Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 1

Question:

Find the set of values of x for which

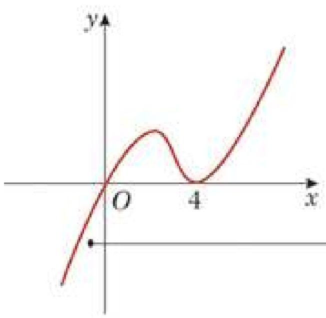
$$16x \leqslant 8x^2 - x^3.$$

Solution:

$$16x \leqslant 8x^2 - x^3$$
$$x^3 - 8x^2 + 16x \leqslant 0$$
$$x(x^2 - 8x + 16) \leqslant 0$$
$$x(x - 4)^2 \leqslant 0$$

You can usually start inequality questions, if there are no modulus signs, by collecting terms together on one side of the equation, and factorising the resulting expression.

Sketching $y = x(x - 4)^2$



The cubic passes through the origin and touches the x-axis at $x = 4$.

You can see from the sketch that $y = x(x - 4)^2$ is negative for $x < 0$.

The solution of $16x \leqslant 8x^2 - x^3$ is

$$x \leqslant 0, x = 4$$

This inequality includes the equality, so you must include the solutions of $x(x - 4)^2 = 0$, which are $x = 0$ and $x = 4$.

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Exercise A, Question 2

Question:

Find the set of values of x for which

$$\frac{2}{x-2} < \frac{1}{x+1}.$$

Solution:

$$\frac{2}{x-2} < \frac{1}{x+1}$$

$$\frac{2}{x-2} - \frac{1}{x+1} < 0$$

$$\frac{2(x+1) - 1(x-2)}{(x-2)(x+1)} = \frac{2x+2-x+2}{(x-2)(x+1)} < 0$$

$$\frac{x+4}{(x-2)(x+1)} < 0$$

You can start by collecting together the terms on one side reducing the expression to a single fraction. In this case no further factorisation is possible.

Considering $f(x) = \dfrac{x+4}{(x-2)(x+1)}$

the critical values are $x = -4, -1, 2$

You find the critical values by solving the numerator equal to zero and the denominator equal to zero. In this case the numerator $= 0$, gives $x = -4$ and the denominator $= 0$ gives $x = -1, 2$.

	$x < -4$	$-4 < x < -1$	$-1 < x < 2$	$2 < x$
Sign of $f(x)$	$-$	$+$	$-$	$+$

For example if $x < -4$, then

$$\frac{x+4}{(x-2)(x+1)} = \frac{\text{negative}}{\text{negative} \times \text{negative}},$$

which is $\dfrac{\text{negative}}{\text{positive}}$, which is negative.

The solution of $\dfrac{2}{x-2} < \dfrac{1}{x+1}$ is

$x < -4$, $-1 < x < 2$.

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Exercise A, Question 3

Question:

Find the set of values of x for which

$$\frac{x^2}{x-2} > 2x.$$

Solution:

$$\frac{x^2}{x-2} > 2x$$

$$\frac{x^2}{x-2} - 2x > 0$$

$$\frac{x^2 - 2x(x-2)}{x-2} > 0$$

$$\frac{4x - x^2}{x-2} > 0$$

$$\frac{x(4-x)}{x-2} > 0$$

You collect the terms together on one side of the inequality, write the expression as a single fraction and factorise the result as far as possible.

Considering $\mathrm{f}(x) = \frac{x(4-x)}{x-2}$,

the critical values are $x = 0$, 2 and 4.

You find the critical values by solving the numerator equal to zero and the denominator equal to zero. In this case the numerator $= 0$, gives $x = 0$, 4 and the denominator gives $x = 2$.

	$x < 0$	$0 < x < 2$	$2 < x < 4$	$4 < x$
Sign of $\mathrm{f}(x)$	+	−	+	−

For example if $4 < x$, then

$$\frac{x(4-x)}{x-2} = \frac{\text{positive} \times \text{negative}}{\text{positive}},$$

which is negative.

The solution of $\frac{x^2}{x-2} > 2x$ is

$x < 0$, $2 < x < 4$.

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Exercise A, Question 4

Question:

Find the set of values of x for which

$$\frac{x^2 - 12}{x} > 1.$$

Solution:

$$\frac{x^2 - 12}{x} > 1$$

Multiply both sides by x^2.

x cannot be zero as $\frac{x^2 - 12}{x}$ would be undefined, so x^2 is positive and you can multiply both sides of an inequality by a positive number or expression without changing the inequality. You could **not** multiply both sides of the inequality by x as x could be positive or negative.

$$\frac{x^2 - 12}{\cancel{x}} \times x^{\cancel{2}} > x^2$$

$$x(x^2 - 12) - x^2 > 0$$

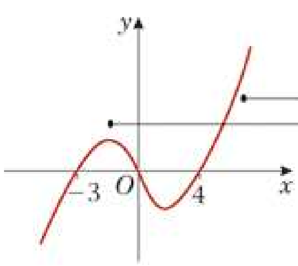
$$x^3 - 12x - x^2 > 0$$

$$x(x^2 - x - 12) > 0$$

$$x(x - 4)(x + 3) > 0$$

Sketching $y = x(x - 4)(x + 3)$

The graph of $y = x(x - 4)(x + 3)$ crosses the y axis at $x = -3$, 0 and 4.



You can see from the sketch that the graph is above the x-axis for $-3 < x < 0$ and $x > 4$. You can then just write down this answer.

The solution of $\frac{x^2 - 12}{x} > 1$ is $-3 < x < 0$, $x > 4$.

If you preferred, you could solve this question using the method illustrated in the solutions to questions 2 and 3 above.

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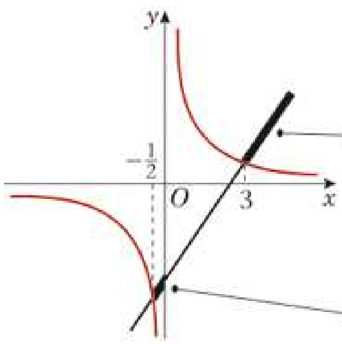
Exercise A, Question 5

Question:

Find the set of values of x for which

$$2x - 5 > \frac{3}{x}.$$

Solution:



Both $y = 2x - 5$ and $y = \frac{3}{x}$ are straightforward graphs to sketch and so this is a suitable question for a graphical method. The question, however, specifies no method and so you can use any method which gives an exact answer.

After sketching the two graphs, $2x - 5 > \frac{3}{x}$ is the set of values of x for which the line is above the curve. These parts of the line have been drawn thickly on the sketch.

$$2x - 5 = \frac{3}{x}$$

$$x(2x - 5) = 3$$

$$2x^2 - 5x - 3 = 0$$

$$(2x + 1)(x - 3) = 0$$

$$x = -\tfrac{1}{2}, 3$$

You need to find the x-coordinates of the points where the line and curve meet to find two end points of the intervals. The other end point ($x = 0$) can be seen by inspecting the sketch.

The solution to $2x - 5 > \frac{3}{x}$ is $-\frac{1}{2} < x < 0, x > 3$.

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Exercise A, Question 6

Question:

Given that k is a constant and that $k > 0$, find, in terms of k, the set of values of x for which $\frac{x+k}{x+4k} > \frac{k}{x}$.

Solution:

$$\frac{x+k}{x+4k} > \frac{k}{x}$$

$$\frac{x+k}{x+4k} - \frac{k}{x} > 0$$

$$\frac{(x+k)x - k(x+4k)}{(x+4k)x} > 0$$

$$\frac{x^2 - 4k^2}{(x+4k)x} > 0$$

$$\frac{(x+2k)(x-2k)}{(x+4k)x} > 0$$

Considering $f(x) = \frac{(x+2k)(x-2k)}{(x+4k)x}$,

the critical values are $x = -4k, -2k, 0$ and $2k$.

For example, when k is positive, in the interval $0 < x < 2k$,
$\frac{(x+2k)(x-2k)}{(x+4k)x} = \frac{\text{positive} \times \text{negative}}{\text{positive} \times \text{positive}}$,
which is negative.

	$x < -4k$	$-4k < x < -2k$	$-2k < x < 0$	$0 < x < 2k$	$2k < x$
Sign of $f(x)$	+	−	+	−	+

The solution of $\frac{x+k}{x+4k} > \frac{k}{x}$ is $x < -4k$, $-2k < x < 0$, $2k < x$.

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Exercise A, Question 7

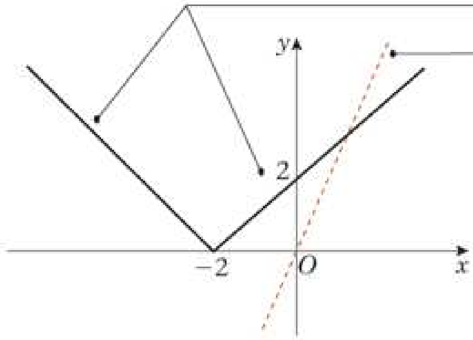
Question:

a Sketch the graph of $y = |x + 2|$.

b Use algebra to solve the inequality $2x > |x + 2|$.

Solution:

a



Inequalities which contain both an expression in x with a modulus sign and an expression in x without a modulus sign, are usually best answered by drawing a sketch. In this case, you have been instructed to draw the sketch first. The continuous line is the graph of $y = |x + 2|$. You should mark the coordinates of the points where the graph cuts the axis.

You should now add the graph of $y = 2x$ to your sketch. This has been done with a dotted line. You find the solution to the inequality by identifying the values of x where the dotted line is above the continuous line.

b The intersection occurs when $x > -2$.

When $x > -2$, $|x + 2| = x + 2$

$$2x = x + 2$$

$$x = 2$$

When $f(x)$ is positive, $|f(x)| = f(x)$.

The solution of $2x > |x + 2|$ is $x > 2$.

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Exercise A, Question 8

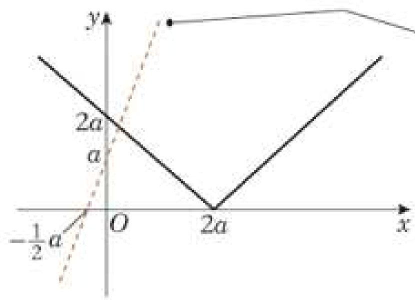
Question:

a Sketch the graph of $y = |x - 2a|$, given that $a > 0$.

b Solve $|x - 2a| > 2x + a$, where $a > 0$.

Solution:

a



The dotted line is added to the sketch in part **a** to help you to solve part **b**. The dotted line is the graph of $y = 2x + a$ and the solution to the inequality in part **b** is found by identifying where the continuous line, which corresponds to $|x - 2a|$, is above the dotted line, which corresponds to $2x + a$.

b The intersection occurs when $x < 2a$.

When $x < 2a$, $|x - 2a| = 2a - x$

If $f(x)$ is negative, then $|f(x)| = -f(x)$.

$$2a - x = 2x + a$$

$$-3x = -a \Rightarrow x = \frac{1}{3}a$$

The solution of $|x - 2a| > 2x + a$ is $x < \frac{1}{3}a$.

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Exercise A, Question 9

Question:

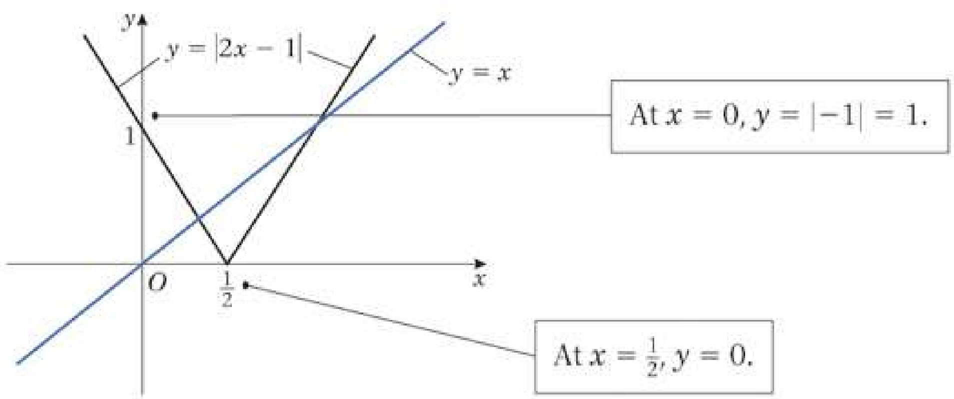
a On the same axes, sketch the graphs of $y = x$ and $y = |2x - 1|$.

b Use algebra to find the coordinates of the points of intersection of the two graphs.

c Hence, or otherwise, find the set of values of x for which $|2x - 1| > x$.

Solution:

a



b There are two points of intersection.
At the right hand point of intersection,

$x > \frac{1}{2} \Rightarrow |2x - 1| = 2x - 1$

$2x - 1 = x \Rightarrow x = 1$

At the left hand point of intersection,

$x < \frac{1}{2} \Rightarrow |2x - 1| = 1 - 2x$

$1 - 2x = x \Rightarrow x = \frac{1}{3}$

If $f(x) > 0$, then $|f(x)| = f(x)$.
If $f(x) < 0$, then $|f(x)| = -f(x)$.

The points of intersection of the two graphs are

$\left(\frac{1}{3}, \frac{1}{3}\right)$ and $(1, 1)$

You need to give both the x-coordinates and the y-coordinates.

c The solution of $|2x - 1| > x$ is $x < \frac{1}{3}, x > 1$.

You identify the regions on the graph where the V shape representing $y = |2x - 1|$ is above the line representing $y = x$.

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Exercise A, Question 10

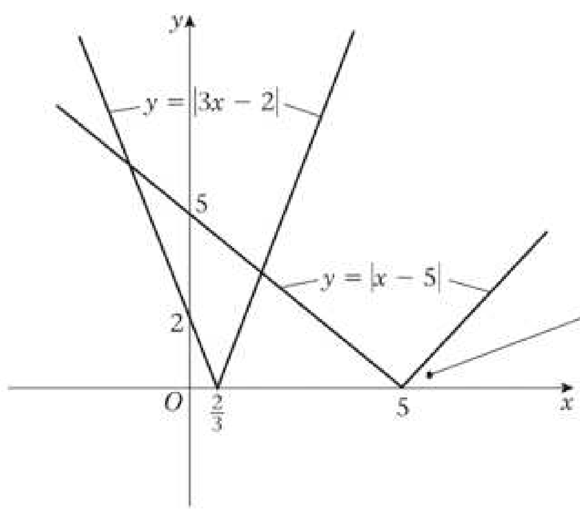
Question:

a On the same axes, sketch the graphs of $y = |x - 5|$ and $y = |3x - 2|$ distinguishing between them clearly.

b Find the set of values of x for which $|x - 5| < |3x - 2|$.

