

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise A, Question 1

Question:

Express the following in the form $r(\cos \theta + i \sin \theta)$, where $-\pi < \theta \leq \pi$. Give the exact values of r and θ where possible, or values to 2 d.p. otherwise.

a 7

b $-5i$

c $\sqrt{3} + i$

d $2 + 2i$

e $1 - i$

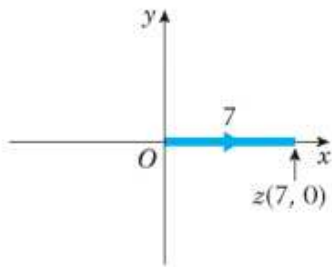
f -8

g $3 - 4i$

h $-8 + 6i$

i $2 - \sqrt{3}i$

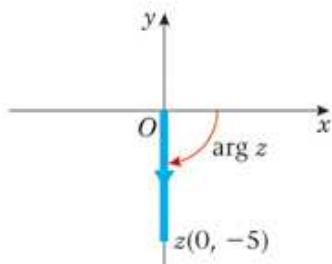
Solution:

a 7

$$r = 7$$

$$\theta = \arg z = 0$$

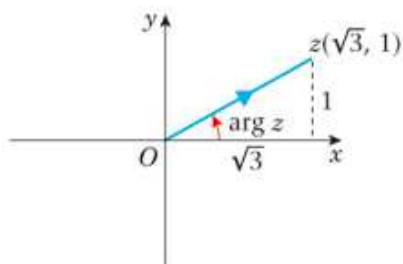
$$\therefore 7 = 7(\cos \theta + i \sin \theta)$$

b $-5i$ 

$$r = 5$$

$$\theta = \arg z = -\frac{\pi}{2}$$

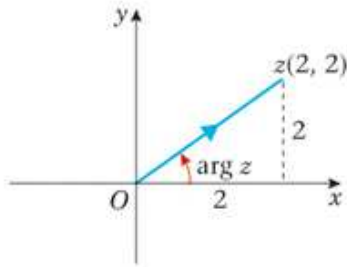
$$\therefore -5i = 5\left(\cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right)\right)$$

c $\sqrt{3} + i$ 

$$r = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$$

$$\theta = \arg z = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

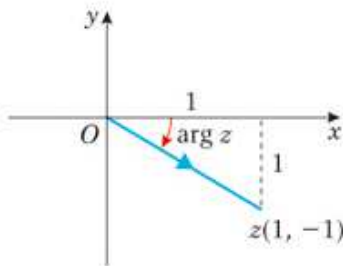
$$\therefore \sqrt{3} + i = 2\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$$

d $2 + 2i$ 

$$r = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

$$\theta = \arg z = \tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$$

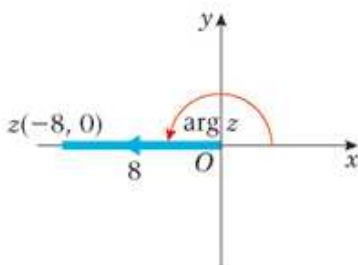
$$\therefore 2 + 2i = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

e $1 - i$ 

$$r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\theta = \arg z = \tan^{-1}\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$$

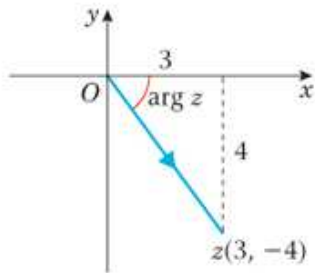
$$\therefore 1 - i = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

f -8 

$$r = 8$$

$$\theta = \arg z = \pi$$

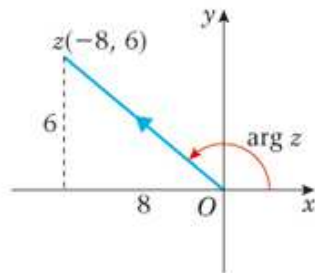
$$\therefore -8 = 8(\cos \pi + i \sin \pi)$$

g $3 - 4i$ 

$$r = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$$

$$\theta = \arg z = -\tan^{-1}\left(\frac{4}{3}\right) = -0.93^\circ \text{ (2 d.p.)}$$

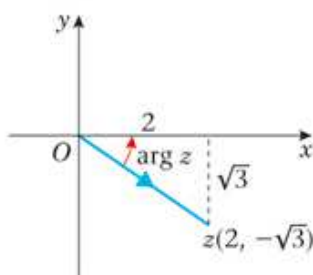
$$\therefore 3 - 4i = 5(\cos(-0.93^\circ) + i \sin(-0.93^\circ))$$

h $-8 + 6i$ 

$$r = \sqrt{(-8)^2 + 6^2} = \sqrt{100} = 10$$

$$\theta = \arg z = \pi - \tan^{-1}\left(\frac{6}{8}\right) = 2.50^\circ \text{ (2 d.p.)}$$

$$\therefore -8 + 6i = 10(\cos(2.50^\circ) + i \sin(2.50^\circ))$$

i $2 - \sqrt{3}i$ 

$$r = \sqrt{2^2 + (-\sqrt{3})^2} = \sqrt{7}$$

$$\theta = \arg z = -\tan^{-1}\left(\frac{\sqrt{3}}{2}\right) = -0.71^\circ \text{ (2 d.p.)}$$

$$\therefore 2 - \sqrt{3}i = \sqrt{7} (\cos(-0.71^\circ) + i \sin(-0.71^\circ))$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise A, Question 2

Question:

Express the following in the form $x + iy$, where $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

a $5\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)$

b $\frac{1}{2}\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$

c $6\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)$

d $3\left(\cos \left(-\frac{2\pi}{3}\right) + i \sin \left(-\frac{2\pi}{3}\right)\right)$

e $2\sqrt{2}\left(\cos \left(-\frac{\pi}{4}\right) + i \sin \left(-\frac{\pi}{4}\right)\right)$

f $-4\left(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6}\right)$

Solution:

$$\begin{aligned}\mathbf{a} \quad & 5\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right) \\ & = 5(0 + i) \\ & = 5i\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad & \frac{1}{2}\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right) \\ & = \frac{1}{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \\ & = \frac{\sqrt{3}}{4} + \frac{1}{4}i\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad & 6\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right) \\ & = 6\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \\ & = -3\sqrt{3} + 3i\end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad & 3\left(\cos \left(-\frac{2\pi}{3}\right) + i \sin \left(-\frac{2\pi}{3}\right)\right) \\ & = 3\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \\ & = -\frac{3}{2} - \frac{3\sqrt{3}}{2}i\end{aligned}$$

$$\begin{aligned}\mathbf{e} \quad & 2\sqrt{2}\left(\cos \left(-\frac{\pi}{4}\right) + i \sin \left(-\frac{\pi}{4}\right)\right) \\ & = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) \\ & = 2 - 2i\end{aligned}$$

$$\begin{aligned}\mathbf{f} \quad & -4\left(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6}\right) \\ & = -4\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) \\ & = 2\sqrt{3} + 2i\end{aligned}$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise A, Question 3

Question:

Express the following in the form $re^{i\theta}$, where $-\pi < \theta \leq \pi$. Give the exact values of r and θ where possible, or values to 2 d.p. otherwise.

