with coordinates (x_{-1}, y_{-1}), (x_0, y_0) and (x_1, y_1) respectively lie on the curve $y = \mathrm{f}(x)$ with $x_1 - x_0 = x_0 - x_{-1} = h$.

(a) Use the first three terms of the Taylor series expansion in ascending powers of $(x - x_0)$ to show that

$$\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)_0 \approx \frac{y_1 - 2y_0 + y_{-1}}{h^2}.$$

(5 marks)

The variable y satisfies the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (x+2)\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 0 \quad \text{with } y = 1 \text{ at } x = 0 \text{ and } y = 1.2 \text{ at } x = 0.1$$

(b) Use the approximations $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 \approx \frac{y_1 - y_{-1}}{2h}$ and $\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)_0 \approx \frac{y_1 - 2y_0 + y_{-1}}{h^2}$ with a step length of 0.1 to estimate the value of y at $x = 0.2$

(6 marks)

5.

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 3 \\ 4 & q & 1 \\ 1 & 2 & -1 \end{pmatrix}, \quad q \neq 4\tfrac{1}{4}.$$

(a) Find $\mathbf{A}^{-1}$ in terms of q. **(7 marks)**

(b) Hence, or otherwise, solve the simultaneous equations

$$x - y + 3z = 1,$$
$$4x + y + z = 2,$$
$$x + 2y - z = 5,$$

showing your working clearly. **(4 marks)**

Turn over

6. Given that

$$y = \sqrt{1 - x^2} \arccos x \,,$$

(a) show that

$$(1 - x^2)\frac{\mathrm{d}y}{\mathrm{d}x} + xy - x^2 + 1 = 0 \,. \qquad \text{(I)}$$

(5 marks)

(b) By differentiating equation (I) twice, or otherwise, obtain the Maclaurin expansion of $y = \sqrt{1 - x^2} \arccos x$ up to and including the term in x^3.

(8 marks)

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7. The plane Π_1 has vector equation

$$\mathbf{r} = 3\mathbf{i} + \mathbf{j} - 4\mathbf{k} + \lambda(\mathbf{j} + 2\mathbf{k}) + \mu(\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

(a) Find a vector $\mathbf{n}$ which is normal to Π_1. **(3 marks)**

(b) Hence find a vector equation of Π_1 in the form $\mathbf{r}.\mathbf{n} = \mathrm{p}$. **(2 marks)**

(c) Find the perpendicular distance between Π_1 and the point A with position vector $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$, giving your answer in the form $a\sqrt{6}$, where $a \in \mathbb{Q}$.

(4 marks)

The plane Π_2 has equation $\mathbf{r}.(\mathbf{i} + b\mathbf{j}) = -4$. The angle between Π_1 and Π_2 is 30°.

(d) Find the possible values of the constant b. **(6 marks)**

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