Solution:

a



You should mark the coordinates of the points where the graphs meet the axes.

b From the graph both intersections are in the region where $x < 5$ and $x - 5$ is negative.
Hence, $|x - 5| = 5 - x$

For $x > \frac{2}{3}$, $|3x - 2| = 3x - 2$

$$3x - 2 = 5 - x$$

$$4x = 7 \Rightarrow x = \frac{7}{4}$$

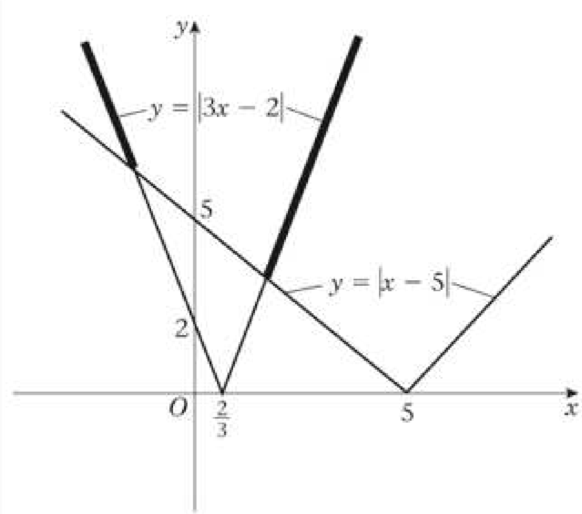
For $x < \frac{2}{3}$, $|3x - 2| = 2 - 3x$

$$2 - 3x = 5 - x$$

$$-2x = 3 \Rightarrow x = -\frac{3}{2}$$

The solution of $|x - 5| < |3x - 2|$ is

$$x < -\frac{3}{2}, x > \frac{7}{4}.$$



You identify the regions in which the lines representing $y = |x - 5|$ are below the lines representing $y = |3x - 2|$.
These are shown with heavy lines above.

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Exercise A, Question 11

Question:

Use algebra to find the set of real values of x for which $|x - 3| > 2|x + 1|$.

Solution:

$$|x - 3| > 2|x + 1|$$
$$(x - 3)^2 > 4(x + 1)^2$$
$$x^2 - 6x + 9 > 4x^2 + 8x + 4$$
$$0 > 3x^2 + 14x - 5$$
$$(x + 5)(3x - 1) < 0$$

As both $|x - 3|$ and $2|x + 1|$ are positive you can square both sides of the inequality without changing the direction of the inequality sign. If a and b are both positive, it is true that $a > b \Rightarrow a^2 > b^2$. You cannot make this step if either or both of a and b are negative.

Considering $f(x) = (x + 5)(3x - 1)$, the critical values are $x = -5$ and $\frac{1}{3}$.

Alternatively you can draw a sketch of $y = (x + 5)(3x - 1)$ and identify the region where the curve is below the y-axis.

	$x < -5$	$-5 < x < \frac{1}{3}$	$\frac{1}{3} < x$
Sign of $f(x)$	$+$	$-$	$+$

The solution of $|x - 3| > 2|x + 1|$ is

$-5 < x < \frac{1}{3}$.

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Exercise A, Question 12

Question:

Find the set of real values of x for which

a $\frac{3x+1}{x-3} < 1$,

b $\left|\frac{3x+1}{x-3}\right| < 1$.

Solution:

a $$\frac{3x+1}{x-3} < 1$$

$$\frac{3x+1}{x-3} - 1 < 0$$

$$\frac{3x+1-1(x-3)}{x-3} < 0$$

$$\frac{2x+4}{x-3} = \frac{2(x+2)}{x-3} < 0$$

Considering $f(x) = \dfrac{2(x+2)}{x-3}$,

the critical values are $x = -2, 3$.

You should compare the solutions to parts **a** and **b**. The questions look similar but the algebraic methods of solution used here are quite different.

	$x < -2$	$-2 < x < 3$	$3 < x$
Sign of $f(x)$	+	−	+

The solution of $\dfrac{3x+1}{x-3} < 1$ is $-2 < x < 3$.

b $$\left|\frac{3x+1}{x-3}\right| < 1$$

$$\left(\frac{3x+1}{x-3}\right)^2 < 1$$

As both $\left|\dfrac{3x+1}{x-3}\right|$ and 1 are positive you can square both sides of the inequality without changing the direction of the inequality sign.

$$(3x+1)^2 < (x-3)^2$$

$$9x^2 + 6x + 1 < x^2 - 6x + 9$$

$$8x^2 + 12x - 8 < 0$$

As 4 is a positive number, you can divide throughout the inequality by 4.

$$2x^2 + 3x - 2 = (x+2)(2x-1) < 0$$

Considering $f(x) = (x+2)(2x-1)$,

the critical values are $x = -2$ and $x = \frac{1}{2}$.

	$x < -2$	$-2 < x < \frac{1}{2}$	$\frac{1}{2} < x$
Sign of $f(x)$	+	−	+

The solution of $\left|\dfrac{3x+1}{x-3}\right| < 1$ is $-2 < x < \frac{1}{2}$.

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Exercise A, Question 13

Question:

Solve, for x, the inequality $|5x + a| \leqslant |2x|$, where $a > 0$.

Solution:

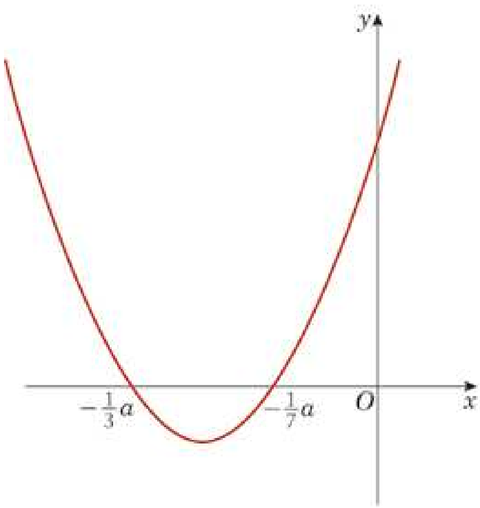
$$|5x + a| \leqslant |2x|$$
$$(5x + a)^2 \leqslant (2x)^2$$

As a is positive, both $|5x + a|$ and $|2x|$ are positive and you can square both sides of the inequality.

$$25x^2 + 10ax + a^2 \leqslant 4x^2$$
$$21x^2 + 10ax + a^2 \leqslant 0$$
$$(3x + a)(7x + a) \leqslant 0$$

Sketching $y = (3x + a)(7x + a)$

The graph is a parabola intersecting the x-axis at $x = -\frac{1}{3}a$ and $x = -\frac{1}{7}a$.



A common error here is not to realise that, for a positive a, $-\frac{1}{3}a$ is a smaller number than $-\frac{1}{7}a$. It is very easy to get the inequality the wrong way round.

The solution of $|5x + a| \leqslant |2x|$ is $-\frac{1}{3}a \leqslant x \leqslant -\frac{1}{7}a$.

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Exercise A, Question 14

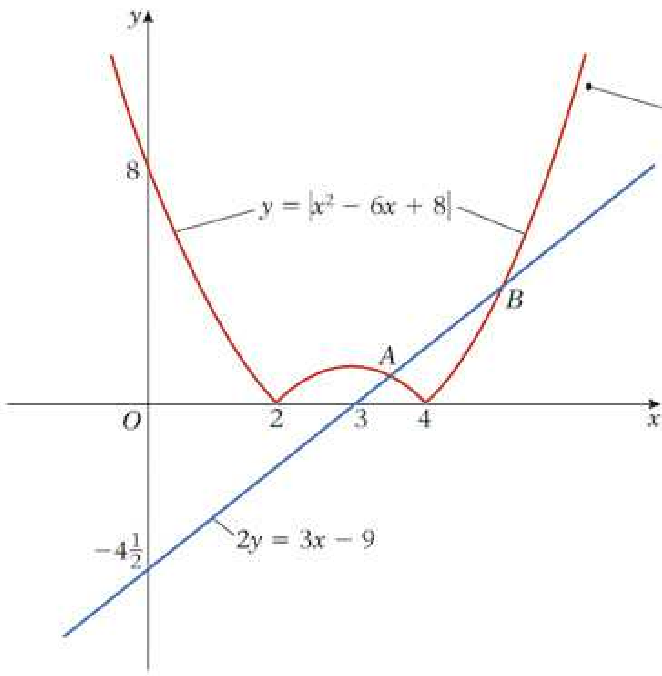
Question:

a Using the same axes, sketch the curve with equation $y = |x^2 - 6x + 8|$ and the line with equation $2y = 3x - 9$. State the coordinates of the points where the curve and the line meet the x-axis.

b Use algebra to find the coordinates of the points where the curve and the line intersect and, hence, solve the inequality $2|x^2 - 6x + 8| > 3x - 9$.

Solution:

a



As $x^2 - 6x + 8 = (x - 2)(x - 4)$ the curve meets the x-axis at $x = 2$ and $x = 4$. The sketching of the graphs of modulus functions is in Chapter 5 of book C3.

The curve meets the x-axis at (2, 0) and (4, 0).

The line meets the x-axis at (3, 0).

b To find the coordinates of A
The x-coordinate of A is in the interval $2 < x < 4$
In this interval $x^2 - 6x + 8$ is negative and, hence,

$$|x^2 - 6x + 8| = -x^2 + 6x - 8$$

If $f(x) < 0$, then $|f(x)| = -f(x)$.

$$-x^2 + 6x - 8 = \frac{3x - 9}{2}$$

$$-2x^2 + 12x - 16 = 3x - 9$$

$$2x^2 - 9x + 7 = 0$$

$$(2x - 7)(x - 1) = 0$$

$$x = \frac{7}{2}, \cancel{1}$$

As the x-coordinate of A is in the interval $2 < x < 4$, the solution $x = 1$ must be rejected.

$$y = \frac{3 \times \frac{7}{2} - 9}{2} = \frac{3}{4}$$

The coordinates of A are $\left(\frac{7}{2}, \frac{3}{4}\right)$.

To find the coordinates of B
The x-coordinate of B is in the interval $x > 4$
In this interval $x^2 - 6x + 8$ is positive and, hence,

$$|x^2 - 6x + 8| = x^2 - 6x - 8$$

If $f(x) > 0$, then $|f(x)| = f(x)$.

$$x^2 + 6x + 8 = \frac{3x - 9}{2}$$

$$2x^2 - 12x + 16 = 3x - 9$$

$$2x^2 - 15x + 25 = 0$$

$$(x - 5)(2x - 5) = 0$$

$$x = 5, \cancel{2\tfrac{1}{2}}$$

As the x-coordinate of B is in the interval $x > 4$, the solution $x = 2\frac{1}{2}$ must be rejected.

$$y = \frac{3 \times 5 - 9}{2} = 3$$

The coordinates of B are (5, 3).

c The solution of $2|x^2 - 6x + 8| > 3x - 9$ is $x < 3\frac{1}{2}, x > 5.$

You solve the inequality by inspecting the graphs. You look for the values of x where the curve is above the line.

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Exercise A, Question 15

Question:

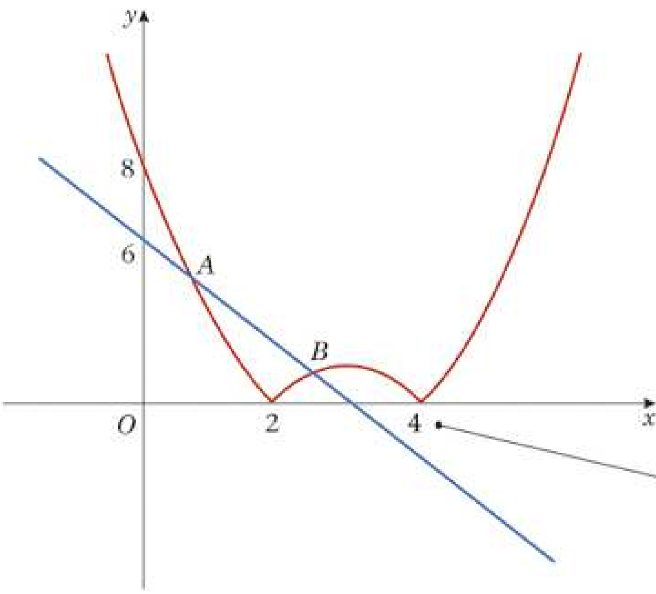
a Sketch, on the same axes, the graph of $y = |(x-2)(x-4)|$, and the line with equation $y = 6 - 2x$.

b Find the exact values of x for which $|(x-2)(x-4)| = 6 - 2x$.

c Hence solve the inequality $|(x-2)(x-4)| < 6 - 2x$.

Solution:

a



You should mark the coordinates of the points where the graphs meet the axes.

b Let the points where the graphs intersect be A and B.

For A, $(x-2)(x-4)$ is positive

$$(x-2)(x-4) = 6-2x$$
$$x^2-6x+8 = 6-2x$$
$$x^2-4x = -2$$
$$x^2-4x+4 = 2$$
$$(x-2)^2 = 2$$
$$x = 2-\sqrt{2}$$

The quadratic equations have been solved by completing the square. You could use the formula for solving a quadratic but the conditions of the question require exact solutions and you should not use decimals.

The quadratic equation has another solution $2+\sqrt{2}$ but the diagram shows that the x-coordinate of A is less than 2, so this solution is rejected.

For B, $(x-2)(x-4)$ is negative

$$-(x-2)(x-4) = 6-2x$$
$$-x^2+6x-8 = 6-2x$$
$$x^2-8x = -14$$
$$x^2-8x+16 = 2$$
$$(x-4)^2 = 2$$
$$x = 4-\sqrt{2}$$

The quadratic equation has another solution $4+\sqrt{2}$ but the diagram shows that the x-coordinate of B is less than 4, so this solution is rejected.

The values of x for which $|(x-2)(x-4)| = 6-2x$ are $2-\sqrt{2}$ and $4-\sqrt{2}$.

c The solution of $|(x-2)(x-4)| < 6-2x$ is $2-\sqrt{2} < x < 4-\sqrt{2}$.

You look for the values of x where the curve is below the line.

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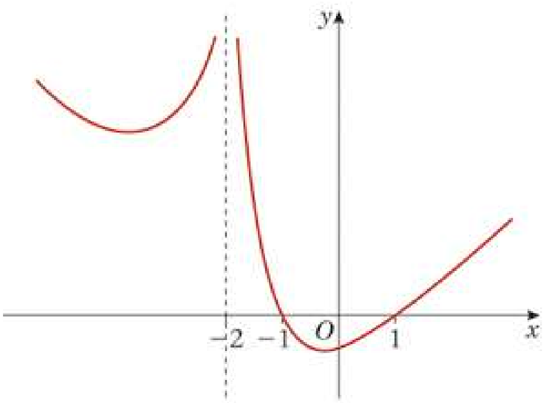
Exercise A, Question 16

Question:



The figure above shows a sketch of the curve with equation

$$y = \frac{x^2 - 1}{|x + 2|}, \quad x \neq -2.$$

The curve crosses the x-axis at $x = 1$ and $x = -1$ and the line $x = -2$ is an asymptote of the curve.

a Use algebra to solve the equation

$$\frac{x^2 - 1}{|x + 2|} = 3(1 - x).$$

b Hence, or otherwise, find the set of values of x for which

$$\frac{x^2 - 1}{|x + 2|} < 3(1 - x).$$

Solution:

a For $x > -2$, $x + 2$ is positive and the equation is

$$\frac{x^2 - 1}{x + 2} = 3(1 - x)$$

$$x^2 - 1 = 3(1 - x)(x + 2) = -3x^2 - 3x + 6$$

$$4x^2 + 3x - 7 = (4x + 7)(x - 1) = 0$$

$$x = -\tfrac{7}{4}, 1$$

As both of these answers are greater than -2 both are valid.