a -3

b $6i$

c $-2\sqrt{3} - 2i$

d $-8 + i$

e $2 - 5i$

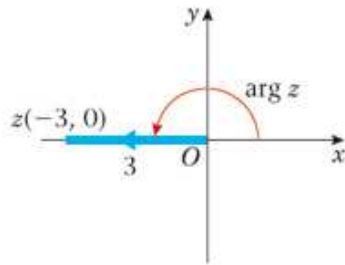
f $-2\sqrt{3} + 2\sqrt{3}i$

g $\sqrt{8}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$

h $8\left(\cos \frac{\pi}{6} - i \sin \frac{\pi}{6}\right)$

i $2\left(\cos \frac{\pi}{5} - i \sin \frac{\pi}{5}\right)$

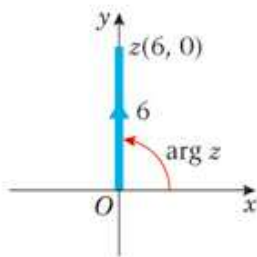
Solution:

a -3 

$$r = 3$$

$$\theta = \arg z = \pi$$

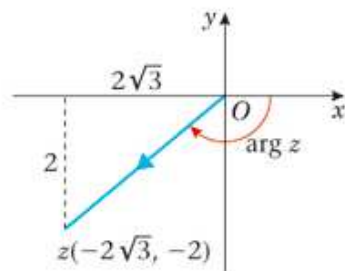
$$\therefore -3 = 3e^{\pi i}$$

b $6i$ 

$$r = 6$$

$$\theta = \arg z = \frac{\pi}{2}$$

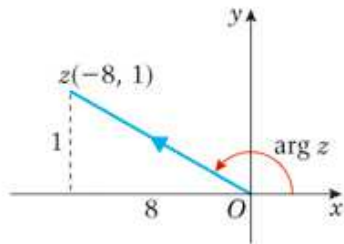
$$\therefore 6i = 6e^{\frac{\pi}{2}i}$$

c $-2\sqrt{3} - 2i$ 

$$r = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{12 + 4} = \sqrt{16} = 4$$

$$\theta = \arg z = -\pi + \tan^{-1}\left(\frac{2}{2\sqrt{3}}\right) = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$$

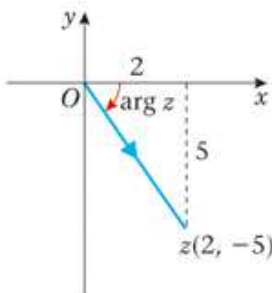
$$\therefore -2\sqrt{3} - 2i = 4e^{\frac{-5\pi}{6}i}$$

d $-8 + i$ 

$$r = \sqrt{(-8)^2 + 1^2} = \sqrt{65}$$

$$\theta = \pi - \tan^{-1}\left(\frac{1}{8}\right) = 3.02^c \text{ (2 d.p.)}$$

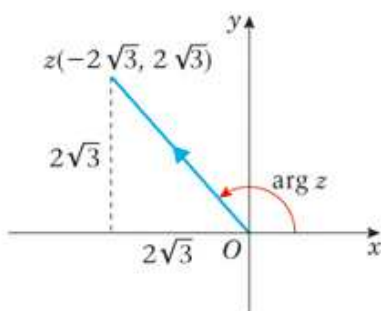
$$\therefore -8 + i = \sqrt{65} e^{3.02i}$$

e $2 - 5i$ 

$$r = \sqrt{2^2 + (-5)^2} = \sqrt{29}$$

$$\theta = -\tan^{-1}\left(\frac{5}{2}\right) = -1.19^c \text{ (2 d.p.)}$$

$$\therefore 2 - 5i = \sqrt{29} e^{-1.19i}$$

f $-2\sqrt{3} + 2\sqrt{3}i$ 

$$r = \sqrt{(-2\sqrt{3})^2 + (2\sqrt{3})^2} = \sqrt{12 + 12} = \sqrt{24}$$

$$= \sqrt{4\sqrt{6}} = 2\sqrt{6}$$

$$\theta = \pi - \tan^{-1}\left(\frac{2\sqrt{3}}{2\sqrt{3}}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$\therefore -2\sqrt{3} + 2\sqrt{3}i = 2\sqrt{6}e^{\frac{3\pi i}{4}}$$

$$\mathbf{g} \quad \sqrt{8}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) \\ = 2\sqrt{2}e^{\frac{\pi i}{4}}$$

$$r = \sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2} \\ \theta = \frac{\pi}{4}$$

$$\mathbf{h} \quad 8\left(\cos \frac{\pi}{6} - i \sin \frac{\pi}{6}\right) \\ = 8\left(\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)\right) \\ = 8e^{-\frac{\pi i}{6}}$$

$$r = 8, \theta = -\frac{\pi}{6}$$

$$\mathbf{i} \quad 2\left(\cos \frac{\pi}{5} - i \sin \frac{\pi}{5}\right) \\ = 2\left(\cos\left(-\frac{\pi}{5}\right) + i \sin\left(-\frac{\pi}{5}\right)\right) \\ = 2e^{-\frac{\pi i}{5}}$$

$$r = 2, \theta = -\frac{\pi}{5}$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise A, Question 4

Question:

Express the following in the form $x + iy$ where $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

a $e^{\frac{\pi}{3}i}$ **b**

$$4e^{\pi i}$$

c $3\sqrt{2} e^{\frac{\pi}{4}i}$

d $8e^{\frac{\pi}{6}i}$ **e**

$$3e^{-\frac{\pi}{2}i}$$

f $e^{\frac{5\pi}{6}i}$

g $e^{-\pi i}$ **h**

$$3\sqrt{2}e^{-\frac{3\pi}{4}i}$$

i $8e^{-\frac{4\pi}{3}i}$

Solution:

$$\mathbf{a} \quad e^{\frac{\pi}{3}i} = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$\mathbf{b} \quad 4e^{\pi i} = 4(\cos \pi + i \sin \pi)$$

$$= 4(-1 + i(0))$$

$$= -4$$

$$\mathbf{c} \quad 3\sqrt{2} e^{\frac{\pi}{4}i} = 3\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

$$= 3\sqrt{2} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right)$$

$$= 3 + 3i$$

$$\mathbf{d} \quad 8e^{\frac{\pi}{6}i} = 8 \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)$$

$$= 8 \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i \right)$$

$$= 4\sqrt{3} + 4i$$

$$\mathbf{e} \quad 3e^{-\frac{\pi}{2}i} = 3 \left(\cos \left(-\frac{\pi}{2} \right) + i \sin \left(-\frac{\pi}{2} \right) \right)$$

$$= 3(0 - i)$$

$$= -3i$$

$$\mathbf{f} \quad e^{\frac{5\pi}{6}i} = \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}$$

$$= -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$\begin{aligned}\mathbf{g} \quad e^{-\pi i} &= \cos(-\pi) + i \sin(-\pi) \\ &= -1 + i(0) \\ &= -1\end{aligned}$$

$$\begin{aligned}\mathbf{h} \quad 3\sqrt{2}e^{-\frac{3\pi i}{4}} &= 3\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i \sin\left(-\frac{3\pi}{4}\right)\right) \\ &= 3\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) \\ &= -3 - 3i\end{aligned}$$

$$\begin{aligned}\mathbf{i} \quad 8e^{-\frac{4\pi i}{3}} &= 8\left(\cos\left(-\frac{4\pi}{3}\right) + i \sin\left(-\frac{4\pi}{3}\right)\right) \\ &= 8\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \\ &= -4 + 4\sqrt{3}i\end{aligned}$$

© Pearson Education Ltd 2009

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise A, Question 5

Question:

Express the following in the form $r(\cos \theta + i \sin \theta)$, where $-\pi < \theta \leq \pi$.

a $e^{\frac{16\pi}{13}i}$

b $4e^{\frac{17\pi}{5}i}$

c $5e^{-\frac{9\pi}{8}i}$

Solution:

$$\begin{aligned} \mathbf{a} \quad e^{\frac{16\pi}{13}i} &= \cos\left(\frac{16\pi}{13}\right) + i \sin\left(\frac{16\pi}{13}\right) \\ &= \cos\left(-\frac{10\pi}{13}\right) + i \sin\left(-\frac{10\pi}{13}\right) \end{aligned}$$

↷ -2π from the argument.

$$\begin{aligned} \mathbf{b} \quad 4e^{\frac{17\pi}{5}i} &= 4\left(\cos\left(\frac{17\pi}{5}\right) + i \sin\left(\frac{17\pi}{5}\right)\right) \\ &= 4\left(\cos\left(\frac{7\pi}{5}\right) + i \sin\left(\frac{7\pi}{5}\right)\right) \\ &= 4\left(\cos\left(-\frac{3\pi}{5}\right) + i \sin\left(-\frac{3\pi}{5}\right)\right) \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad 5e^{-\frac{9\pi}{8}i} &= 5\left(\cos\left(-\frac{9\pi}{8}\right) + i \sin\left(-\frac{9\pi}{8}\right)\right) \\ &= 5\left(\cos\left(\frac{7\pi}{8}\right) + i \sin\left(\frac{7\pi}{8}\right)\right) \end{aligned}$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise A, Question 6

Question:

Use $e^{i\theta} = \cos \theta + i \sin \theta$ to show that $\sin \theta = \frac{1}{2i} (e^{i\theta} - e^{-i\theta})$.