For $x < -2$, $x + 2$ is negative and the equation is

$$\frac{x^2 - 1}{-(x + 2)} = 3(1 - x)$$

$$x^2 - 1 = -3(1 - x)(x + 2) = 3x^2 + 3x - 6$$

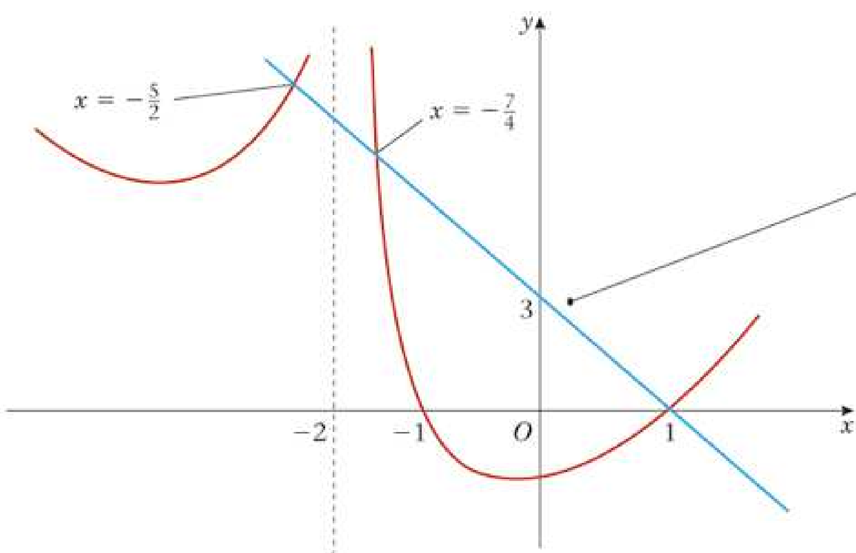
$$2x^2 + 3x - 5 = (2x + 5)(x - 1) = 0$$

$$x = -\tfrac{5}{2}, \cancel{1}$$

As 1 is not less than -2 the answer 1 should be 'rejected' here. However, the earlier working has already shown 1 to be a correct solution.

The solutions are $-\frac{5}{2}$, $-\frac{7}{4}$ and 1.

b



To complete the question, you add the graph of $y = 3(1 - x)$ to the graph which has already been drawn for you. You know the x-coordinates of the points of intersection from part **a**.

The solution of $\dfrac{x^2 - 1}{|x + 2|} < 3(1 - x)$ is

$x < -\frac{5}{2}$, $-\frac{7}{4} < x < 1$.

You look for the values of x on the graph where the curve is below the line.

Solutionbank FP2
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Exercise A, Question 17

Question:

a Express $\frac{1}{(x+1)(x+2)}$ in partial fractions.

b Hence, or otherwise, show that

$$\sum_{r=1}^{n} \frac{2}{(r+1)(r+2)} = \frac{n}{n+2}.$$

Solution:

a $$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

The methods used for partial fractions are in Chapter 1 of book C4. You may use any of the methods which can be used to split complex fractions into partial fractions. Here the substitution method is used.

Multiply throughout by $(x + 1)(x + 2)$

$$1 = A(x+2) + B(x+1)$$

Substitute $x = -1$

$$1 = A(-1+2) + B(-1+1) = A \Rightarrow A = 1$$

Substitute $x = -2$

$$1 = A(-2+2) + B(-2+1) = -B \Rightarrow B = -1$$

Hence

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$

b Substituting r for x and multiplying by 2

$$\frac{2}{(r+1)(r+2)} = \frac{2}{r+1} - \frac{2}{r+2}$$

The summation involves twice the fraction you worked on in part **a** with r substituted for x. So you begin by making the substitution and multiplying every term by 2.

Hence

$$\sum_{r=1}^{n} \frac{2}{(r+1)(r+2)} = \sum_{r=1}^{n} \left(\frac{2}{r+1} - \frac{2}{r+2}\right)$$

$$= \frac{2}{2} - \cancel{\frac{2}{3}}$$

This is the first term of the summation, with $r = 1$, broken up using the partial fractions. Here

$$\frac{2}{(1+1)(1+2)} = \frac{2}{1+1} - \frac{2}{1+2} = \frac{2}{2} - \frac{2}{3}.$$

$$+ \cancel{\frac{2}{3}} - \cancel{\frac{2}{4}}$$

$$+ \cancel{\frac{2}{4}} - \cancel{\frac{2}{5}}$$

This is the third term of the summation, with $r = 3$, broken up using the partial fractions. Here

$$\frac{2}{(3+1)(3+2)} = \frac{2}{3+1} - \frac{2}{3+2} = \frac{2}{4} - \frac{2}{5}.$$

$$\vdots$$

$$+ \cancel{\frac{2}{n}} - \cancel{\frac{2}{n+1}}$$

$$+ \cancel{\frac{2}{n+1}} - \frac{2}{n+2}$$

$$= \frac{2}{2} - \frac{2}{n+2} = 1 - \frac{2}{n+2}$$

$$= \frac{n+2-2}{n+2} = \frac{n}{n+2}, \text{ as required}$$

You write out two or three terms at the beginning and end of the summation and show, by crossing through the fractions, how the fractions cancel each other out. In this case all of the terms are cancelled except the first and last and these are the only two left.

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Exercise A, Question 18

Question:

a Express $\frac{2}{(r+1)(r+3)}$ in partial fractions.

b Hence prove that

$$\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} = \frac{n(5n+13)}{6(n+2)(n+3)}.$$

Solution:

a $\dfrac{2}{(r+1)(r+3)} = \dfrac{A}{r+1} + \dfrac{B}{r+3}$

Multiply throughout by $(r+1)(r+3)$

$2 = A(r+3) + B(r+1)$

Equating the coefficients of r

$0 = A + B$ ①

Equating the constant coefficients

$2 = 3A + B$ ②

The methods used for partial fractions are in Chapter 1 of book C4. You may use any of the methods which can be used to split complex fractions into partial fractions. Here the method used is equating coefficients and solving the resulting simultaneous equations.

Subtracting ① from ②

$2 = 2A \Rightarrow A = 1$

Substituting $A = 1$ into ①

$0 = 1 + B \Rightarrow B = -1$

Hence

$\dfrac{2}{(r+1)(r+3)} = \dfrac{1}{r+1} - \dfrac{1}{r+3}$

You use the partial fractions in part **a** to break up each term in the summation into two parts.

b $\displaystyle\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} = \sum_{r=1}^{n} \left(\frac{1}{r+1} - \frac{1}{r+3}\right)$

$= \frac{1}{2} - \cancel{\frac{1}{4}}$

This is the first term of the summation, with $r = 1$, broken up using the partial fractions. Here $\dfrac{2}{(1+1)(1+3)} = \dfrac{1}{1+1} - \dfrac{1}{1+3} = \dfrac{1}{2} - \dfrac{1}{4}$.

$+ \frac{1}{3} - \cancel{\frac{1}{5}}$

$+ \cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}$

$+ \cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}$

$\vdots$

$+ \cancel{\frac{1}{n}} - \dfrac{1}{n+2}$

You write out some terms at the beginning and end of the summation and show, by crossing through the fractions, how the fractions cancel each other out. In this case two terms are left at the start of the summation and two at the end.

$+ \cancel{\dfrac{1}{n+1}} - \dfrac{1}{n+3}$

This is the nth term of the summation, with $r = n$, broken up using the partial fractions. Here $\dfrac{2}{(n+1)(n+3)} = \dfrac{1}{n+1} - \dfrac{1}{n+3}$.

$= \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{n+2} - \dfrac{1}{n+3}$

$= \dfrac{5}{6} - \dfrac{1}{n+2} - \dfrac{1}{n+3}$

$= \dfrac{5(n+2)(n+3) - 6(n+3) - 6(n+2)}{6(n+2)(n+3)}$

$= \dfrac{5n^2 + 25n + 30 - 6n - 18 - 6n - 12}{6(n+2)(n+3)}$

$= \dfrac{5n^2 + 13n}{6(n+2)(n+3)} = \dfrac{n(5n+13)}{6(n+2)(n+3)}$, as required.

You complete the question by expressing your answer as a single fraction and simplifying it to the answer exactly as it is printed on the question paper.

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Exercise A, Question 19

Question:

a Show that

$$\frac{r+1}{r+2} - \frac{r}{r+1} \equiv \frac{1}{(r+1)(r+2)}, \quad r \in \mathbb{Z}^+.$$

b Hence, or otherwise, find

$\displaystyle\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)}$, giving your answer as a single fraction in terms of n.

Solution:

a LHS $= \dfrac{r+1}{r+2} - \dfrac{r}{r+1}$

$= \dfrac{(r+1)^2 - r(r+2)}{(r+1)(r+2)}$

$= \dfrac{r^2 + 2r + 1 - r^2 - 2r}{(r+1)(r+2)}$

$= \dfrac{1}{(r+1)(r+2)}$

$=$ RHS, as required

To show that an algebraic identity is true, you should start from one side of the identity, here the left hand side (LHS), and use algebra to show that it is equal to the other side of the identity, here the right hand side (RHS).

b $\displaystyle\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \sum_{r=1}^{n}\left(\frac{r+1}{r+2} - \frac{r}{r+1}\right)$

You use the identity that you proved in part **a** to break up each term in the summation into two parts.

$= \cancel{\tfrac{2}{3}} - \tfrac{1}{2}$

This is the LHS of the identity with $r = 1$.

$+ \cancel{\tfrac{3}{4}} - \cancel{\tfrac{2}{3}}$

This is the LHS of the identity with $r = 2$.

$+ \cancel{\tfrac{4}{5}} - \cancel{\tfrac{3}{4}}$

This is the LHS of the identity with $r = 3$.

$\vdots$

$+ \cancel{\dfrac{n}{n+1}} - \cancel{\dfrac{n-1}{n}}$

This is the LHS of the identity with $r = n - 1$.

$\dfrac{r+1}{r+2} - \dfrac{r}{r+1} = \dfrac{n-1+1}{n-1+2} - \dfrac{n-1}{n-1+1}$

$= \dfrac{n}{n+1} - \dfrac{n-1}{n}$

$+ \dfrac{n+1}{n+2} - \cancel{\dfrac{n}{n+1}}$

This is the LHS of the identity with $r = n$.

$= \dfrac{n+1}{n+2} - \dfrac{1}{2}$

The only terms which have not cancelled one another out are the $-\frac{1}{2}$ in the first line of the summation and the $\dfrac{n+1}{n+2}$ in the last line.

$= \dfrac{2(n+1) - (n+2)}{2(n+2)} = \dfrac{2n + 2 - n - 2}{2(n+2)}$

$= \dfrac{n}{2(n+2)}$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 20

Question:

$$f(x) = \frac{2}{(x+1)(x+2)(x+3)}$$

a Express f(x) in partial fractions.

b Hence find $\sum_{r=1}^{n} f(r)$.

Solution:

a Let $\dfrac{2}{(x+1)(x+2)(x+3)} = \dfrac{A}{x+1} + \dfrac{B}{x+2} + \dfrac{C}{x+3}$

Multiplying throughout by $(x+1)(x+2)(x+3)$

$2 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$

Substitute $x = -1$

$2 = A \times 1 \times 2 \Rightarrow A = 1$

Substitute $x = -2$

$2 = B \times -1 \times 1 \Rightarrow B = -2$

Substitute $x = -3$

$2 = C \times -2 \times -1 \Rightarrow C = 1$

Hence

$$f(x) = \frac{1}{x+1} - \frac{2}{x+2} + \frac{1}{x+3}$$

When -1 is substituted for x then both $B(x+1)(x+3)$ and $C(x+1)(x+2)$ become zero.

b Using the result in part **a** with $x = r$

$$\sum_{r=1}^{n} f(r) = \frac{1}{r+1} - \frac{2}{r+2} + \frac{1}{r+3}$$

You use the partial fractions in part **a** to break up each term in the summation into three parts.

$$\begin{aligned}
&= \tfrac{1}{2} - \tfrac{2}{3} + \cancel{\tfrac{1}{4}} \\
&\quad + \tfrac{1}{3} - \cancel{\tfrac{2}{4}} + \cancel{\tfrac{1}{5}} \\
&\quad + \cancel{\tfrac{1}{4}} - \cancel{\tfrac{2}{5}} + \cancel{\tfrac{1}{6}} \\
&\quad \vdots \\
&\quad + \cancel{\frac{1}{n-1}} - \cancel{\frac{2}{n}} + \cancel{\frac{1}{n+1}} \\
&\quad + \cancel{\frac{1}{n}} - \cancel{\frac{2}{n+1}} + \frac{1}{n+2} \\
&\quad + \cancel{\frac{1}{n+1}} - \frac{2}{n+2} + \frac{1}{n+3} \\
&= \frac{1}{2} - \frac{2}{3} + \frac{1}{3} + \frac{1}{n+2} - \frac{2}{n+2} + \frac{1}{n+3} \\
&= \frac{1}{6} - \frac{1}{n+2} + \frac{1}{n+3}
\end{aligned}$$

Three terms at the beginning of the summation and three terms at the end have not been cancelled out.

This question asks for no particular form of the answer. You should collect together like terms but, otherwise, the expression can be left as it is. You do not have to express your answer as a single fraction unless the question asks you to do this.

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Exercise A, Question 21

Question:

a Express as a simplified single fraction $\frac{1}{(r-1)^2} - \frac{1}{r^2}$.

b Hence prove, by the method of differences, that

$$\sum_{r=2}^{n} \frac{2r-1}{r^2(r-1)^2} = 1 - \frac{1}{n^2}.$$

Solution:

a $$\frac{1}{(r-1)^2} - \frac{1}{r^2} = \frac{r^2-(r-1)^2}{r^2(r-1)^2}$$

$$= \frac{r^2-(r^2-2r+1)}{r^2(r-1)^2}$$

$$= \frac{2r-1}{r^2(r-1)^2}$$

Methods for simplifying algebraic fractions can be found in Chapter 1 of book C3.

b $$\sum_{r=2}^{n} \frac{2r-1}{r^2(r-1)^2} = \sum_{r=2}^{n}\left(\frac{1}{(r-1)^2} - \frac{1}{r^2}\right)$$

This summation starts from $r = 2$ and not from the more common $r = 1$. It could not start from $r = 1$ as $\frac{1}{(r-1)^2}$ is not defined for that value.

$$= \frac{1}{1^2} - \cancel{\frac{1}{2^2}}$$

$$+ \cancel{\frac{1}{2^2}} - \cancel{\frac{1}{3^2}}$$

$$+ \cancel{\frac{1}{3^2}} - \cancel{\frac{1}{4^2}}$$

$$\vdots$$

$$+ \cancel{\frac{1}{(n-2)^2}} - \cancel{\frac{1}{(n-1)^2}}$$

$$+ \cancel{\frac{1}{(n-1)^2}} - \frac{1}{n^2}$$

In this summation all of the terms cancel out with one another except for one term at the beginning and one term at the end.

$$= \frac{1}{1^2} - \frac{1}{n^2} = 1 - \frac{1}{n^2}, \text{ as required}$$

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Exercise A, Question 22

Question:

Find the sum of the series

$$\ln\frac{1}{2} + \ln\frac{2}{3} + \ln\frac{3}{4} + \ldots + \ln\frac{n}{n+1}.$$

Solution:

Let $S = \ln\frac{1}{2} + \ln\frac{2}{3} + \ln\frac{3}{4} + \ldots + \ln\frac{n}{n+1}$

The general term of this series is $\ln\frac{r}{r+1}$.

Using a law of logarithms

$$\ln\frac{r}{r+1} = \ln r - \ln(r+1)$$

$$S = \sum_{r=1}^{n} \ln\frac{r}{r+1} = \sum_{r=1}^{n} (\ln r - \ln(r+1))$$

For logarithms to any base $\ln\frac{a}{b} = \ln a - \ln b$. This law gives a difference and so you can use the method of differences to sum the series.

$$\begin{aligned}
&= \ln 1 - \cancel{\ln 2} \\
&\quad + \cancel{\ln 2} - \cancel{\ln 3} \\
&\quad + \cancel{\ln 3} - \cancel{\ln 4} \\
&\quad \vdots \\
&\quad + \cancel{\ln(n-1)} - \cancel{\ln n} \\
&\quad + \cancel{\ln n} - \ln(n+1) \\
&= \ln 1 - \ln(n+1) = -\ln(n+1)
\end{aligned}$$

$\ln 1 = 0$

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Exercise A, Question 23

Question:

a Express $\frac{1}{r(r+2)}$ in partial fractions.

b Hence prove, by the method of differences, that

$$\sum_{r=1}^{n} \frac{4}{r(r+2)} = \frac{n(3n+5)}{(n+1)(n+2)}.$$

c Find the value of $\sum_{r=50}^{100} \frac{4}{r(r+2)}$, to 4 decimal places.

Solution:

a Let $\frac{1}{r(r+2)} = \frac{A}{r} + \frac{B}{r+2}$

You may use any appropriate method to find the partial fractions. If you know an abbreviated method, often called the 'cover up rule', this is accepted at this level.

Multiply throughout by $r(r+2)$

$$1 = A(r+2) + Br$$

Equating constant coefficients

$$1 = 2A \Rightarrow A = \tfrac{1}{2}$$

Equating coefficients of r

$$0 = A + B \Rightarrow B = -A = -\tfrac{1}{2}$$

Hence

$$\frac{1}{r(r+2)} = \frac{1}{2r} - \frac{1}{2(r+2)}$$

b

$$\frac{4}{r(r+2)} = \frac{2}{r} - \frac{2}{r+2}$$

You need to multiply the result of part **a** throughout by 4 to apply the result to part **b**. Remember to multiply every term by 4.

$$\sum_{r=1}^{n} \frac{4}{r(r+2)} = \sum_{r=1}^{n} \left(\frac{2}{r} - \frac{2}{r+2}\right)$$

$$= \frac{2}{1} - \cancel{\frac{2}{3}}$$

$$+ \frac{2}{2} - \cancel{\frac{2}{4}}$$

$$+ \cancel{\frac{2}{3}} - \cancel{\frac{2}{5}}$$

$$\vdots$$

$$+ \cancel{\frac{2}{n-2}} - \cancel{\frac{2}{n}}$$

$$+ \cancel{\frac{2}{n-1}} - \frac{2}{n+1}$$

$$+ \cancel{\frac{2}{n}} - \frac{2}{n+2}$$

Each right hand term is cancelled out by the left hand term two rows below it.

$$= \frac{2}{1} + \frac{2}{2} - \frac{2}{n+1} - \frac{2}{n+2}$$

Four terms are left. Two from the beginning of the summation and two from the end.

$$= 3 - \frac{2}{n+1} - \frac{2}{n+2}$$

$$= \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{(n+1)(n+2)}$$

$$= \frac{3n^2 + 9n + 6 - 2n - 4 - 2n - 2}{(n+1)(n+2)}$$

$$= \frac{3n^2 + 5n}{(n+1)(n+2)} = \frac{n(3n+5)}{(n+1)(n+2)}, \text{ as required.}$$

You have to get your answer exactly into the form printed in the question. Put all three terms over the common denominator $(n+1)(n+2)$ and simplify the numerator.

c $$\sum_{r=50}^{100} \frac{4}{r(r+2)} = \sum_{r=1}^{100} \frac{4}{r(r+2)} - \sum_{r=1}^{49} \frac{4}{r(r+2)}$$

$$= \frac{100 \times 305}{101 \times 102} - \frac{49 \times 152}{50 \times 51}$$

$$= 2.960\,590\ldots - 2.920\,784$$

$$= 0.0398 \text{ (4 d.p.)}$$

$$\sum_{r=50}^{100} \mathrm{f}(r) = \sum_{r=1}^{100} \mathrm{f}(r) - \sum_{r=1}^{49} \mathrm{f}(r)$$

You find the sum from the 50th to the 100th term by subtracting the sum from the first to the 49th term from the sum from the first to the 100th term.
It is a common error to subtract one term too many, in this case the 50th term. The sum you are finding starts with the 50th term. You must not subtract it from the series – you have to leave it in the series.

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Exercise A, Question 24

Question:

a By expressing $\frac{2}{4r^2 - 1}$ in partial fractions, or otherwise, prove that

$$\sum_{r=1}^{n} \frac{2}{4r^2 - 1} = 1 - \frac{1}{2n + 1}.$$

b Hence find the exact value of

$$\sum_{r=11}^{20} \frac{2}{4r^2 - 1}.$$

Solution:

a $4r^2 - 1 = (2r-1)(2r+1)$

Let

$$\frac{2}{4r^2-1} = \frac{2}{(2r-1)(2r+1)} = \frac{A}{2r-1} + \frac{B}{2r+1}$$

Multiply throughout by $(2r-1)(2r+1)$

$$2 = A(2r+1) + B(2r-1)$$

Substitute $r = \frac{1}{2}$

$$2 = 2A \Rightarrow A = 1$$

Substitute $r = -\frac{1}{2}$

$$2 = -2B \Rightarrow B = -1$$

Hence

$$\frac{2}{4r^2-1} = \frac{1}{2r-1} - \frac{1}{2r+1}$$

This question gives you the option to choose your own method (the questions has 'or otherwise') and, as you are given the answer, you could, if you preferred, use the method of mathematical induction which you learnt in module FP1.
If the method of differences is used, you begin by factorising $4r^2 - 1$, using the difference of two squares, and then express

$\frac{2}{(2r-1)(2r+1)}$ in partial fractions.

$$\sum_{r=1}^{n} \frac{2}{4r^2-1} = \sum_{r=1}^{n}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$$

$$= \frac{1}{1} - \cancel{\frac{1}{3}}$$
$$+ \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}$$
$$+ \cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}$$
$$\vdots$$
$$+ \cancel{\frac{1}{2n-3}} - \cancel{\frac{1}{2n-1}}$$
$$+ \cancel{\frac{1}{2n-1}} - \frac{1}{2n+1}$$

$$= 1 - \frac{1}{2n+1}, \text{ as required}$$

With $r = 1$,

$$\frac{1}{2r-1} - \frac{1}{2r+1} = \frac{1}{2\times1-1} - \frac{1}{2\times1+1} = \frac{1}{1} - \frac{1}{3}$$

With $r = n-1$,

$$\frac{1}{2r-1} - \frac{1}{2r+1} = \frac{1}{2\times(n-1)-1} - \frac{1}{2\times(n-1)+1}$$
$$= \frac{1}{2n-2-1} - \frac{1}{2n-2+1} = \frac{1}{2n-3} - \frac{1}{2n-1}$$

The only terms which are not cancelled out in the summation are the $\frac{1}{1}$ at the beginning and the $-\frac{1}{2n+1}$ at the end.

b $$\sum_{r=11}^{20} \frac{2}{4r^2-1} = \sum_{r=1}^{20} \frac{2}{4r^2-1} - \sum_{r=1}^{10} \frac{2}{4r^2-1}$$

$$= \left(1 - \frac{1}{41} - 1 + \frac{1}{21}\right)$$

$$= -\frac{1}{41} + \frac{1}{21} = \frac{-21+41}{41\times21}$$

$$= \frac{20}{861}$$

You find the sum from the 11th to the 20th term by subtracting the sum from the first to the 10th term from the sum from the first to the 20th term.

The conditions of the question require an exact answer, so you must not use decimals.

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Exercise A, Question 25

Question:

Given that for all real values of r,

$$(2r+1)^3 - (2r-1)^3 = Ar^2 + B,$$

where A and B are constants,

a find the value of A and the value of B.

b Hence show that

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

c Calculate $\sum_{r=1}^{40} (3r-1)^2$.

Solution:

a Using the binomial expansion

$(2r+1)^3 = 8r^3 + 12r^2 + 6r + 1$ ①

$(2r-1)^3 = 8r^3 - 12r^2 + 6r - 1$ ②

Subtracting ② from ①

$(2r+1)^3 - (2r-1)^3 - 24r^2 + 2$ ③

$A = 24,\ B = 2$

Subtracting the two expansions gives an expression in r^2. This enables you to sum r^2 using the method of differences.

b Using identity ③ in part **a**

$$\sum_{r=1}^{n}(24r^2+2) = \sum_{r=1}^{n}\left((2r+1)^3 - (2r-1)^3\right)$$

$$24\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} 2 = \sum_{r=1}^{n}\left((2r+1)^3 - (2r-1)^3\right)$$

$$\sum_{r=1}^{n} 2 = \underbrace{2+2+2+\ldots+2}_{n \text{ times}} = 2n$$

It is a common error to write $\sum_{r=1}^{n} 2 = 2.$

$$\begin{aligned}24\sum_{r=1}^{n} r^2 + 2n = {} & \cancel{3^3} - 1^3 \\ & + \cancel{5^3} - \cancel{3^3} \\ & + \cancel{7^3} - \cancel{5^3} \\ & \vdots \\ & + \cancel{(2n-1)^3} - \cancel{(2n-3)^3} \\ & + (2n+1)^3 - \cancel{(2n-1)^3}\end{aligned}$$

This expression is $(2r+1)^3 - (2r-1)^3$ with $n-1$ substituted for r.

$(2(n-1)+1)^3 - (2(n-1)-1)^3$

$= (2n-1)^3 - (2n-3)^3$

$$24\sum_{r=1}^{n} r^2 + 2n = (2n+1)^3 - 1$$

Summing gives you an equation in $\sum r^2$, which you solve. You then factorise the result to give the answer in the form required by the question.

$$24\sum_{r=1}^{n} r^2 = 8n^3 + 12n^2 + 6n + 1 - 1 - 2n$$

$$= 8n^3 + 12n^2 + 4n = 4n(2n^2 + 3n + 1)$$

$$= 4n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^2 = \frac{4n(n+1)(2n+1)}{24} = \tfrac{1}{6}n(n+1)(2n+1), \text{ as required.}$$

c $(3r-1)^2 = 9r^2 - 6r + 1$

Hence

$$\sum_{r=1}^{40} (3r-1)^2 = 9\sum_{r=1}^{40} r^2 - 6\sum_{r=1}^{40} r + \sum_{r=1}^{40} 1$$

In the formula proved in part **b**, you replace the n by 40.

Using the result in part **b**.

$$9\sum_{r=1}^{40} r^2 = 9 \times \tfrac{1}{6} \times 40 \times 41 \times 81 = 199\,260$$

Using the standard result $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$,

This is a standard formula from the FP1 specification. The FP2 specification requires you to know the material in the FP1 specification.

$$6\sum_{r=1}^{40} r = 6 \times \frac{40 \times 41}{2} = 4920$$

$$\sum_{r=1}^{40} 1 = 40$$

$\sum_{r=1}^{40} 1 = 40$ is 40 ones added together which is, of course, 40.

Combining these results

$$\sum_{r=1}^{40} (3r-1)^2 = 199\,260 - 4920 + 40 = 194\,380$$

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Exercise A, Question 26

Question:

$$f(r) = \frac{1}{r(r+1)}, \; r \in \mathbb{Z}^+$$

a Show that

$$f(r) - f(r+1) = \frac{k}{r(r+1)(r+2)},$$

stating the value of k.

b Hence show, by the method of differences, that

$$\sum_{r=1}^{2n} \frac{1}{r(r+1)(r+2)} = \frac{n(2n+3)}{4(n+1)(2n+1)}.$$

Solution:

a $f(r) - f(r+1) = \dfrac{1}{r(r+1)} - \dfrac{1}{(r+1)(r+2)}$

$= \dfrac{r+2-r}{r(r+1)(r+2)} = \dfrac{2}{r(r+1)(r+2)}$

which is the required result with $k = 2$.

$f(r+1)$ is $f(r) = \dfrac{1}{r(r+1)}$ with r replaced by $r+1$. This gives $\dfrac{1}{(r+1)(r+1+1)} = \dfrac{1}{(r+1)(r+2)}$

b Using the result in part **a**

$$\sum_{r=1}^{2n} \frac{2}{r(r+1)(r+2)} = \sum_{r=1}^{2n} \left(\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}\right)$$

$$= \frac{1}{1\times 2} - \cancel{\frac{1}{2\times 3}}$$
$$+ \cancel{\frac{1}{2\times 3}} - \cancel{\frac{1}{3\times 4}}$$
$$+ \cancel{\frac{1}{3\times 4}} - \cancel{\frac{1}{4\times 5}}$$
$$\vdots$$
$$+ \cancel{\frac{1}{(2n-1)2n}} - \cancel{\frac{1}{2n(2n+1)}}$$
$$+ \cancel{\frac{1}{2n(2n+1)}} - \frac{1}{(2n+1)(2n+2)}$$
$$= \frac{1}{2} - \frac{1}{(2n+1)(2n+2)}$$

As $k = 2$, this is twice the summation you were asked to work out. You must remember to divide by 2 later.

Most summations in this topic have an upper limit of n but this question has an upper limit of $2n$. So the last two pairs of terms are the differences with $r = 2n - 1$ and $r = 2n$.

Hence

$$\sum_{r=1}^{2n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(2n+1)(2n+2)}$$

Dividing throughout by 2.

$$= \frac{1}{4} - \frac{1}{4(2n+1)(n+1)}$$
$$= \frac{(n+1)(2n+1) - 1}{4(n+1)(2n+1)}$$
$$= \frac{2n^2+3n+1-1}{4(n+1)(2n+1)}$$
$$= \frac{2n^2+3n}{4(n+1)(2n+1)}$$
$$= \frac{n(2n+3)}{4(n+1)(2n+1)}, \text{ as required}$$

You complete the question by putting the fractions over a common denominator and simplifying to the printed answer.

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Edexcel AS and A Level Modular Mathematics

Exercise A, Question 27

Question:

a Show that

$$\frac{r^3 - r + 1}{r(r+1)} \equiv r - 1 + \frac{1}{r} - \frac{1}{r+1},$$

for $r \neq 0, -1$.

b Find $\sum_{r=1}^{n} \frac{r^3 - r + 1}{r(r+1)}$, expressing your answer as a single fraction in its simplest form.

Solution:

a RHS $= r - 1 + \frac{1}{r} - \frac{1}{r+1}$

$= \frac{(r-1)r(r+1) + (r+1) - r}{r(r+1)}$

$= \frac{r(r^2-1)+1}{r(r+1)}$

$= \frac{r^3 - r + 1}{r(r+1)}$ = LHS, as required

To show that an algebraic identity is true, you should start from one side of the identity, here the right hand side (RHS), and use algebra to show that it is equal to the other side of the identity, here the left hand side (LHS).

b Using the result in part **a**

This summation is broken up into 3 separate summations. Only the third of these uses the method of differences.

$$\sum_{r=1}^{n} \frac{r^3 - r + 1}{r(r+1)} = \sum_{r=1}^{n} r - \sum_{r=1}^{n} 1 + \sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+1}\right)$$

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

This is a standard formula from the FP1 specification. The FP2 specification requires you to know the material in the FP1 specification.

$$\sum_{r=1}^{n} 1 = n$$

$$\begin{aligned}\sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+1}\right) &= \frac{1}{1} - \cancel{\frac{1}{2}}\\ &+ \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\\ &+ \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\\ &\vdots\\ &+ \cancel{\frac{1}{n-1}} + \cancel{\frac{1}{n}}\\ &+ \cancel{\frac{1}{n}} - \frac{1}{n+1}\\ &= 1 + \frac{1}{n+1}\end{aligned}$$

In the summation, using the method of differences, all of the terms cancel out with one another except for one term at the beginning and one term at the end.