Solution:

$$e^{i\theta} = \cos \theta + i \sin \theta \quad \text{①}$$

$$e^{-i\theta} = \cos(-\theta) + i \sin(-\theta) = \cos \theta - i \sin \theta \quad \text{②}$$

$$\text{①} - \text{②} \Rightarrow e^{i\theta} - e^{-i\theta} = 2i \sin \theta$$

$$\frac{1}{2i} (e^{i\theta} - e^{-i\theta}) = \sin \theta$$

$$\therefore \sin \theta = \frac{1}{2i} (e^{i\theta} - e^{-i\theta}) \text{ (as required)}$$

© Pearson Education Ltd 2009

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise B, Question 1

Question:

Express the following in the form $x + iy$.

a $(\cos 2\theta + i \sin 2\theta)(\cos 3\theta + i \sin 3\theta)$

b $\left(\cos \frac{3\pi}{11} + i \sin \frac{3\pi}{11}\right)\left(\cos \frac{8\pi}{11} + i \sin \frac{8\pi}{11}\right)$

c $3\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) \times 2\left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12}\right)$

d $\sqrt{6}\left(\cos \left(\frac{-\pi}{12}\right) + i \sin \left(\frac{-\pi}{12}\right)\right) \times \sqrt{3}\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$

e $4\left(\cos \left(\frac{-5\pi}{9}\right) + i \sin \left(\frac{-5\pi}{9}\right)\right) \times \frac{1}{2}\left(\cos \left(\frac{-5\pi}{18}\right) + i \sin \left(\frac{-5\pi}{18}\right)\right)$

f $6\left(\cos \frac{\pi}{10} + i \sin \frac{\pi}{10}\right) \times 5\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right) \times \frac{1}{3}\left(\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}\right)$

g $(\cos 4\theta + i \sin 4\theta)(\cos \theta - i \sin \theta)$

h $3\left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12}\right) \times \sqrt{2}\left(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3}\right)$

Solution:

$$\begin{aligned}
 \mathbf{a} \quad & (\cos 2\theta + i \sin 2\theta)(\cos 3\theta + i \sin 3\theta) \\
 & = \cos(2\theta + 3\theta) + i \sin(2\theta + 3\theta) \\
 & = \cos 5\theta + i \sin 5\theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \left(\cos \frac{3\pi}{11} + i \sin \frac{3\pi}{11} \right) \left(\cos \frac{8\pi}{11} + i \sin \frac{8\pi}{11} \right) \\
 & = \cos \left(\frac{3\pi}{11} + \frac{8\pi}{11} \right) + i \sin \left(\frac{3\pi}{11} + \frac{8\pi}{11} \right) \\
 & = \cos \pi + i \sin \pi \\
 & = -1 + i(0) \\
 & = -1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & 3 \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \times 2 \left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right) \\
 & = 3(2) \left(\cos \left(\frac{\pi}{4} + \frac{\pi}{12} \right) + i \sin \left(\frac{\pi}{4} + \frac{\pi}{12} \right) \right) \\
 & = 6 \left(\cos \left(\frac{\pi}{3} \right) + i \sin \left(\frac{\pi}{3} \right) \right) \\
 & = 6 \left(\frac{1}{2} + i \frac{\sqrt{3}}{2} \right) \\
 & = 3 + 3\sqrt{3}i
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \sqrt{6} \left(\cos \left(-\frac{\pi}{12} \right) + i \sin \left(-\frac{\pi}{12} \right) \right) \times \sqrt{3} \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \\
 & = (\sqrt{6})(\sqrt{3}) \left(\cos \left(-\frac{\pi}{12} + \frac{\pi}{3} \right) + i \sin \left(-\frac{\pi}{12} + \frac{\pi}{3} \right) \right) \\
 & = \sqrt{18} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \\
 & = 3 \left(\frac{\sqrt{2}}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right) \\
 & = 3 + 3i
 \end{aligned}$$

$ \begin{aligned} \sqrt{18} &= \sqrt{9}\sqrt{2} \\ &= 3\sqrt{2} \end{aligned} $
--

$$\begin{aligned}
 \mathbf{e} \quad & 4\left(\cos\left(-\frac{5\pi}{9}\right) + i\sin\left(-\frac{5\pi}{9}\right)\right) \times \frac{1}{2}\left(\cos\left(-\frac{5\pi}{18}\right) + i\sin\left(-\frac{5\pi}{18}\right)\right) \\
 &= 4\left(\frac{1}{2}\right)\left(\cos\left(-\frac{5\pi}{9} + -\frac{5\pi}{18}\right) + i\sin\left(-\frac{5\pi}{9} + -\frac{5\pi}{18}\right)\right) \\
 &= 2\left(\cos\left(-\frac{15\pi}{18}\right) + i\sin\left(-\frac{15\pi}{18}\right)\right) \\
 &= 2\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right) \\
 &= 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) \\
 &= -\sqrt{3} - i
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & 6\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right) \times 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \times \frac{1}{3}\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right) \\
 &= 6(5)\left(\frac{1}{3}\right)\left(\cos\left(\frac{\pi}{10} + \frac{\pi}{3} + \frac{2\pi}{5}\right) + i\sin\left(\frac{\pi}{10} + \frac{\pi}{3} + \frac{2\pi}{5}\right)\right) \\
 &= 10\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) \\
 &= 10\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \\
 &= -5\sqrt{3} + 5i
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & (\cos 4\theta + i\sin 4\theta)(\cos \theta - i\sin \theta) \\
 &= (\cos 4\theta + i\sin 4\theta)(\cos(-\theta) + i\sin(-\theta)) \\
 &= \cos(4\theta + -\theta) + i\sin(4\theta + -\theta) \\
 &= \cos 3\theta + i\sin 3\theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & 3\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) \times \sqrt{2}\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) \\
 &= 3\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) \times \sqrt{2}\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) \\
 &= 3(\sqrt{2})\left(\cos\left(\frac{\pi}{12} - \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{12} - \frac{\pi}{3}\right)\right) \\
 &= 3\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) \\
 &= 3\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) \\
 &= 3 - 3i
 \end{aligned}$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise B, Question 2

Question:

Express the following in the form $x + iy$.

a $\frac{\cos 5\theta + i \sin 5\theta}{\cos 2\theta + i \sin 2\theta}$

b $\frac{\sqrt{2}\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)}{\frac{1}{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)}$

c $\frac{3\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)}{4\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)}$

d $\frac{\cos 2\theta - i \sin 2\theta}{\cos 3\theta + i \sin 3\theta}$

Solution:

$$\begin{aligned}
 \mathbf{a} \quad & \frac{\cos 5\theta + i \sin 5\theta}{\cos 2\theta + i \sin 2\theta} \\
 &= \cos(5\theta - 2\theta) + i \sin(5\theta - 2\theta) \\
 &= \cos 3\theta + i \sin 3\theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \frac{\sqrt{2}\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)}{\frac{1}{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)} \\
 &= \frac{\sqrt{2}}{\left(\frac{1}{2}\right)}\left(\cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right) \\
 &= 2\sqrt{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) \\
 &= 2\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) \\
 &= 2 + 2i
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \frac{3\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)}{4\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)} \\
 &= \frac{3}{4}\left(\cos\left(\frac{\pi}{3} - \frac{5\pi}{6}\right) + i \sin\left(\frac{\pi}{3} - \frac{5\pi}{6}\right)\right) \\
 &= \frac{3}{4}\left(\cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right)\right) \\
 &= \frac{3}{4}(0 - i) \\
 &= -\frac{3}{4}i
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \frac{\cos 2\theta - i \sin 2\theta}{\cos 3\theta + i \sin 3\theta} \\
 &= \frac{\cos(-2\theta) + i \sin(-2\theta)}{\cos 3\theta + i \sin 3\theta} \\
 &= \cos(-2\theta - 3\theta) + i \sin(-2\theta - 3\theta) \\
 &= \cos(-5\theta) + i \sin(-5\theta) \text{ or } \cos 5\theta - i \sin 5\theta
 \end{aligned}$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise B, Question 3

Question:

z and w are two complex numbers where

$$z = -9 + 3\sqrt{3}i, |w| = \sqrt{3} \text{ and } \arg w = \frac{7\pi}{12}.$$

Express the following in the form $r(\cos \theta + i \sin \theta)$,

a z ,

b w ,

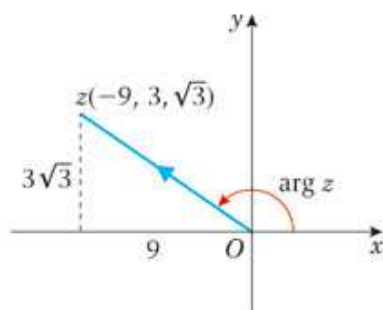
c zw ,

d $\frac{z}{w}$,

where $-\pi < \theta \leq \pi$.