Combining the three summations

$$\begin{aligned}\sum_{r=1}^{n} \frac{r^3 - r + 1}{r(r+1)} &= \frac{n(n+1)}{2} - n + 1 - \frac{1}{n+1}\\ &= \frac{n(n+1)^2 - 2n(n+1) + 2(n+1) - 2}{2(n+1)}\\ &= \frac{n^3 + 2n^2 + n - 2n^2 - 2n + 2n + 2 - 2}{2(n+1)}\\ &= \frac{n^3 + n}{2(n+1)} = \frac{n(n^2+1)}{2(n+1)}\end{aligned}$$

To complete the question, you put the results of the three summations over a common denominator and simplify the resulting expression as far as possible.

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Exercise A, Question 28

Question:

a Express $\frac{2r+3}{r(r+1)}$ in partial fractions.

b Hence find $\sum_{r=1}^{n} \frac{2r+3}{r(r+1)} \times \frac{1}{3^r}$.

Solution:

a Let $\frac{2r+3}{r(r+1)} = \frac{A}{r} + \frac{B}{r+1}$

Multiply throughout by $r(r+1)$

$$2r + 3 = A(r+1) + Br$$

Substitute $r = 0$

$$3 = A$$

Substitute $r = -1$

$$1 = -B \Rightarrow B = -1$$

Hence

$$\frac{2r+3}{r(r+1)} = \frac{3}{r} - \frac{1}{r+1}$$

The partial fractions in part **a** form only part of the expression which you have to sum in part **b**; the $\frac{1}{3^r}$ is omitted. Before part **b** can be done, further work has to be carried out on the general term of the summation.

b Using the result in part **a**, the general term of the summation can be written

$$\frac{2r+3}{r(r+1)} \times \frac{1}{3^r} = \frac{3}{r} \times \frac{1}{3^r} - \frac{1}{r+1} \times \frac{1}{3^r} = \frac{1}{3^{r-1}r} - \frac{1}{3^r(r+1)}$$

$\frac{3}{3^r} = \frac{1}{3^{r-1}}$ is an important step here.

$$\sum_{r=1}^{n} \frac{2r+3}{r(r+1)} \times \frac{1}{3^r} = \frac{1}{3^0 \times 1} - \cancel{\frac{1}{3^1 \times 2}}$$

$$+ \cancel{\frac{1}{3^1 \times 2}} - \cancel{\frac{1}{3^2 \times 3}}$$

$$+ \cancel{\frac{1}{3^2 \times 3}} - \cancel{\frac{1}{3^4 \times 4}}$$

$$\vdots$$

$$+ \cancel{\frac{1}{3^{n-2}(n-1)}} - \cancel{\frac{1}{3^{n-1} \times n}}$$

$$+ \cancel{\frac{1}{3^{n-1} \times n}} - \frac{1}{3^n(n+1)}$$

This is $\frac{1}{3^{r-1}r} - \frac{1}{3^r(r+1)}$ with $r = 1$.

This is $\frac{1}{3^{r-1}r} - \frac{1}{3^r(r+1)}$ with $r = n - 1$.

$$\sum_{r=1}^{n} \frac{2r+3}{r(r+1)} \times \frac{1}{3^r} = 1 - \frac{1}{3^n(n+1)}$$

After summing, only the first and last terms are left. The first term $\frac{1}{3^0 \times 1} = 1$.

Solutionbank FP2
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Exercise A, Question 29

Question:

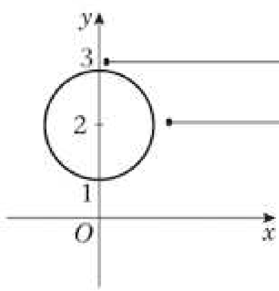
a Sketch, in an Argand diagram, the curve with equation $|z - 2i| = 1$.

Given that the point representing the complex number z lies on this curve,

b find the maximum value of $|z|$.

Solution:

a



You need to know that the locus of z in the Argand plane, when $|z - a| = b$, is a circle centre a, radius b. So this is a circle with centre (0, 2) on the complex axis and with radius 1.

b The maximum value of $|z|$ is 3.

$|z|$ is the distance of the point representing z from the origin. You can see from the diagram that the point furthest from the origin on the circle is the point representing 3i.

Solutionbank FP2
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Exercise A, Question 30

Question:

Solve the equation $z^5 = i$, giving your answers in the form $\cos\theta + i\sin\theta$.

Solution:

a $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\frac{\pi}{2}}$

The modulus of the complex number i is 1 and its argument is $\frac{\pi}{2}$. So $i = 1e^{i\frac{\pi}{2}}$.

$z^5 = e^{i\frac{\pi}{2}}, e^{i\frac{5\pi}{2}}, e^{i\frac{9\pi}{2}}, e^{i\frac{13\pi}{2}}, e^{i\frac{17\pi}{2}}$

Additional solutions are found by increasing the arguments in steps of 2π. As the equation is of degree 5, there are exactly 5 distinct answers.

$z = e^{i\frac{\pi}{10}}, e^{i\frac{5\pi}{10}}, e^{i\frac{9\pi}{10}}, e^{i\frac{13\pi}{10}}, e^{i\frac{17\pi}{10}}$

For example, if $z^5 = e^{i\frac{9\pi}{2}}$ then $z = \left(e^{i\frac{9\pi}{2}}\right)^{\frac{1}{5}} = e^{i\frac{9\pi}{10}}$.

Hence

$z = \cos\theta + i\sin\theta$, where

$\theta = \frac{\pi}{10}, \frac{5\pi}{10}\left(= \frac{\pi}{2}\right), \frac{9\pi}{10}, \frac{13\pi}{10}, \frac{17\pi}{10}$

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Exercise A, Question 31

Question:

Show that

$$\frac{\cos 2x + i \sin 2x}{\cos 9x - i \sin 9x}$$

can be expressed in the form $\cos nx + i \sin nx$, where n is an integer to be found.

Solution:

Using Euler's solution $e^{i\theta} = \cos\theta + i\sin\theta$,

$$\cos 2x + i\sin 2x = e^{i2x}$$

$$\cos 9x - i\sin 9x = \cos(-9x) + i\sin(-9x) = e^{i(-9x)}$$

For any angle θ, $\cos\theta = \cos(-\theta)$ and $-\sin\theta = \sin(-\theta)$

You will find these relations useful when finding the arguments of complex numbers.

Hence

$$\frac{\cos 2x + i\sin 2x}{\cos 9x - i\sin 9x} = \frac{e^{i2x}}{e^{i(-9x)}} = e^{i(2x+9x)} = e^{i11x}$$

$$= \cos 11x + i\sin 11x$$

Manipulating the arguments in $e^{i\theta}$ you use the ordinary laws of indices.

This is the required form with $n = 11$.

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Exercise A, Question 32

Question:

The transformation T from the z-plane to the w-plane is given by

$$w = \frac{z+1}{z-1}, \quad z \neq 1.$$

Find the image in the w-plane of the circle $|z| = 1$, $z \neq 1$ under the transformation.

Solution:

$$w = \frac{z+1}{z-1}$$

$$w(z-1) = wz - w = z + 1$$

$$wz - z = w + 1 \Rightarrow z(w-1) = w+1$$

$$z = \frac{w+1}{w-1}$$

As $|z| = 1$, $\left|\frac{w+1}{w-1}\right| = 1$

and

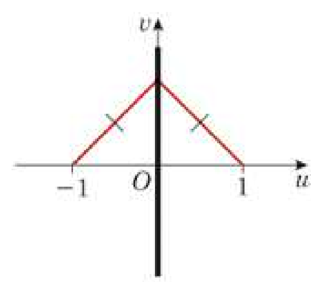
$$\frac{|w+1|}{|w-1|} = 1 \Rightarrow |w+1| = |w-1|$$

The question gives information about $|z|$ and you are trying to show something about w. It is a good idea to change the subject of the formula to z. You can then put the modulus of the right hand side of the new formula, which contains w, equal to 1.

For any complex numbers a and b, $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$.

The locus of w is the line equidistant from the points representing the real numbers -1 and 1. This line is the imaginary axis.
Hence, the image of $|z| = 1$ under T is the imaginary axis.

You need to know that the locus of z in the Argand plane, when $|z - a| = |z - b|$, is the line equidistant from the points representing the complex numbers a and b. That is the perpendicular bisector of the line joining the points. In this case;



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Exercise A, Question 33

Question:

a Express $z = 1 + i\sqrt{3}$ in the form $r(\cos\theta + i\sin\theta)$, $r > 0$, $-\pi < \theta \leqslant \pi$.

b Hence, or otherwise, show that the two solutions of

$$w^2 = (1 + i\sqrt{3})^3$$

are $(2\sqrt{2})i$ and $(-2\sqrt{2})i$.

Solution:

a $z = 1 + i\sqrt{3} = r(\cos\theta + i\sin\theta) = r\cos\theta + i\,r\sin\theta$

Equating real parts

$$1 = r\cos\theta \quad ①$$

Equating complex parts

$$\sqrt{3} = r\sin\theta \quad ②$$

Squaring both ① and ② and adding the results

$$r^2\cos^2\theta + r^2\sin^2\theta = r^2 = 1^2 + (\sqrt{3})^2 = 4$$

$$r = 2$$

Substituting into ①

$$1 = 2\cos\theta \Rightarrow \cos\theta = \tfrac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

Unless the question clearly specifies otherwise, in this topic you should give all arguments in radians and exact answers should be given wherever possible.

Hence $1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

b From part **a**

$$1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2e^{i\frac{\pi}{3}}$$

$$(1 + i\sqrt{3})^3 = \left(2e^{i\frac{\pi}{3}}\right)^3 = 8e^{i\pi}$$

You use part **a** to put the right hand side of the equation into a form from which the square roots can be found.

Hence the equation can be written

$$w^2 = 8e^{i\pi}, 8e^{i3\pi}$$

$$w = \sqrt{8}e^{i\frac{\pi}{2}}, \sqrt{8}e^{i\frac{3\pi}{2}}$$

Additional solutions are found by increasing the arguments in steps of 2π. As the equation is a quadratic, there are just 2 distinct answers.

The two solutions are

$$w = \sqrt{8}e^{i\frac{\pi}{2}} = \sqrt{8}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2\sqrt{2}i$$

and

$$w = \sqrt{8}e^{i\frac{3\pi}{2}} = \sqrt{8}\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = (-2\sqrt{2})i,$$

as required.

Using $\cos\frac{\pi}{2} = \cos\frac{3\pi}{2} = 0$, $\sin\frac{\pi}{2} = 1$ and $\sin\frac{3\pi}{2} = -1$.

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Exercise A, Question 34

Question:

The transformation from the z-plane to the w-plane is given by

$$w = \frac{2z-1}{z-2}.$$

Show that the circle $|z| = 1$ is mapped onto the circle $|w| = 1$.

Solution:

$$w = \frac{2z-1}{z-2} \Rightarrow wz - 2w = 2z - 1$$

$$wz - 2z = 2w - 1 \Rightarrow z(w-2) = 2w - 1$$

$$z = \frac{2w-1}{w-2}$$

$$|z| = 1 \Rightarrow \left|\frac{2w+1}{w-2}\right| = 1$$

You know that $|z| = 1$ and you are trying to find out about w. So it is a good idea to change the subject of the formula to z. You can then put the modulus of the right hand side of the new formula, which contains w, equal to 1.

$$|2w-1| = |w-2|$$

Let $w = u + \mathrm{i}v$

$$|2(u+\mathrm{i}v) - 1| = |u + \mathrm{i}v - 2|$$

$$|(2u-1) + \mathrm{i}2v| = |(u-2) + \mathrm{i}v|$$

$$|(2u-1) + \mathrm{i}2v|^2 = |(u-2) + \mathrm{i}v|^2$$

It is not easy to interpret this locus geometrically and so it is sensible to transform the problem into algebra, using the rule that if $z = x + \mathrm{i}y$, then $|z|^2 = x^2 + y^2$.

$$(2u-1)^2 + 4v^2 = (u-2)^2 + v^2$$

$$4u^2 - 4u + 1 + 4v^2 = u^2 - 4u + 4 + v^2$$

$$3u^2 + 3v^2 = 3 \Rightarrow u^2 + v^2 = 1$$

This is a circle centre O, radius 1 and has the equation $|w| = 1$ in the Argand plane.

Hence, the circle $|z| = 1$ is mapped onto the circle $|w| = 1$, as required.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 35

Question:

a Solve the equation

$$z^5 = 4 + 4\mathrm{i}$$

giving your answers in the form $z = r\,\mathrm{e}^{\mathrm{i}k\pi}$, where r is the modulus of z and k is a rational number such that $0 \leqslant k \leqslant 2$.

b Show on an Argand diagram the points representing your solutions.

Solution:

a Let $4 + 4\mathrm{i} = r(\cos\theta + \mathrm{i}\sin\theta) = r\cos\theta + \mathrm{i}r\sin\theta$

Equating real parts

$$4 = r\cos\theta \quad ①$$

Equating imaginary parts

$$4 = r\sin\theta \quad ②$$

Dividing ② by ①

$$\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$$

Substituting $\theta = \frac{\pi}{4}$ into ①

$$4 = r\cos\frac{\pi}{4} \Rightarrow 4 = r \times \frac{1}{\sqrt{2}} \Rightarrow r = 4\sqrt{2}$$

Hence

$$4 + 4\mathrm{i} = 4\sqrt{2}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)$$
$$= 2^{\frac{5}{2}}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$$

$$z^5 = 2^{\frac{5}{2}}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}},\ 2^{\frac{5}{2}}\mathrm{e}^{\mathrm{i}\frac{9\pi}{4}},\ 2^{\frac{5}{2}}\mathrm{e}^{\mathrm{i}\frac{17\pi}{4}},\ 2^{\frac{5}{2}}\mathrm{e}^{\mathrm{i}\frac{25\pi}{4}},\ 2^{\frac{5}{2}}\mathrm{e}^{\mathrm{i}\frac{33\pi}{4}}$$

$$z = 2^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\frac{\pi}{20}},\ 2^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\frac{9\pi}{20}},\ 2^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\frac{17\pi}{20}},\ 2^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\frac{25\pi}{20}},\ 2^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\frac{33\pi}{20}}$$

This is the required form with $r = \sqrt{2}$ and

$k = \frac{1}{20}, \frac{9}{20}, \frac{17}{20}, \frac{25}{20}\left(= \frac{5}{4}\right), \frac{33}{20}.$

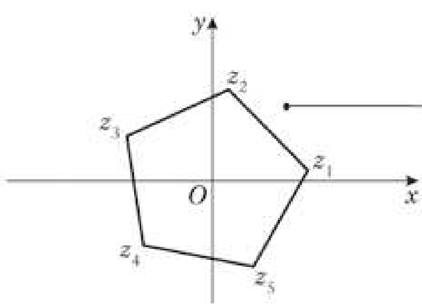
Finding the roots of a complex number is usually easier if you obtain the number in the form $r\mathrm{e}^{\mathrm{i}\theta}$. As you will use Euler's relation, the first step towards this is to get the complex number into the form $r(\cos\theta + \mathrm{i}\sin\theta)$.

To take the fifth root, write $4\sqrt{2} = 2^{\frac{5}{2}}$.

For example, if $z^5 = 2^{\frac{5}{2}}\mathrm{e}^{\mathrm{i}\frac{9\pi}{4}}$ then

$z = \left(2^{\frac{5}{2}}\mathrm{e}^{\mathrm{i}\frac{9\pi}{4}}\right)^{\frac{1}{5}} = 2^{\frac{5}{2}\times\frac{1}{5}}\mathrm{e}^{\mathrm{i}\frac{9\pi}{4}\times\frac{1}{5}} = 2^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\frac{9\pi}{20}}.$

b



The points representing the 5 roots are the vertices of a regular pentagon.

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Exercise A, Question 36

Question:

The point P represents the complex number z in an Argand diagram. Given that

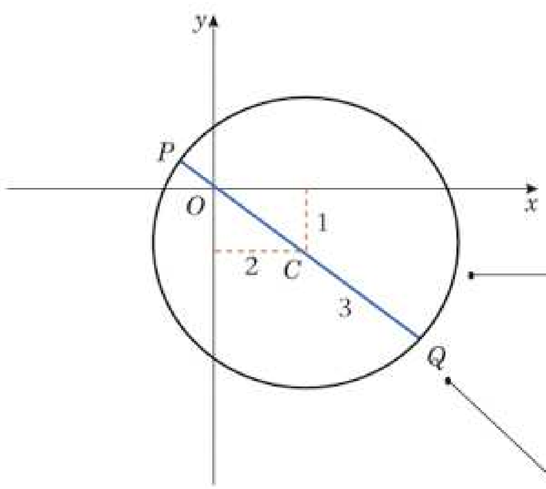
$$|z - 2 + i| = 3,$$

a sketch the locus of P in an Argand diagram,

b find the exact values of the maximum and minimum of $|z|$.

Solution:

a



The locus of $|z - a| = k$, where a is a complex number and k is a real number, is a circle with radius k and centre the point representing a. Rewriting the relation in the question as $|z - (2 - i)| = 3$, this locus is a circle of radius 3 with centre $(2, -1)$.

b $OC^2 = 1^2 + 2^2 = 5 \Rightarrow OC = \sqrt{5}$

$OQ = OC + CQ = \sqrt{5} + 3$

Hence the maximum value of $|z|$ is $3 + \sqrt{5}$.

$OP = CP - CO = 3 - \sqrt{5}$

Hence the minimum value of $|z|$ is $3 - \sqrt{5}$.

$|z|$ is the distance of the point representing z from the origin. The point on the circle furthest from O is marked by Q on the diagram and the point closest to O by P. The distances of Q and P from O represent the maximum and minimum values of $|z|$ respectively.

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Exercise A, Question 37

Question:

The transformation T from the z-plane to the w-plane is given by

$$w = \frac{1}{z-2},\ z \neq 2,$$

where $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$.
Show that under T the straight line with equation

$$2x + y = 5$$

is transformed to a circle in the w-plane with centre $\left(1, -\frac{1}{2}\right)$ and radius $\frac{\sqrt{5}}{2}$.

Solution:

$$w = \frac{1}{z-2}$$

$$z - 2 = \frac{1}{w}$$

$$x + \mathrm{i}y - 2 = \frac{1}{u + \mathrm{i}v}$$

$$x - 2 + \mathrm{i}y = \frac{1}{u + \mathrm{i}v} \times \frac{u - \mathrm{i}v}{u - \mathrm{i}v} = \frac{u}{u^2 + v^2} - \frac{\mathrm{i}v}{u^2 + v^2}$$

Multiplying the numerator and denominator by the conjugate complex of the denominator.

Equating real parts

$$x - 2 = \frac{u}{u^2 + v^2} \Rightarrow x = 2 + \frac{u}{u^2 + v^2}$$

Equating imaginary parts

$$y = -\frac{v}{u^2 + v^2}$$

Hence $\quad 2x + y = 5$

maps to $\quad 2\left(2 + \frac{u}{u^2 + v^2}\right) - \frac{v}{u^2 + v^2} = 5$

This is the equation of the curve in the w-plane. The rest of the solution is showing that this is the equation of a circle, using the method of completing the square.

$$\frac{2u}{u^2 + v^2} - \frac{v}{u^2 + v^2} = 1$$

$$2u - v = u^2 + v^2$$

$$u^2 - 2u + v^2 + v = 0$$

$$u^2 - 2u + 1 + v^2 + v + \tfrac{1}{4} = \tfrac{5}{4}$$

$$(u-1)^2 + \left(v + \tfrac{1}{2}\right)^2 = \left(\tfrac{1}{2}\sqrt{5}\right)^2$$

This is a circle in the w-plane with centre $\left(1, -\frac{1}{2}\right)$

and radius $\frac{1}{2}\sqrt{5}$, as required.

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Exercise A, Question 38

Question:

a Use de Moivre's theorem to show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

b Hence find 3 distinct solutions of the equation $16x^5 - 20x^3 + 5x + 1 = 0$, giving your answers to 3 decimal places where appropriate.

Solution:

a By de Moivre's theorem

$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5 = (c + is)^5$, say

$= c^5 + 5c^4 is + 10c^3 i^2 s^2 + 10c^2 i^3 s^3 + 5c i^4 s^4 + i^5 s^5$

$= c^5 + i5c^4 s - 10c^3 s^2 - i10c^2 s^3 + 5cs^4 - is^5$

It is sensible to abbreviate $\cos\theta$ and $\sin\theta$ as c and s respectively when you have as many powers of $\cos\theta$ and $\sin\theta$ to write out as you have in this question.

Use the binomial expansion.

Use $i^2 = -1$, $i^3 = -i$, $i^4 = 1$ and $i^5 = i \times i^4 = i \times 1 = i$.

Equating real parts

$\cos 5\theta = c^5 - 10c^3 s^2 + 5cs^4$

Using $\cos^2\theta + \sin^2\theta = 1$

$\cos 5\theta = c^5 - 10c^3(1 - c^2) + 5c(1 - c^2)^2$

$= c^5 - 10c^3 + 10c^5 + 5c - 10c^3 + 5c^5$

$= 16c^5 - 20c^3 + 5c$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$, as required

b Substitute $x = \cos\theta$ into $16x^5 - 20x^3 + 5x + 1 = 0$

$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta + 1 = 0$

$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = -1$

Using the result of part **a**

$\cos 5\theta = -1$

$5\theta = \pi, 3\pi, 5\pi$

$\theta = \frac{\pi}{5}, \frac{3\pi}{5}, \frac{5\pi}{5}$

$x = \cos\theta = \cos\frac{\pi}{5}, \cos\frac{3\pi}{5}, \cos\pi$

$= 0.809, -0.309, -1$

Additional solutions are found by increasing the angles in steps of 2π. You are asked for 3 answers, so you need 3 angles at this stage.

The two approximate answers are given to 3 decimal places, as the question specified; the remaining answer -1 is exact.

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Edexcel AS and A Level Modular Mathematics

Exercise A, Question 39

Question:

a Use de Moivre's theorem to show that $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$.

b Hence, or otherwise, show that

$$\int_0^{\frac{\pi}{2}} \sin^5\theta \, d\theta = \frac{8}{15}.$$

Solution:

a $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

Let $z = e^{i\theta}$

Putting $z = e^{i\theta}$ shortens the working.

then $\sin\theta = \dfrac{z - z^{-1}}{2i}$

Use Pascal's triangle to remember the coefficients in $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

$$\sin^5\theta = \left(\frac{z - z^{-1}}{2i}\right)^5$$

$$= \frac{1}{(2i)^5}(z^5 - 5z^4 \times z^{-1} + 10z^3 \times z^{-2} - 10z^2 \times z^{-3} + 5z \times z^{-4} - z^{-5})$$

$$= \frac{1}{32i}(z^5 - 5z^3 + 10z - 10z^{-1} + 5z^{-3} - z^{-5})$$

$$= \frac{1}{16}\left(\frac{z^5 - z^{-5}}{2i} - \frac{5(z^3 - z^{-3})}{2i} + \frac{10(z - z^{-1})}{2i}\right)$$

$$= \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta), \text{ as required}$$

The general relation is

$$\sin n\theta = \frac{e^{in\theta} - e^{-in\theta}}{2i} = \frac{z^n - z^{-n}}{2i}$$

b $$\int_0^{\frac{\pi}{2}} \sin^5\theta \, d\theta = \frac{1}{16}\int_0^{\frac{\pi}{2}} (\sin 5\theta - 5\sin 3\theta + 10\sin\theta)\, d\theta$$

Each term on the right hand side of the identity shown in part **a** can be integrated using the formula

$$\int \sin n\theta \, d\theta = -\frac{\cos n\theta}{n}.$$

$$= \frac{1}{16}\left[-\tfrac{1}{5}\cos 5\theta + \tfrac{5}{3}\cos 3\theta - 10\cos\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{16}\left(0 - \left(-\tfrac{1}{5} + \tfrac{5}{3} - 10\right)\right)$$

$$= \frac{1}{16} \times \frac{128}{15} = \frac{8}{15}, \text{ as required}$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 40

Question:

The transformation from the z-plane to the w-plane is given by

$$w = \frac{z - \mathrm{i}}{z}.$$

a Show that under this transformation the line $\mathrm{Im}\, z = \frac{1}{2}$ is mapped to the circle with equation $|w| = 1$.

b Hence, or otherwise, find, in the form $w = \frac{az + b}{cz + d}$, where a, b, c and $d \in \mathbb{C}$, the transformation that maps the line $\mathrm{Im}\, z = \frac{1}{2}$ to the circle, centre $(3 - \mathrm{i})$ and radius 2.

Solution:

a $z = x + \frac{1}{2}\text{i}$

The real part of a complex number on $\text{Im}\, z = \frac{1}{2}$ can have any real value, which you can represent by the symbol x, but the imaginary part must be $\frac{1}{2}$.

$$w = \frac{z - \text{i}}{z}$$

$$zw = z - \text{i} \Rightarrow z - wz = \text{i}$$

$$z = \frac{\text{i}}{1 - w}$$

Let $w = u + \text{i}v$

$$x + \tfrac{1}{2}\text{i} = \frac{\text{i}}{1 - u - \text{i}v}$$

Multiplying the numerator and denominator by $1 - u + \text{i}v$

Multiply the numerator and the denominator of the right hand side by the conjugate complex of $1 - u - \text{i}v$ which is $1 - u + \text{i}v$.

$$x + \tfrac{1}{2}\text{i} = \frac{\text{i}(1 - u + \text{i}v)}{(1 - u)^2 + v^2},$$

$$= \frac{-v}{(1 - u)^2 + v^2} + \frac{1 - u}{(1 - u)^2 + v^2}\text{i}$$

Equating imaginary parts

$$\frac{1}{2} = \frac{1 - u}{u^2 - 2u + 1 + v^2}$$

You are aiming at $|w| = 1$. If $w = u + \text{i}v$, this is the equivalent to $u^2 + v^2 = 1$. So that is the expression you are looking for.

$$u^2 - 2u + 1 + v^2 = 2 - 2u$$

$$u^2 + v^2 = 1$$

$u^2 + v^2 = 1$ is a circle centre O, radius 1.

Hence the line, $\text{Im}\, z = \frac{1}{2}$ is mapped onto the circle with equation $|w| = 1$.

b The transformation $w' = \frac{z - \text{i}}{z}$ maps the line $\text{Im}\, z = \frac{1}{2}$ onto the circle with centre O and radius 1.

The first transformation is the transformation in part **a**.

The transformation $w'' = 2w'$ maps the circle with centre O and radius 1 onto the circle with centre O and radius 2.

The transformation $z \mapsto kz$ increases the radius of the circle by a factor of k. This transformation is an enlargement, factor k, centre of enlargement O.

The transformation $w = w'' + 3 - \text{i}$ maps the circle with centre O and radius 2 onto the circle with centre $3 - \text{i}$ and radius 2.

The transformation $z \mapsto z + a$ maps a circle centre O to a circle centre a. This transformation is a translation.

Combining the transformations

$$w = 2\left(\frac{z - \text{i}}{z}\right) + 3 - \text{i}$$

$$= \frac{2z - 2\text{i} + 3z - \text{i}z}{z}$$

$$= \frac{(5 - \text{i})z - 2\text{i}}{z}$$

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Exercise A, Question 41

Question:

a Solve the equation

$$z^3 = 32 + 32\sqrt{3}\mathrm{i},$$

giving your answers in the form $r\,\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$, $-\pi < \theta \leqslant \pi$.

b Show that your solutions satisfy the equation

$$z^9 + 2^k = 0,$$

for an integer k, the value of which should be stated.

Solution:

a Let $32 + 32\sqrt{3}i = r(\cos\theta + i\sin\theta) = r\cos\theta + i r\sin\theta$

Equating real parts

$$32 = r\cos\theta \quad ①$$

Equating imaginary parts

$$32\sqrt{3} = r\sin\theta \quad ②$$

Dividing ② by ①

$$\tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$$

Substituting $\theta = \frac{\pi}{3}$ into ①

$$32 = r\cos\frac{\pi}{3} \Rightarrow 32 = r \times \frac{1}{2} \Rightarrow r = 64$$

Hence

$$32 + 32\sqrt{3}i = 64\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$= 64e^{i\frac{\pi}{3}}$$

$$z^3 = 64e^{i\frac{\pi}{3}},\ 64e^{i\frac{7\pi}{3}},\ 64e^{i\frac{-5\pi}{3}}$$

$$z = 4e^{i\frac{\pi}{9}},\ 4e^{i\frac{7\pi}{9}},\ 4e^{-i\frac{5\pi}{9}}$$

The solutions are $re^{i\theta}$ where $r = 4$ and

$$\theta = -\frac{5\pi}{9}, \frac{\pi}{9}, \frac{7\pi}{9}$$

Finding the roots of a complex number is usually easier if you obtain the number in the form $re^{i\theta}$. As you will use Euler's relation, the first step towards this is to get the complex number into the form $r(\cos\theta + i\sin\theta)$.

Additional solutions are found by increasing or decreasing the arguments in steps of 2π. You are asked for 3 answers, so you need 3 arguments. Had you increased the argument $\frac{7\pi}{3}$ by 2π to $\frac{13\pi}{3}$, this would have given a correct solution to the equation but it would lead to $\theta = \frac{13\pi}{9}$, which does not satisfy the condition $\theta \leqslant \pi$ in the question. So the third argument has to be found by subtracting 2π from $\frac{\pi}{3}$.

b

$$z = 4e^{i\frac{\pi}{9}},\ 4e^{i\frac{7\pi}{9}},\ 4e^{-i\frac{-5\pi}{9}}$$

$$z^9 = \left(4e^{i\frac{\pi}{9}}\right)^9, \left(4e^{i\frac{7\pi}{9}}\right)^9, \left(4e^{-i\frac{-5\pi}{9}}\right)^9$$

$$= 4^9e^{i\pi},\ 4^9e^{i7\pi},\ 4^9e^{-i5\pi}$$

$e^{i\pi} = \cos\pi + i\sin\pi = -1 + 0 = -1$.
Similarly for the arguments 7π and -5π.

The value of all three of these expressions is $-4^9 = -2^{18}$

Hence the solutions satisfy $z^9 + 2^k = 0$, where $k = 18$.