Solution:

a $z = -9 + 3\sqrt{3}i$



$$r = \sqrt{(-9)^2 + (3\sqrt{3})^2} = \sqrt{81 + 27} = \sqrt{108}$$

$$= \sqrt{36} \sqrt{3} = 6\sqrt{3}$$

$$\theta = \arg z = \pi - \tan^{-1}\left(\frac{3\sqrt{3}}{9}\right) = \frac{5\pi}{6}$$

$$z = 6\sqrt{3} \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)$$

b $r = |w| = \sqrt{3}$

$$\theta = \arg w = \frac{7\pi}{12}$$

$$w = \sqrt{3} \left(\cos \frac{7\pi}{12} + i \sin \frac{7\pi}{12} \right)$$

$$\begin{aligned}\mathbf{c} \quad zw &= 6\sqrt{3}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) \times \sqrt{3}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right) \\ &= (6\sqrt{3})(\sqrt{3})\left(\cos\left(\frac{5\pi}{6} + \frac{7\pi}{12}\right) + i\sin\left(\frac{5\pi}{6} + \frac{7\pi}{12}\right)\right) \\ &= 18\left(\cos\left(\frac{17\pi}{12}\right) + i\sin\left(\frac{17\pi}{12}\right)\right) \\ &= 18\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad \frac{z}{w} &= \frac{6\sqrt{3}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}{\sqrt{3}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)} \\ &= \frac{6\sqrt{3}}{\sqrt{3}}\left(\cos\left(\frac{5\pi}{6} - \frac{7\pi}{12}\right) + i\sin\left(\frac{5\pi}{6} - \frac{7\pi}{12}\right)\right) \\ &= 6\left(\cos\left(\frac{3\pi}{12}\right) + i\sin\left(\frac{3\pi}{12}\right)\right) \\ &= 6\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\end{aligned}$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise C, Question 1

Question:

Use de Moivre's theorem to simplify each of the following:

a $(\cos \theta + i \sin \theta)^6$

b $(\cos 3\theta + i \sin 3\theta)^4$

c $\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)^5$

d $\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)^8$

e $\left(\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}\right)^5$

f $\left(\cos \frac{\pi}{10} - i \sin \frac{\pi}{10}\right)^{15}$

g $\frac{\cos 5\theta + i \sin 5\theta}{(\cos 2\theta + i \sin 2\theta)^2}$

h $\frac{(\cos 2\theta + i \sin 2\theta)^7}{(\cos 4\theta + i \sin 4\theta)^3}$

i $\frac{1}{(\cos 2\theta + i \sin 2\theta)^3}$

j $\frac{(\cos 2\theta + i \sin 2\theta)^4}{(\cos 3\theta + i \sin 3\theta)^3}$

k $\frac{\cos 5\theta + i \sin 5\theta}{(\cos 3\theta - i \sin 3\theta)^2}$

l $\frac{\cos \theta - i \sin \theta}{(\cos 2\theta - i \sin 2\theta)^3}$

Solution:

$$\mathbf{a} \quad (\cos \theta + i \sin \theta)^6 \\ = \cos 6\theta + i \sin 6\theta$$

$$\mathbf{b} \quad (\cos 3\theta + i \sin 3\theta)^4 \\ = \cos(4(3\theta)) + i \sin(4(3\theta)) \\ = \cos 12\theta + i \sin 12\theta$$

$$\mathbf{c} \quad \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)^5 \\ = \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \\ = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$\mathbf{d} \quad \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)^8 \\ = \cos \frac{8\pi}{3} + i \sin \frac{8\pi}{3} \\ = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \\ = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$\mathbf{e} \quad \left(\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}\right)^5 \\ = \cos \frac{10\pi}{5} + i \sin \frac{10\pi}{5} \\ = \cos 2\pi + i \sin 2\pi \\ = \cos 0 + i \sin 0 \\ = 1 + i(0) \\ = 1$$

$$\mathbf{f} \quad \left(\cos \frac{\pi}{10} - i \sin \frac{\pi}{10}\right)^{15} \\ = \left(\cos \left(-\frac{\pi}{10}\right) + i \sin \left(-\frac{\pi}{10}\right)\right)^{15} \\ = \cos \left(-\frac{15\pi}{10}\right) + i \sin \left(-\frac{15\pi}{10}\right) \\ = \cos \left(\frac{5\pi}{10}\right) + i \sin \left(\frac{5\pi}{10}\right) \\ = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \\ = 0 + i \\ = i$$

$$\begin{aligned} \mathbf{g} \quad & \frac{\cos 5\theta + i \sin 5\theta}{(\cos 2\theta + i \sin 2\theta)^2} \\ &= \frac{\cos 5\theta + i \sin 5\theta}{\cos 4\theta + i \sin 4\theta} \\ &= \cos(5\theta - 4\theta) + i \sin(5\theta - 4\theta) \\ &= \cos \theta + i \sin \theta \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & \frac{(\cos 2\theta + i \sin 2\theta)^7}{(\cos 4\theta + i \sin 4\theta)^3} \\ &= \frac{\cos 14\theta + i \sin 14\theta}{\cos 12\theta + i \sin 12\theta} \\ &= \cos(14\theta - 12\theta) + i \sin(14\theta - 12\theta) \\ &= \cos 2\theta + i \sin 2\theta \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & \frac{1}{(\cos 2\theta + i \sin 2\theta)^3} \\ &= (\cos 2\theta + i \sin 2\theta)^{-3} \\ &= \cos(-6\theta) + i \sin(-6\theta) \\ &= \cos 6\theta - i \sin 6\theta \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad & \frac{(\cos 2\theta + i \sin 2\theta)^4}{(\cos 3\theta + i \sin 3\theta)^3} \\ &= \frac{\cos 8\theta + i \sin 8\theta}{\cos 9\theta + i \sin 9\theta} \\ &= \cos(8\theta - 9\theta) + i \sin(8\theta - 9\theta) \\ &= \cos(-\theta) + i \sin(-\theta) \\ &= \cos \theta - i \sin \theta \end{aligned}$$

$$\begin{aligned} \mathbf{k} \quad & \frac{\cos 5\theta + i \sin 5\theta}{(\cos 3\theta + i \sin 3\theta)^2} \\ &= \frac{\cos 5\theta + i \sin 5\theta}{(\cos(-3\theta) + i \sin(-3\theta))^2} \\ &= \frac{\cos 5\theta + i \sin 5\theta}{\cos(-6\theta) + i \sin(-6\theta)} \\ &= \cos(5\theta - -6\theta) + i \sin(5\theta - -6\theta) \\ &= \cos 11\theta - i \sin 11\theta \end{aligned}$$

$$\begin{aligned} \mathbf{l} \quad & \frac{\cos \theta - i \sin \theta}{(\cos 2\theta - i \sin 2\theta)^3} \\ &= \frac{\cos(-\theta) - i \sin(-\theta)}{(\cos(-2\theta) - i \sin(-2\theta))^3} \\ &= \frac{\cos(-\theta) - i \sin(-\theta)}{\cos(-6\theta) - i \sin(-6\theta)} \\ &= \cos(-\theta - -6\theta) + i \sin(-\theta - -6\theta) \\ &= \cos 5\theta - i \sin 5\theta \end{aligned}$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise C, Question 2

Question:

Evaluate $\frac{\left(\cos \frac{7\pi}{13} + i \sin \frac{7\pi}{13}\right)^4}{\left(\cos \frac{4\pi}{13} - i \sin \frac{4\pi}{13}\right)^6}$.