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Exercise A, Question 42

Question:

a Use de Moivre's theorem to show that $\sin 5\theta = \sin\theta\,(16\cos^4\theta - 12\cos^2\theta + 1)$.

b Hence, or otherwise, solve, for $0 \leqslant \theta < \pi$,

$$\sin 5\theta + \cos\theta \sin 2\theta = 0.$$

Solution:

a By de Moivre's theorem

$\cos 5\theta + \mathrm{i}\sin 5\theta = (\cos\theta + \mathrm{i}\sin\theta)^5 = (c + \mathrm{i}s)^5$, say

$= c^5 + 5c^4\mathrm{i}s + 10c^3\mathrm{i}^2s^2 + 10c^2\mathrm{i}^3s^3 + 5c\mathrm{i}^4s^4 + \mathrm{i}^5s^5$

$= c^5 + \mathrm{i}5c^4s - 10c^3s^2 - \mathrm{i}10c^2s^3 + 5cs^4 - \mathrm{i}s^5$

Equating imaginary parts

$\sin 5\theta = 5c^4s - 10c^2s^3 + s^5$

$= s(5c^4 - 10c^2s^2 + s^4)$

Repeatedly using the identity $\cos^2\theta + \sin^2\theta = 1$, which in this context is $s^2 = 1 - c^2$.

$= s(5c^4 - 10c^2(1 - c^2) + (1 - c^2)^2)$

$= s(5c^4 - 10c^2 + 10c^4 + 1 - 2c^2 + c^4)$

$= s(16c^4 - 12c^2 + 1)$

$= \sin\theta(16\cos^4\theta - 12\cos^2\theta + 1)$, as required

b $\sin 5\theta + \cos\theta\sin 2\theta = 0$

$\sin\theta(16\cos^4\theta - 12\cos^2\theta + 1) + 2\sin\theta\cos^2\theta = 0$

Using the identity proved in part **a** and the identity $\sin 2\theta = 2\sin\theta\cos\theta$.

$\sin\theta(16\cos^4\theta - 10\cos^2\theta + 1) = 0$

$\sin\theta(2\cos^2\theta - 1)(8\cos^2\theta - 1) = 0$

Hence $\sin\theta = 0$, $\cos^2\theta = \frac{1}{2}$, $\cos^2\theta = \frac{1}{8}$

$\sin\theta = 0 \Rightarrow \theta = 0$

$\cos^2\theta = \frac{1}{2} \Rightarrow \cos\theta = \pm\frac{1}{\sqrt{2}}$

$\cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

$\cos\theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{3\pi}{4}$

You must consider the negative as well as the positive square roots.

$\cos^2\theta = \frac{1}{8} \Rightarrow \cos\theta = \pm\frac{1}{2\sqrt{2}}$

$\cos\theta = \frac{1}{2\sqrt{2}} \Rightarrow \theta = 1.209$ (3 d.p.)

$\cos\theta = -\frac{1}{2\sqrt{2}} \Rightarrow \theta = 1.932$ (3 d.p.)

The question has specified no accuracy and any sensible accuracy would be accepted for the approximate answers.

The solutions of the equation are

$0, \frac{\pi}{4}, \frac{3\pi}{4}$, 1.209 (3 d.p.) and 1.932 (3 d.p.).

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Exercise A, Question 43

Question:

a Given that $z = \cos\theta + i\sin\theta$, show that $z^n + z^{-n} = 2\cos n\theta$.

b Express $\cos^6\theta$ in terms of cosines of multiples of θ.

c Hence show that

$$\int_0^{\frac{\pi}{2}} \cos^6\theta \, d\theta = \frac{5\pi}{32}.$$

Solution:

a $z = \cos\theta + \mathrm{i}\sin\theta$

Using de Moivre's theorem

$$z^n = (\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta \quad ①$$

From ①

$$z^{-n} = \frac{1}{z^n} = \frac{1}{\cos n\theta + \mathrm{i}\sin n\theta}$$

$$= \frac{1}{\cos n\theta + \mathrm{i}\sin n\theta} \times \frac{\cos n\theta - \mathrm{i}\sin n\theta}{\cos n\theta - \mathrm{i}\sin n\theta}$$

Multiply the numerator and denominator by $\cos n\theta - \mathrm{i}\sin n\theta$, the conjugate complex number of $\cos n\theta + \mathrm{i}\sin n\theta$.

$$= \frac{\cos n\theta - \mathrm{i}\sin n\theta}{\cos^2 n\theta + \sin^2 n\theta} = \cos n\theta - \mathrm{i}\sin n\theta \quad ②$$

Use $\cos^2 n\theta + \sin^2 n\theta = 1$.

$$z^n + z^{-n} = \cos n\theta + \mathrm{i}\sin n\theta + \cos n\theta - \mathrm{i}\sin n\theta$$

$= 2\cos n\theta$, as required.

b $\cos\theta = \dfrac{e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta}}{2} = \dfrac{z + z^{-1}}{2}$

$$\cos^6\theta = \left(\frac{z + z^{-1}}{2}\right)^6$$

$$= \tfrac{1}{64}(z^6 + 6z^5z^{-1} + 15z^4z^{-2} + 20z^3z^{-3} + 15z^2z^{-4} + 6z^1z^{-5} + z^{-6})$$

Pair the terms as shown.

$$= \tfrac{1}{64}(z^6 + 6z^4 + 15z^2 + 20 + 15z^{-2} + 6z^{-4} + z^{-6})$$

$$= \tfrac{1}{32}\left(\frac{z^6 + z^{-6}}{2} + \frac{6(z^4 + z^{-4})}{2} + \frac{15(z^2 + z^{-2})}{2} + \frac{20}{2}\right)$$

$$= \tfrac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$$

You use $\dfrac{z^n + z^{-n}}{2} = \cos n\theta$ with $n = 6$, 4 and 2.

c $\displaystyle\int_0^{\frac{\pi}{2}} \cos^6\theta\,\mathrm{d}\theta = \tfrac{1}{32}\int_0^{\frac{\pi}{2}} (\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)\,\mathrm{d}\theta$

$$= \tfrac{1}{32} - \left[\tfrac{1}{6}\sin 6\theta + \tfrac{6}{4}\sin 4\theta + \tfrac{15}{2}\sin 2\theta + 10\,\theta\right]_0^{\frac{\pi}{2}}$$

$$= \tfrac{1}{32} \times 10 \times \frac{\pi}{2} = \frac{5\pi}{32}, \text{ as required}$$

With the exception of 10θ all of these terms have value 0 at both the upper and the lower limit.

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Exercise A, Question 44

Question:

a Prove that

$$(z^n - e^{i\theta})(z^n - e^{-i\theta}) = z^{2n} - 2z^n \cos\theta + 1.$$

b Hence, or otherwise, find the roots of the equation

$$z^6 - z^3\sqrt{2} + 1 = 0,$$

in the form $\cos\alpha + i\sin\alpha$, where $-\pi < \alpha \leqslant \pi$.

Solution:

a $(z^n - e^{i\theta})(z^n - e^{-i\theta}) = z^{2n} - z^n e^{-i\theta} - z^n e^{i\theta} + e^{i\theta}e^{-i\theta}$

$$= z^{2n} - 2z^n\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right) + 1$$

$$= z^{2n} - 2z^n \cos\theta + 1, \text{ as required}$$

The specification requires you to be familiar with

$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and

$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$.

b Using the result of part **a** with $n = 3$ and $\theta = \frac{\pi}{4}$,

$$z^6 - 2z^3\cos\frac{\pi}{4} + 1 = \left(z^3 - e^{i\frac{\pi}{4}}\right)\left(z^3 - e^{-i\frac{\pi}{4}}\right)$$

$$z^6 - z^3\sqrt{2} + 1 = \left(z^3 - e^{i\frac{\pi}{4}}\right)\left(z^3 - e^{-i\frac{\pi}{4}}\right) = 0$$

As $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $2z^3\cos\frac{\pi}{4} = z^3\sqrt{2}$

$$z^3 = e^{i\frac{\pi}{4}}, e^{-i\frac{\pi}{4}}$$

$$z^3 = e^{i\frac{\pi}{4}}, e^{i\frac{9\pi}{4}}, e^{-i\frac{7\pi}{4}} \Rightarrow z = e^{i\frac{\pi}{12}}, e^{i\frac{9\pi}{12}}, e^{-i\frac{7\pi}{12}}$$

$$z^3 = e^{-i\frac{\pi}{4}}, e^{i\frac{7\pi}{4}}, e^{-i\frac{9\pi}{4}} \Rightarrow z = e^{-i\frac{\pi}{12}}, e^{i\frac{7\pi}{12}}, e^{-i\frac{9\pi}{12}}$$

The solutions of $z^6 - z^3\sqrt{2} + 1 = 0$ are $\cos\alpha + i\sin\alpha$,

where $\alpha = -\frac{3\pi}{4}, -\frac{7\pi}{12}, -\frac{\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}$

Each of these two expressions give rise to 3 distinct expressions when multiples of 2π are added and subtracted from the arguments. The original equation is of degree 6 and there will, usually, be 6 distinct answers.

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Exercise A, Question 45

Question:

The transformation

$$w = \frac{z+2}{z+\mathrm{i}},$$

where $z \neq \mathrm{i}$, $w \neq \mathrm{i}$, maps the complex number $z = x + \mathrm{i}y$ onto the complex number $w = u + \mathrm{i}v$.

a Show that, if the point representing w lies on the real axis, the point representing z lies on a straight line.

b Show further that, if the point representing w lies on the imaginary axis, the point representing z lies on the circle

$$\left|z + 1 + \tfrac{1}{2}\mathrm{i}\right| = \frac{\sqrt{5}}{2}.$$

Solution:

a On the real axis, $w = u$

If the point lies on the real axis in the w-plane, the imaginary part of the associated complex number is zero. So $w = u + 0\text{i} = u$.

$$w = u = \frac{z+2}{z+\text{i}}$$

$$uz + u\text{i} = z + 2 \Rightarrow uz - z = 2 - u\text{i} \Rightarrow z = \frac{2 - u\text{i}}{u-1}$$

Hence

$$z = x + \text{i}y = \frac{2}{u-1} - \frac{u\text{i}}{u-1}$$

Equating real and imaginary parts

$$x = \frac{2}{u-1} \quad ①$$

$$y = -\frac{u}{u-1} \quad ②$$

After equating real and imaginary parts, you obtain x and y in terms of the parameter u. Eliminating u gives the Cartesian equation of the locus of the point in the z-plane.

From ① $xu - x = 2$ ③

Dividing ② by ①

$$\frac{y}{x} = -\tfrac{1}{2}u \Rightarrow u = -\frac{2y}{x}$$

Substituting for u in ③

$$x \times -\frac{2y}{x} - x = 2$$

$$-2y - x = 2 \Rightarrow x + 2y + 2 = 0$$

This is the equation of a straight line in the z-plane, as required.

b On the imaginary axis, $w = \text{i}v$

If the point lies on the imaginary axis in the z-plane, then the real part of the associated complex number is zero. So $w = 0 + \text{i}v = \text{i}v$.

$$w = \text{i}v = \frac{z+2}{z+\text{i}}$$

$$\text{i}vz - v = z + 2 \Rightarrow \text{i}vz - z = v + 2 \Rightarrow z = \frac{v+2}{-1+\text{i}v}$$

$$z = \frac{v+2}{-1+\text{i}v} \times \frac{-1-\text{i}v}{-1-\text{i}v} = \frac{-(v+2) - v(v+2)\text{i}}{v^2+1}$$

$$z = x + \text{i}y = -\frac{v+2}{v^2+1} - \frac{v(v+2)\text{i}}{v^2+1}$$

Equating real and imaginary parts

$$x = -\frac{v+2}{v^2+1} \quad ①$$

$$y = -\frac{v(v+2)\mathrm{i}}{v^2+1} \quad ②$$

As in part **a**, after equating real and imaginary parts, you obtain x and y in terms of a parameter; in this case v. Eliminating v gives the Cartesian equation of the locus of the point in the z-plane.

From ① $xv^2 + x = -v - 2$ ③

Dividing ② by ①

$$\frac{y}{x} = v$$

Substituting for v in ③

$$x \times \frac{y^2}{x^2} + x = -\frac{y}{x} - 2$$

Multiplying by x

$$y^2 + x^2 = -y - 2x$$

$$x^2 + 2x + y^2 + y = 0$$

$$x^2 + 2x + 1 + y^2 + y + \tfrac{1}{4} = \tfrac{5}{4}$$

$$(x+1)^2 + \left(y + \tfrac{1}{2}\right)^2 = \left(\tfrac{1}{2}\sqrt{5}\right)^2$$

Completing squares gives you the centre and radius of the circle.

This is the Cartesian equation of a circle with centre $\left(-1, -\frac{1}{2}\right)$ and radius $= \frac{1}{2}\sqrt{5}$.

In the z-plane, z lies on the circle $\left|z + 1 + \frac{1}{2}\mathrm{i}\right| = \frac{1}{2}\sqrt{5}$, as required.

The locus of $|z - a| = k$, where a is a complex number and k is a real number, is a circle with radius k and centre the point representing a. As you know the centre and the radius, you can write down the locus of z without further working.

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Exercise A, Question 46

Question:

A complex number z is represented by the point P in the Argand diagram.
Given that

$$|z - 3\text{i}| = 3,$$

a sketch the locus of P.

b Find the complex number z which satisfies both $|z - 3\text{i}| = 3$ and $\arg(z - 3\text{i}) = \frac{3\pi}{4}$.

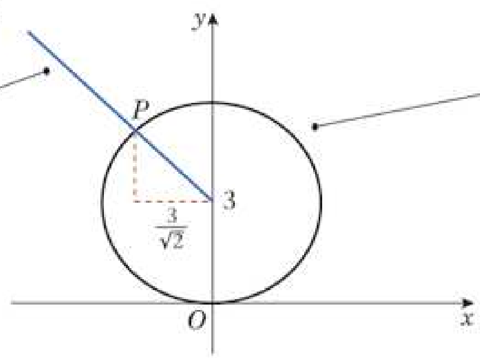
The transformation T from the z-plane to the w-plane is given by

$$w = \frac{2\text{i}}{z}.$$

c Show that T maps $|z - 3\text{i}| = 3$ to a line in the w-plane, and give the Cartesian equation of this line.

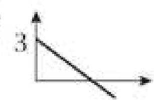
Solution:

a



The locus of P is the circle with centre (0, 3) and radius 3. The coordinates (0, 3) represent the complex number 3i in the Argand diagram.

The half-line representing $\arg(z - 3\text{i}) = \frac{3\pi}{4}$ has been added to the diagram. This starts at (0, 3) and makes an angle of $\frac{3\pi}{4}$ with the positive x-direction. It is a common error to turn this half into a full line. The half line

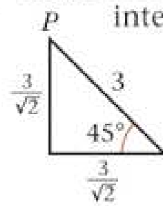


has a different equation, $\arg(z - 3\text{i}) = -\frac{\pi}{4}$.

b From the diagram, z is the intersection of the circle and the half line marked P in the diagram.

$$z = -\frac{3}{\sqrt{2}} + \text{i}\left(3 + \frac{3}{\sqrt{2}}\right)$$

The geometry of the point of intersection is shown here.



The coordinates of P can then be just written down.

c The circle $|z - 3\text{i}| = 3$ has Cartesian equation

$$x^2 + (y - 3)^2 = 9$$

$$x^2 + y^2 - 6y + 9 = 9$$

$$x^2 + y^2 = 6y \qquad (1)$$

If $z = x + \text{i}y$,

$$w = \frac{2\text{i}}{z} = \frac{2\text{i}}{x + \text{i}y} = \frac{2\text{i}}{x + \text{i}y} \times \frac{x - \text{i}y}{x - \text{i}y}$$

$$= \frac{2y + \text{i}2x}{x^2 + y^2}$$

Multiplying the numerator and the denominator by the conjugate complex of $x + \text{i}y$ which is $x - \text{i}y$. $(x + \text{i}y)(x - \text{i}y) = x^2 + y^2$.

$$u + \text{i}v = \frac{2y}{x^2 + y^2} + \text{i}\frac{2x}{x^2 + y^2}$$

From (1) above, $x^2 + y^2 = 6y$

Hence $u + \text{i}v = \frac{2y}{6y} + \text{i}\frac{2x}{6y} = \frac{1}{3} + \text{i}\frac{x}{3y}$

Equating real parts

$$u = \tfrac{1}{3}$$

A 'simple' equation like $u = \frac{1}{3}$ is quite difficult to recognise in this context. This is the equation of the straight line parallel to the v (imaginary) axis in the w-plane.

The circle maps to the straight line with equation $u = \frac{1}{3}$ in the w-plane.

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Exercise A, Question 47

Question:

The point P on the Argand diagram represents the complex number z.

a Given that $|z| = 1$, sketch the locus of P.

The point Q is the image of P under the transformation

$$w = \frac{1}{z-1}.$$

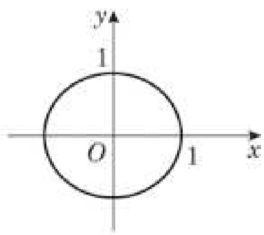
b Given that $z = e^{i\theta}$, $0 < \theta < 2\pi$, show that

$$w = -\tfrac{1}{2} - \tfrac{1}{2}i\cot\tfrac{1}{2}\theta.$$

c Make a separate sketch of the locus Q.

Solution:

a



b If $z = e^{i\theta}$

Using Euler's relation $e^{i\theta} = \cos\theta + i\sin\theta$.

$$w = \frac{1}{z-1} = \frac{1}{e^{i\theta}-1}$$

$$= \frac{1}{\cos\theta + i\sin\theta - 1} = \frac{1}{\cos\theta - 1 + i\sin\theta}$$

$$= \frac{1}{\cos\theta - 1 + i\sin\theta} \times \frac{\cos\theta - 1 - i\sin\theta}{\cos\theta - 1 - i\sin\theta}$$

Multiply the numerator and denominator by $\cos\theta - 1 - i\sin\theta$, the conjugate complex of $\cos\theta - 1 + i\sin\theta$.

$$= \frac{\cos\theta - 1 - i\sin\theta}{(\cos\theta - 1)^2 + \sin^2\theta}$$

$$= \frac{\cos\theta - 1 - i\sin\theta}{\cos^2\theta - 2\cos\theta + 1 + \sin^2\theta}$$

$$= \frac{\cos\theta - 1}{2 - 2\cos\theta} - \frac{i\sin\theta}{2 - 2\cos\theta}$$

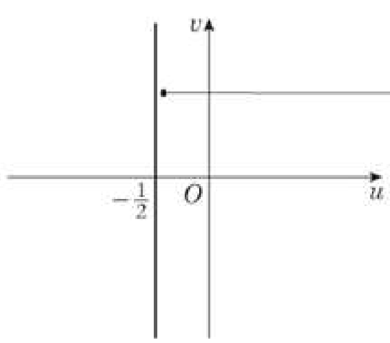
Using $\sin\theta = 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$.

Using $\cos\theta = 1 - 2\sin^2\frac{1}{2}\theta$.

$$= \frac{\cos\theta - 1}{2(1 - \cos\theta)} - i\frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{4\sin^2\frac{1}{2}\theta}$$

$= -\frac{1}{2} - \frac{1}{2}i\cot\frac{1}{2}\theta$, as required

c



$w = u + iv = -\frac{1}{2} - \frac{1}{2}i\cot\frac{1}{2}\theta$. Equating real parts gives $u = -\frac{1}{2}$. This is the equation of a straight line parallel to the v (imaginary) axis.

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Exercise A, Question 48

Question:

In an Argand diagram the point P represents the complex number z.

Given that $\arg\left(\frac{z - 2\text{i}}{z + 2}\right) = \frac{\pi}{2}$,

a sketch the locus of P,

b deduce the value of $|z + 1 - \text{i}|$.

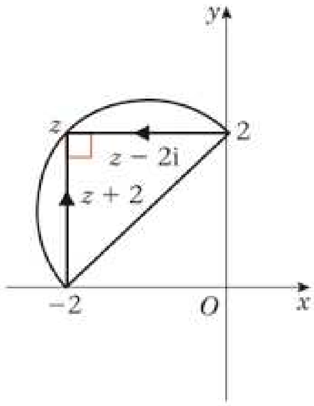
The transformation T from the z-plane to the w-plane is defined by

$$w = \frac{2(1 + \text{i})}{z + 2}, \; z \neq 2.$$

c Show that the locus of P in the z-plane is mapped to part of a straight line in the w-plane, and show this in an Argand diagram.

Solution:

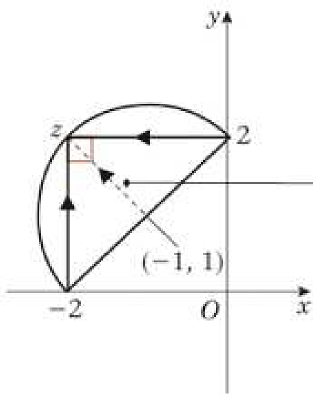
a



$\arg\left(\frac{z-2i}{z+2}\right) = \arg(z-2i) - \arg(z+2) = \frac{\pi}{2}$.

The angles which the vectors make with the positive x-axis differ by a right angle. As drawn here, the difference is $\pi - \frac{\pi}{2} = \frac{\pi}{2}$. The locus of the points, where the difference is a right angle, is a semi-circle, with the line joining -2 on the real axis to 2 on the imaginary axis as diameter.
It is a common error to complete the circle. The lower right hand completion of the circle has equation $\arg\left(\frac{z-2i}{z+2}\right) = -\frac{\pi}{2}$.

b



The dotted line represents the complex number $z + 1 - i = z - (-1 + i)$. The length of this vector is the radius of the circle.

The diameter of the circle is given by

$$d^2 = 2^2 + 2^2 = 8$$

$$|z + 1 - i| = \frac{\sqrt{8}}{2} = \sqrt{2}$$

c $$w = \frac{2(1+i)}{z+2}$$

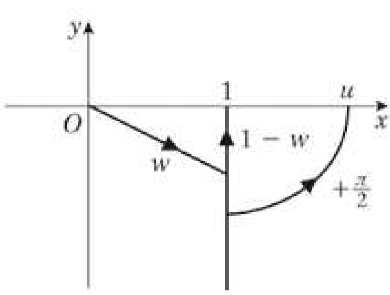
$$z = \frac{2(1+i)}{w} - 2$$

You find the transformation of $\arg\left(\frac{z-2i}{z+2}\right) = \frac{\pi}{2}$ under T by making z 'the subject of the transformation' and using this to substitute for z in the expression $\frac{z-2i}{z+2}$.

$$\frac{z-2i}{z+2} = \frac{\frac{2(1+i)}{w} - 2 - 2i}{\frac{2(1+i)}{w} - 2 + 2} = \frac{2(1+i) - 2(1+i)w}{2(1+i)} = 1 - w$$

Hence the transformation of $\arg\left(\frac{z-2i}{z+2}\right) = \frac{\pi}{2}$

under T is $\arg(1 - w) = \frac{\pi}{2}$. This is a half-line as shown in the following diagram.



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Exercise A, Question 49

Question:

The transformation T from the complex z-plane to the complex w-plane is given by

$$w = \frac{z+1}{z+\mathrm{i}}, \; z \neq \mathrm{i}.$$

a Show that T maps points on the half-line $\arg z = \frac{\pi}{4}$ in the z-plane into points on the circle $|w| = 1$ in the w-plane.

b Find the image under T in the w-plane of the circle $|z| = 1$ in the z-plane.

c Sketch on separate diagrams the circle $|z| = 1$ in the z-plane and its image under T in the w-plane.

d Mark on your sketches the point P, where $z = \mathrm{i}$, and its image Q under T in the w-plane.

Solution:

a If $z = x + \mathrm{i}y$, then $\arg z = \frac{\pi}{4} \Rightarrow \frac{y}{x} = 1$

Let $x = y = \lambda$

$$w = \frac{\lambda + \lambda\mathrm{i} + 1}{\lambda + \lambda\mathrm{i} + \mathrm{i}} = \frac{(\lambda + 1) + \lambda\mathrm{i}}{\lambda + (\lambda + 1)\mathrm{i}}$$

$$|w| = \left|\frac{(\lambda + 1) + \lambda\mathrm{i}}{\lambda + (\lambda + 1)\mathrm{i}}\right| = \frac{|(\lambda + 1) + \lambda\mathrm{i}|}{|\lambda + (\lambda + 1)\mathrm{i}|}$$

For all complex numbers a and b, $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

$$= \frac{((\lambda + 1)^2 + \lambda^2)^{\frac{1}{2}}}{(\lambda^2 + (\lambda + 1)^2)^{\frac{1}{2}}} = 1$$

Hence the points on $\arg z = \frac{\pi}{4}$ map, under T, onto points on the circle $|w| = 1$.

As $\lambda = 0$, the image would only be part of this circle but the wording of the question does not require you to be more specific. You are only required to show that the image points are points on the circle; not all of the points on the circle. (The image is, in fact, just the lower right quadrant of the circle.)

b

$$wz + w\mathrm{i} = z + 1$$

$$wz - z = 1 - \mathrm{i}w$$

$$z = \frac{1 - \mathrm{i}w}{w - 1}$$

$$|z| = \frac{|1 - \mathrm{i}w|}{|w - 1|} = 1$$

Hence $|1 - \mathrm{i}w| = |w - 1|$

This is the image under T of $|z| = 1$ but it is difficult to interpret and part **c** would be difficult without some further working.

$|1 - \mathrm{i}w| = |-\mathrm{i}(w + \mathrm{i})| = |-\mathrm{i}||w + \mathrm{i}| = 1 \times |w + \mathrm{i}| = |w + \mathrm{i}|$

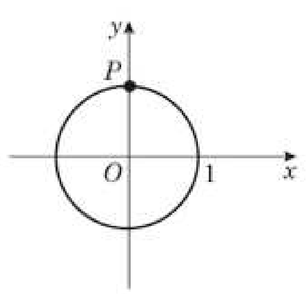
The image of $|z| = 1$ in the z-plane is

$$|w + \mathrm{i}| = |w - 1|$$

in the w-plane.

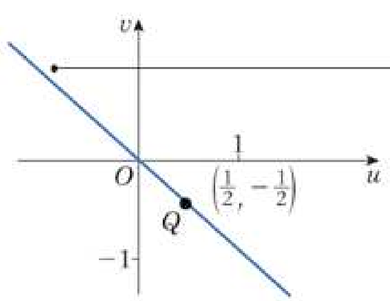
This is the locus of points equidistant from the points in the Argand plane representing $-\mathrm{i}$ and one. That is the perpendicular bisector of $(0, -1)$ and $(1, 0)$.

c d



z

$$z = \mathrm{i} \Rightarrow w = \frac{1 + \mathrm{i}}{2\mathrm{i}} = \frac{\mathrm{i} + 1}{2\mathrm{i}} = \tfrac{1}{2} - \tfrac{1}{2}\mathrm{i}$$



w

The perpendiclar bisector of $(0, -1)$ and $(1, 0)$ is the line $v = -u$.

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Exercise A, Question 50

Question:

a Find the roots z_1 and z_2 of the equation

$$z^2 - 2iz - 2 = 0.$$

The transformation

$$w = \frac{az + b}{z + d}, z \neq -d,$$

where a, b and d are complex numbers, maps the complex number z onto the complex number w. Given that z_1 and z_2 are invariant under this transformation and that $z = 0$ maps to $w = i$,

b find the values of a, b and d.

Using your values of a, b and d,

c show that $|z| = 2\left|\frac{w - i}{w}\right|$.

d Hence, or otherwise, find the radius and centre of the circle described by w when z moves on the unit circle $|z| = 1$.

Solution:

a

$$z^2 - 2iz = 2$$
$$z^2 - 2iz + i^2 = 2 + i^2$$
$$(z - i)^2 = 1$$
$$z - i = \pm 1$$
$$z = 1 + i, -1 + i$$

You can use any method to solve this quadratic. Completing the square is an efficient method in this case.

b For an invariant point $w = z$

$$z = \frac{az + b}{z + d}$$
$$z^2 + dz = az + b$$
$$z^2 + (d - a)z - b = 0$$

An invariant point is a point unchanged by the mapping. So w and z are the same point and the expression can be transformed into a quadratic.

This must be the same equation as that in part **a**, which is

$$z^2 - 2iz - 2 = 0$$

The complex numbers $1 + i$ and $-1 + i$ must be the roots of both this quadratic equation and the quadratic equation in part **a**. So, the two equations must be the same and equating the coefficients of x and the constant coefficients gives a simple relation between a and d and the value of b.

Hence, equating coefficients,

$$d - a = -2i \text{ and } b = 2$$

$z = 0$, $w = i$

$$i = \frac{b}{d} \Rightarrow d = \frac{b}{i} = \frac{ib}{i^2} = -ib$$
$$d = -2i \text{ and } a = 0$$
$$a = 0, b = 2, d = -2i$$

c

$$w = \frac{2}{z - 2\text{i}}$$

$$zw - 2\text{i}\,w = 2 \Rightarrow z = \frac{2 + 2\text{i}\,w}{w}$$

$$z = \frac{2\text{i}(w - \text{i})}{w}$$

$$|z| = |2\text{i}|\left|\frac{(w - \text{i})}{w}\right|$$

For all complex numbers a and b, $|ab| = |a||b|$.

$$|z| = 2\left|\frac{w - \text{i}}{w}\right|, \text{ as required}$$

As $|2\text{i}| = 2$.

d

$$|z| = 1 \Rightarrow 2\left|\frac{w - \text{i}}{w}\right| = 1$$

$$2|w - \text{i}| = |w|$$

$$4|w - \text{i}|^2 = |w|^2$$

Let $w = u + \text{i}v$

$$4|u + \text{i}(v - 1)|^2 = |u + \text{i}v|^2$$

$$4(u^2 + (v - 1)^2) = u^2 + v^2$$

$$4u^2 + 4v^2 - 8v + 4 = u^2 + v^2$$

$$3u^2 + 3v^2 - 8v + 4 = 0$$

$$u^2 + v^2 - \tfrac{8}{3}v = -\tfrac{4}{3}$$

$$u^2 + v^2 - \tfrac{8}{3}v + \tfrac{16}{9} = -\tfrac{4}{3} + \tfrac{16}{9}$$

$$u^2 + \left(v - \tfrac{4}{3}\right)^2 = \tfrac{4}{9} = \left(\tfrac{2}{3}\right)^2$$

Completing the square gives the centre and radius of the circle.

The image is a circle, centre $\left(0, \tfrac{4}{3}\right)$, radius $\tfrac{2}{3}$.