Solution:

$$\begin{aligned} & \frac{\left(\cos \frac{7\pi}{13} + i \sin \frac{7\pi}{13}\right)^4}{\left(\cos \frac{4\pi}{13} - i \sin \frac{4\pi}{13}\right)^6} \\ &= \frac{\left(\cos \frac{7\pi}{13} + i \sin \frac{7\pi}{13}\right)^4}{\left(\cos \left(-\frac{4\pi}{13}\right) - i \sin \left(-\frac{4\pi}{13}\right)\right)^6} \\ &= \frac{\cos \left(\frac{28\pi}{13}\right) + i \sin \left(\frac{28\pi}{13}\right)}{\cos \left(-\frac{24\pi}{13}\right) - i \sin \left(-\frac{24\pi}{13}\right)} \\ &= \cos \left(\frac{28\pi}{13} - -\frac{24\pi}{13}\right) + i \sin \left(\frac{28\pi}{13} - -\frac{24\pi}{13}\right) \\ &= \cos \left(\frac{52\pi}{13}\right) + i \sin \left(\frac{52\pi}{13}\right) \\ &= \cos 4\pi + i \sin 4\pi \\ &= \cos 0 + i \sin 0 \\ &= 1 + i(0) \\ &= 1 \end{aligned}$$

© Pearson Education Ltd 2009

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise C, Question 3

Question:

Express the following in the form $x + iy$ where $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

a $(1 + i)^5$

b $(-2 + 2i)^8$

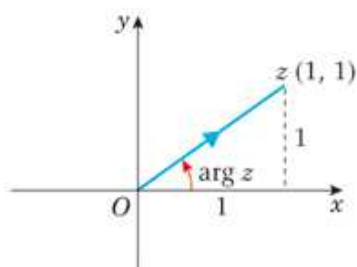
c $(1 - i)^6$

d $(1 - \sqrt{3}i)^6$

e $\left(\frac{3}{2} - \frac{1}{2}\sqrt{3}i\right)^9$

f $(-2\sqrt{3} - 2i)^5$

Solution:

a $(1 + i)^5$ If $z = 1 + i$, then

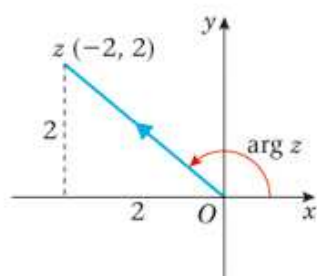
$$r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\theta = \arg z = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$\text{So, } 1 + i = \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

$$\begin{aligned} \therefore (1 + i)^5 &= \left[\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right]^5 \\ &= (\sqrt{2})^5 \left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \right) \\ &= 4\sqrt{2} \left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \right) \\ &= 4\sqrt{2} \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right) \\ &= -4 - 4i \end{aligned}$$

$$\begin{aligned} (\sqrt{2})^5 &= \sqrt{2} \sqrt{2} \sqrt{2} \sqrt{2} \sqrt{2} \\ &= 4\sqrt{2} \end{aligned}$$

Therefore, $(1 + i)^5 = -4 - 4i$ **b** $(-2 + 2i)^8$ If $z = -2 + 2i$, then

$$r = \sqrt{(-2)^2 + 2^2} = \sqrt{4 + 4} = \sqrt{8} = \sqrt{4} \sqrt{2} = 2\sqrt{2}$$

$$\theta = \arg z = \pi - \tan^{-1}\left(\frac{2}{2}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

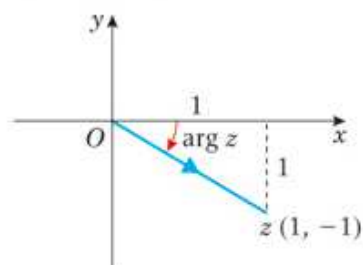
$$\text{So, } -2 + 2i = 2\sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$$

$$\begin{aligned} \therefore (-2 + 2i)^8 &= \left[2\sqrt{2} \left(\cos \left(\frac{3\pi}{4} \right) + i \sin \left(\frac{3\pi}{4} \right) \right) \right]^8 \\ &= (2\sqrt{2})^8 \left(\cos \left(\frac{24\pi}{4} \right) + i \sin \left(\frac{24\pi}{4} \right) \right) \\ &= (256)(16) (\cos 6\pi + i \sin 6\pi) \\ &= 4096 (1 + i(0)) \\ &= 4096 \end{aligned}$$

Therefore, $(-2 + 2i)^8 = 4096$

c $(1 - i)^6$

If $z = 1 - i$, then



$$r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\theta = \arg z = \pi - \tan^{-1}\left(\frac{1}{1}\right) = \pi - \frac{\pi}{4}$$

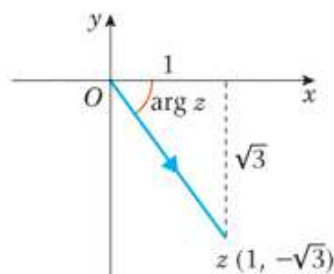
$$\text{So, } 1 - i = \sqrt{2} \left(\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right)$$

$$\begin{aligned} \therefore (1 - i)^6 &= \left[\sqrt{2} \left(\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right) \right]^6 \\ &= (\sqrt{2})^6 \left(\cos \left(-\frac{6\pi}{4} \right) + i \sin \left(-\frac{6\pi}{4} \right) \right) \\ &= 8 \left(\cos \left(-\frac{3\pi}{2} \right) + i \sin \left(-\frac{3\pi}{2} \right) \right) \\ &= 8(0 + i) \\ &= 8i \end{aligned}$$

Therefore, $(1 - i)^6 = 8i$

d $(1 - \sqrt{3}i)^6$

If $z = 1 - \sqrt{3}i$, then



$$r = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{1 + 3} = \sqrt{4} = 2$$

$$\theta = \arg z = -\tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = -\frac{\pi}{3}$$

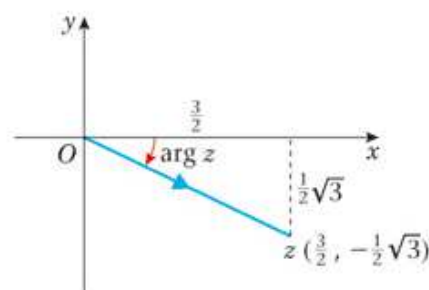
$$\text{So, } 1 - \sqrt{3}i = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$$

$$\begin{aligned} \therefore (1 - \sqrt{3}i)^6 &= \left[2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)\right]^6 \\ &= (2)^6 \left(\cos\left(-\frac{6\pi}{3}\right) + i\sin\left(-\frac{6\pi}{3}\right)\right) \\ &= 64(\cos(-2\pi) + i\sin(-2\pi)) \\ &= 64(1 + i(0)) \\ &= 64 \end{aligned}$$

Therefore, $(1 - \sqrt{3}i)^6 = 64$

e $\left(\frac{3}{2} - \frac{1}{2}\sqrt{3}i\right)^9$

If $z = \frac{3}{2} - \frac{1}{2}\sqrt{3}i$, then



$$r = \sqrt{\left(\frac{3}{2}\right)^2 + \left(-\frac{1}{2}\sqrt{3}\right)^2} = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{\frac{12}{4}} = \sqrt{3}$$

$$\theta = \arg z = -\tan^{-1}\left(\frac{\frac{1}{2}\sqrt{3}}{\frac{3}{2}}\right) = -\tan^{-1}\frac{\sqrt{3}}{3} = -\frac{\pi}{6}$$

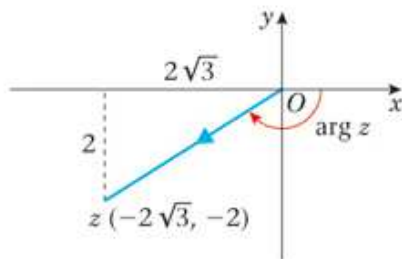
$$\text{So, } \frac{3}{2} - \frac{1}{2}\sqrt{3}i = \sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$

$$\begin{aligned} \therefore \left(\frac{3}{2} - \frac{1}{2}\sqrt{3}i\right)^9 &= \left[\sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)\right]^9 \\ &= (\sqrt{3})^9 \left(\cos\left(-\frac{9\pi}{6}\right) + i\sin\left(-\frac{9\pi}{6}\right)\right) \\ &= 81\sqrt{3} \left(\cos\left(-\frac{3\pi}{2}\right) + i\sin\left(-\frac{3\pi}{2}\right)\right) \\ &= 81\sqrt{3}(0 + i) \\ &= 81\sqrt{3}i \end{aligned}$$

$$\text{Therefore, } \left(\frac{3}{2} - \frac{1}{2}\sqrt{3}i\right)^9 = 81\sqrt{3}i$$

f $(-2\sqrt{3} - 2i)^5$

If $z = -2\sqrt{3} - 2i$, then



$$r = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{12 + 4} = \sqrt{16} = 4$$

$$\theta = \arg z = -\pi - \tan^{-1}\left(\frac{2}{2\sqrt{3}}\right) = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$$

$$\text{So, } -2\sqrt{3} - 2i = 4\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$$

$$\begin{aligned} \therefore (-2\sqrt{3} - 2i)^5 &= \left[4\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)\right]^5 \\ &= 4^5 \left(\cos\left(-\frac{25\pi}{6}\right) + i\sin\left(-\frac{25\pi}{6}\right)\right) \\ &= 1024\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) \\ &= 1024\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) \\ &= 512\sqrt{3} - 512i \end{aligned}$$

$$\text{Therefore, } (-2\sqrt{3} - 2i)^5 = 512\sqrt{3} - 512i$$

© Pearson Education Ltd 2009

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise C, Question 4

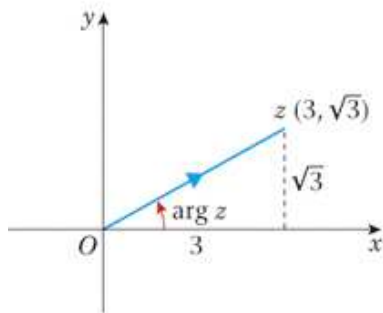
Question:

Express $(3 + \sqrt{3}i)^5$ in the form $a + b\sqrt{3}i$ where a and b are integers.

Solution:

$$(3 + \sqrt{3}i)^5$$

If $z = 3 + \sqrt{3}i$, then



$$r = \sqrt{3^2 + (\sqrt{3})^2} = \sqrt{9 + 3} = \sqrt{12} = \sqrt{4} \sqrt{3} = 2\sqrt{3}$$

$$\theta = \arg z = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$$

$$\text{So, } 3 + \sqrt{3}i = 2\sqrt{3} \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)$$

$$\begin{aligned} \therefore (3 + \sqrt{3}i)^5 &= \left[2\sqrt{3} \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) \right]^5 \\ &= (2\sqrt{3})^5 \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right) \\ &= 32(9\sqrt{3}) \left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i \right) \\ &= 288\sqrt{3} \left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i \right) \\ &= -144\sqrt{3}\sqrt{3} + 144\sqrt{3}i \\ &= -432 + 144\sqrt{3}i \end{aligned}$$

$$\text{Therefore, } (3 + \sqrt{3}i)^5 = -432 + 144\sqrt{3}i$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise D, Question 1

Question:

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$$

Solution:

$$(\cos \theta + i \sin \theta)^3 = \cos^3 \theta + i \sin^3 \theta$$

$$= \cos^3 \theta + {}^3C_1 \cos^2 \theta (i \sin \theta)$$

$$+ {}^3C_2 \cos \theta (i \sin \theta)^2 + (i \sin \theta)^3$$

$$= \cos^3 \theta + 3i \cos^2 \theta \sin \theta + 3i^2 \cos \theta \sin^2 \theta + i^3 \sin^3 \theta$$

$$= \cos^3 \theta + 3i \cos^2 \theta \sin \theta - 3 \cos \theta \sin^2 \theta - i \sin^3 \theta$$

Hence,

$$\cos 3\theta + i \sin 3\theta = \cos^3 \theta + 3i \cos^2 \theta \sin \theta - 3 \cos \theta \sin^2 \theta - i \sin^3 \theta$$

Equating the imaginary parts gives,

$$\sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta$$

$$= 3(1 - \sin^2 \theta) \sin \theta - \sin^3 \theta$$

$$= 3 \sin \theta (1 - \sin^2 \theta) - \sin^3 \theta$$

$$= 3 \sin \theta - 3 \sin^3 \theta - \sin^3 \theta$$

$$= 3 \sin \theta - 4 \sin^3 \theta$$

Hence, $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$ (as required)

de Moivre's Theorem.

Binomial expansion.

Applying $\cos^2 \theta = 1 - \sin^2 \theta$.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise D, Question 2

Question:

$$\sin 5\theta = 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$$

Solution:

$$\begin{aligned} (\cos \theta + i \sin \theta)^5 &= \cos 5\theta + i \sin 5\theta \\ &= \cos^5 \theta + {}^5C_1 \cos^4 \theta (i \sin \theta) + {}^5C_2 \cos^3 \theta (i \sin \theta)^2 \\ &\quad + {}^5C_3 \cos^2 \theta (i \sin \theta)^3 + {}^5C_4 \cos \theta (i \sin \theta)^4 + (i \sin \theta)^5 \\ &= \cos^5 \theta + 5i \cos^4 \theta \sin \theta + 10i^2 \cos^3 \theta \sin^2 \theta + 10i^3 \cos^2 \theta \sin^3 \theta \\ &\quad + 5i^4 \cos \theta \sin^4 \theta + i^5 \sin^5 \theta \end{aligned}$$

de Moivre's Theorem.

Binomial expansion.

Hence,

$$\begin{aligned} \cos 5\theta + i \sin 5\theta &= \cos^5 \theta + 5i \cos^4 \theta \sin \theta - 10 \cos^3 \theta \sin^2 \theta - 10i \cos^2 \theta \sin^3 \theta \\ &\quad + 5 \cos \theta \sin^4 \theta + i \sin^5 \theta \end{aligned}$$

Equating the imaginary parts gives,

$$\begin{aligned} \sin 5\theta &= 5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta \\ &= 5 (\cos^2 \theta)^2 \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta \\ &= 5 (1 - \sin^2 \theta)^2 \sin \theta - 10 (1 - \sin^2 \theta) \sin^3 \theta + \sin^5 \theta \\ &= 5 \sin \theta (1 - 2 \sin^2 \theta + \sin^4 \theta) - 10 \sin^3 \theta (1 - \sin^2 \theta) + \sin^5 \theta \\ &= 5 \sin \theta - 10 \sin^3 \theta + 5 \sin^5 \theta - 10 \sin^3 \theta + 10 \sin^5 \theta + \sin^5 \theta \\ &= 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta \end{aligned}$$

Applying $\cos^2 \theta = 1 - \sin^2 \theta$.

Hence, $\sin 5\theta = 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$ (as required)

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise D, Question 3

Question:

$$\cos 7\theta = 64 \cos^7 \theta - 112 \cos^5 \theta + 56 \cos^3 \theta - 7 \cos \theta$$

Solution:

$$(\cos \theta + i \sin \theta)^7 = \cos 7\theta + i \sin 7\theta$$

$$\begin{aligned} &= \cos^7 \theta + {}^7C_1 \cos^6 \theta (i \sin \theta) + {}^7C_2 \cos^5 \theta (i \sin \theta)^2 \\ &\quad + {}^7C_3 \cos^4 \theta (i \sin \theta)^3 + {}^7C_4 \cos^3 \theta (i \sin \theta)^4 + {}^7C_5 \cos^2 \theta (i \sin \theta)^5 \\ &\quad + {}^7C_6 \cos \theta (i \sin \theta)^6 + (i \sin \theta)^7 \end{aligned}$$

de Moivre's Theorem.

$$\begin{aligned} &= \cos^7 \theta + 7i \cos^6 \theta \sin \theta + 21i^2 \cos^5 \theta \sin^2 \theta \\ &\quad + 35i^3 \cos^4 \theta \sin^3 \theta + 35i^4 \cos^3 \theta \sin^4 \theta + 21i^5 \cos^2 \theta \sin^5 \theta \\ &\quad + 7i^6 \cos \theta \sin^6 \theta + i^7 \sin^7 \theta \end{aligned}$$

Binomial expansion.

Hence,

$$\begin{aligned} \cos 7\theta + i \sin 7\theta &= \cos^7 \theta + 7i \cos^6 \theta \sin \theta - 21 \cos^5 \theta \sin^2 \theta \\ &\quad - 35i^3 \cos^4 \theta \sin^3 \theta + 35i^4 \cos^3 \theta \sin^4 \theta + 21i^5 \cos^2 \theta \sin^5 \theta \\ &\quad - 7 \cos \theta \sin^6 \theta - i \sin^7 \theta \end{aligned}$$

Equating the imaginary parts gives,

$$\begin{aligned} \cos 7\theta &= \cos^7 \theta - 21 \cos^5 \theta \sin^2 \theta + 35 \cos^3 \theta \sin^4 \theta - 7 \cos \theta \sin^6 \theta \\ &= \cos^7 \theta - 21 \cos^5 \theta (1 - \cos^2 \theta) + 35 \cos^3 \theta (1 - \cos^2 \theta)^2 \\ &\quad - 7 \cos \theta (1 - \cos^2 \theta)^3 \\ &= \cos^7 \theta - 21 \cos^5 \theta + 21 \cos^7 \theta + 35 \cos^3 \theta (1 - 2 \cos^2 \theta + \cos^4 \theta) \\ &\quad - 7 \cos \theta (1 - 3 \cos^2 \theta + 3 \cos^4 \theta - \cos^6 \theta) \\ &= \cos^7 \theta - 21 \cos^5 \theta + 21 \cos^7 \theta + 35 \cos^3 \theta - 70 \cos^5 \theta + 35 \cos^7 \theta \\ &\quad - 7 \cos \theta + 21 \cos^3 \theta - 21 \cos^5 \theta + 7 \cos^7 \theta \\ &= 64 \cos^7 \theta - 112 \cos^5 \theta + 56 \cos^3 \theta - 7 \cos \theta \end{aligned}$$

Applying
 $\cos^2 \theta = 1 - \sin^2 \theta$.

Hence, $\cos 7\theta = 64 \cos^7 \theta - 112 \cos^5 \theta + 56 \cos^3 \theta - 7 \cos \theta$ (as required)

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise D, Question 4

Question:

$$\cos^4 \theta = \frac{1}{8} (\cos 4\theta + 4 \cos 2\theta + 3)$$

Solution:

Let $z = \cos \theta + i \sin \theta$

$$\left(z + \frac{1}{z}\right)^4 = (2 \cos \theta)^4 = 16 \cos^4 \theta$$

$$z + \frac{1}{z} = 2 \cos \theta$$

$$= z^4 + {}^4C_1 z^3 \left(\frac{1}{z}\right) + {}^4C_2 z^2 \left(\frac{1}{z}\right)^2 + {}^4C_3 z \left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4$$

$$= z^4 + 4z^3 \left(\frac{1}{z}\right) + 6z^2 \left(\frac{1}{z^2}\right) + 4z \left(\frac{1}{z^3}\right) + \frac{1}{z^4}$$

$$= z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$$

$$= \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$$

$$= 2 \cos 4\theta + 4(2 \cos 2\theta) + 6$$

$$z^n + \frac{1}{z^n} = 2 \cos n\theta$$

So, $16 \cos^4 \theta = 2 \cos 4\theta + 8 \cos 2\theta + 6$

$$16 \cos^4 \theta = 2(\cos 4\theta + 4 \cos 2\theta + 3)$$

$$\cos^4 \theta = \frac{2}{16} (\cos 4\theta + 4 \cos 2\theta + 3)$$

Therefore, $\cos^4 \theta = \frac{1}{8} (\cos 4\theta + 4 \cos 2\theta + 3)$ (as required)

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise D, Question 5

Question:

$$\sin^5 \theta = \frac{1}{16} (\sin 5\theta - 5 \sin 3\theta + 10 \sin \theta)$$

Solution:

Let $z = \cos \theta + i \sin \theta$

$$\begin{aligned} \left(z + \frac{1}{z}\right)^5 &= (2i \sin \theta)^5 = 32i^5 \sin^5 \theta = 32i \sin^5 \theta \quad \left\{ \begin{array}{l} z - \frac{1}{z} = 2i \sin \theta \\ z^n + \frac{1}{z^n} = 2i \sin n\theta \end{array} \right. \\ &= z^5 + {}^5C_1 z^4 \left(-\frac{1}{z}\right) + {}^5C_2 z^3 \left(-\frac{1}{z}\right)^2 + {}^5C_3 z^2 \left(-\frac{1}{z}\right)^3 + {}^5C_4 z \left(-\frac{1}{z}\right)^4 + \left(-\frac{1}{z}\right)^5 \\ &= z^5 + 5z^4 \left(-\frac{1}{z}\right) + 10z^3 \left(-\frac{1}{z}\right)^2 + 10z^2 \left(-\frac{1}{z}\right)^3 + 5z \left(-\frac{1}{z}\right)^4 + \left(-\frac{1}{z}\right)^5 \\ &= z^5 + 5z^4 \left(\frac{1}{z}\right) + 10z^3 \left(\frac{1}{z^2}\right) - 10z^2 \left(\frac{1}{z^3}\right) + 5z \left(\frac{1}{z^4}\right) - \frac{1}{z^5} \\ &= z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5} \\ &= \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right) \\ &= 2i \sin 5\theta - 5(2i \sin 3\theta) + 10(2i \sin \theta) \quad \left\{ \begin{array}{l} z^n + \frac{1}{z^n} = 2i \sin n\theta \end{array} \right. \end{aligned}$$

$$\text{So, } 32i \sin^5 \theta = 2i \sin 5\theta - 10i \sin 3\theta + 20i \sin \theta \quad (\div 2i)$$

$$16 \sin^5 \theta = \sin 5\theta - 5 \sin 3\theta + 10 \sin \theta$$

$$\sin^5 \theta = \frac{1}{16} (\sin 5\theta - 5 \sin 3\theta + 10 \sin \theta)$$

$$\text{Therefore, } \sin^5 \theta = \frac{1}{16} (\sin 5\theta - 5 \sin 3\theta + 10 \sin \theta)$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise D, Question 6

Question:

a Show that $32 \cos^6 \theta = \cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10$.

b Hence find $\int_0^{\frac{\pi}{6}} \cos^6 \theta \, d\theta$ in the form $a\pi + b\sqrt{3}$ where a and b are constants.

Solution:

Let $z = \cos \theta + i \sin \theta$

$$\begin{aligned}
 \mathbf{a} \quad \left(z + \frac{1}{z}\right)^6 &= (2 \cos \theta)^6 = 64 \cos^6 \theta \quad \leftarrow \boxed{z - \frac{1}{z} = 2 \cos \theta} \\
 &= z^6 + {}^6C_1 z^5 \left(\frac{1}{z}\right) + {}^6C_2 z^4 \left(\frac{1}{z}\right)^2 + {}^6C_3 z^3 \left(\frac{1}{z}\right)^3 + {}^6C_4 z^2 \left(\frac{1}{z}\right)^4 + {}^6C_5 z \left(\frac{1}{z}\right)^5 + \left(\frac{1}{z}\right)^6 \\
 &= z^6 + 6z^5 \left(\frac{1}{z}\right) + 15z^4 \left(\frac{1}{z^2}\right) + 20z^3 \left(\frac{1}{z^3}\right) + 15z^2 \left(\frac{1}{z^4}\right) + 6z \left(\frac{1}{z^5}\right) + \frac{1}{z^6} \\
 &= z^6 - 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6} \\
 &= \left(z^6 - \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20 \\
 &= 2 \cos 6\theta + 6(2 \cos 4\theta) + 15(2 \cos 2\theta) + 20 \quad \leftarrow \boxed{z^n + \frac{1}{z^n} = 2 \cos n\theta}
 \end{aligned}$$

$$\text{So, } 64 \cos^6 \theta = 2 \cos 6\theta + 12 \cos 4\theta + 30 \cos 2\theta + 20$$

$$32 \cos^6 \theta = \cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10 \text{ (as required)}$$

$$\begin{aligned}
 \mathbf{b} \quad \int_0^{\frac{\pi}{6}} \cos^6 \theta &= \frac{1}{32} \int_0^{\frac{\pi}{6}} \cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10\theta \\
 &= \frac{1}{32} \left[\frac{\sin^6 \theta}{6} + \frac{6 \sin^4 \theta}{4} + \frac{15 \sin^2 \theta}{2} + 10\theta \right]_0^{\frac{\pi}{6}} \\
 &= \frac{1}{32} \left[\left(\frac{\sin(\frac{\pi}{6})}{6} + \frac{6 \sin^4(\frac{\pi}{6})}{4} + \frac{15 \sin^2(\frac{\pi}{6})}{2} + \frac{10\pi}{6} \right) - (0) \right] \\
 &= \frac{1}{32} \left[0 + \frac{3\sqrt{3}}{2} + \frac{15\sqrt{3}}{2} + \frac{5\pi}{3} \right] \\
 &= \frac{1}{32} \left[\frac{3\sqrt{3}}{4} + \frac{15\sqrt{3}}{4} + \frac{5\pi}{3} \right] \\
 &= \frac{1}{32} \left[\frac{9\sqrt{3}}{2} + \frac{5\pi}{3} \right] \\
 &= \frac{5\pi}{96} + \frac{9}{64} \sqrt{3}
 \end{aligned}$$

$$\therefore \int_0^{\frac{\pi}{6}} \cos^6 \theta = \frac{5\pi}{96} + \frac{9}{64} \sqrt{3}$$

$$\boxed{a = \frac{5}{96}, b = \frac{9}{64}}$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise D, Question 7

Question:

- a** Use de Moivre's theorem to show that $\sin 4\theta = 4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta$.
- b** Hence, or otherwise, show that $\tan 4\theta = \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta}$.
- c** Use your answer to part **b** to find, to 2 d.p., the four solutions of the equation $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$.

Solution:

$$\begin{aligned}
 \mathbf{a} \quad (\cos \theta + i \sin \theta)^4 &= \cos 4\theta + i \sin 4\theta && \text{de Moivre's Theorem.} \\
 &= \cos^4 \theta + {}^4C_1 \cos^3 \theta (i \sin \theta) + {}^4C_2 \cos^2 \theta (i \sin \theta)^2 \\
 &\quad + {}^4C_3 \cos \theta (i \sin \theta)^3 + (i \sin \theta)^4 && \text{Binomial expansion.} \\
 &= \cos^4 \theta + 4i \cos^3 \theta \sin \theta + 6i^2 \cos^2 \theta \sin^2 \theta \\
 &\quad + 4i^3 \cos \theta \sin^3 \theta + i^4 \sin^4 \theta \\
 &= \cos^4 \theta + 4i \cos^3 \theta \sin \theta - 6 \cos^2 \theta \sin^2 \theta - 4i \cos \theta \sin^3 \theta + \sin^4 \theta
 \end{aligned}$$

Hence,

$$\cos 4\theta + i \sin 4\theta = \cos^4 \theta + 4i \cos^3 \theta \sin \theta - 6 \cos^2 \theta \sin^2 \theta - 4i \cos \theta \sin^3 \theta + \sin^4 \theta \quad \textcircled{1}$$

Equating the imaginary parts of $\textcircled{1}$ gives:

$$\sin^4 \theta = 4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta \text{ (as required)}$$

b Equating the real parts of $\textcircled{1}$ gives:

$$\cos 4\theta = \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta$$

$$\tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta} = \frac{4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta}{\cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta} \quad \frac{(\cos 4\theta \div \cos^4 \theta)}{(\cos 4\theta \div \cos^4 \theta)}$$

$$\begin{aligned}
 &= \frac{\frac{4 \cos^3 \theta \sin \theta}{\cos^4 \theta} - \frac{4 \cos \theta \sin^3 \theta}{\cos^4 \theta}}{\frac{\cos^4 \theta}{\cos^4 \theta} - \frac{6 \cos^2 \theta \sin^2 \theta}{\cos^4 \theta} + \frac{\sin^4 \theta}{\cos^4 \theta}}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\frac{4 \cancel{\cos^3} \theta \sin \theta}{\cancel{\cos^3} \theta \cos \theta} - \frac{4 \cancel{\cos} \theta \sin^3 \theta}{\cancel{\cos} \theta \cos^3 \theta}}{\frac{\cancel{\cos^4} \theta}{\cancel{\cos^4} \theta} - \frac{6 \cancel{\cos^2} \theta \sin^2 \theta}{\cancel{\cos^2} \theta \cos^2 \theta} + \frac{\sin^4 \theta}{\cos^4 \theta}}
 \end{aligned}$$

$$= \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta}$$

$$\text{Therefore, } \tan^4 \theta = \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta} \text{ (as required)}$$

$$\text{c } x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$$

$$x^4 - 6x^2 + 1 = 4x - 4x^3$$

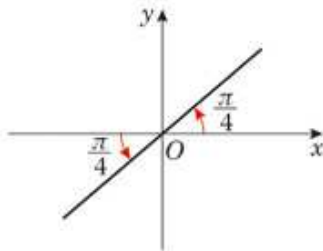
$$1 = \frac{4x - 4x^3}{x^4 - 6x^2 + 1} \quad \textcircled{2}$$

Let $x = \tan \theta$; then

$$\textcircled{2} \Rightarrow \frac{4 \tan \theta - 4 \tan^3 \theta}{\tan^4 \theta - 6 \tan^2 \theta + 1} = 1$$

$$\tan 4\theta = 1 \quad \leftarrow \text{From part b.}$$

$$\alpha = \frac{\pi}{4}$$



$$4\theta = \left\{ \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \dots \right\}$$

$$\theta = \left\{ \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}, \dots \right\}$$

$$\therefore x = \tan \theta = \tan \frac{\pi}{16}, \tan \frac{5\pi}{16}, \tan \frac{9\pi}{16}, \tan \frac{13\pi}{16}$$

$$x = 0.19891\dots, 1.49660\dots, -5.02733\dots, -0.66817\dots,$$

$$x = 0.20, 1.50, -5.03, -0.67 \text{ (2 d.p.)}$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise E, Question 1

Question:

Solve the following equations, expressing your answers for z in the form $x + iy$, where $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

a $z^4 - 1 = 0$

b $z^3 - i = 0$

c $z^3 = 27$

d $z^4 + 64 = 0$

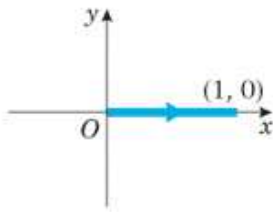
e $z^4 + 4 = 0$

f $z^3 + 8i = 0$

Solution:

a $z^4 - 1 = 0$

$$z^4 = 1$$



for 1, $r = 1$ and $\theta = 0$

So $z^4 = 1 (\cos 0 + i \sin 0)$

$$z^4 = \cos(0 + 2k\pi) + i \sin(0 + 2k\pi) \quad k \in \mathbb{Z}$$

Hence, $z = [\cos(2k\pi) + i \sin(2k\pi)]^{\frac{1}{4}}$

$$z = \cos\left(\frac{2k\pi}{4}\right) + i \sin\left(\frac{2k\pi}{4}\right)$$

de Moivre's Theorem.

$$z = \cos\left(\frac{k\pi}{2}\right) + i \sin\left(\frac{k\pi}{2}\right)$$

$$k = 0, z = \cos 0 + i \sin 0 = 1$$

$$k = 1, z = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = i$$

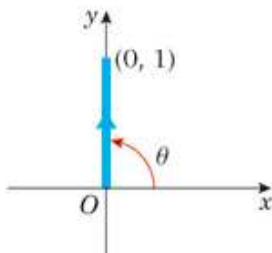
$$k = 2, z = \cos \pi + i \sin \pi = -1$$

$$k = -1, z = \cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right) = -i$$

Therefore, $z = 1, i, -1, -i$

b $z^3 - i = 0$

$$z^3 = i$$



for i , $r = 1$ and $\theta = \frac{\pi}{2}$

So $z^3 = 1 \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)$

$$z^3 = \cos\left(\frac{\pi}{2} + 2k\pi\right) + i \sin\left(\frac{\pi}{2} + 2k\pi\right), \quad k \in \mathbb{Z}$$

$$\text{Hence, } z = \left[\cos\left(\frac{\pi}{2} + 2k\pi\right) + i \sin\left(\frac{\pi}{2} + 2k\pi\right) \right]^{\frac{1}{3}}$$

$$z = \cos\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right) + i \sin\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right)$$

de Moivre's Theorem.

$$z = \cos\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right) + i \sin\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)$$

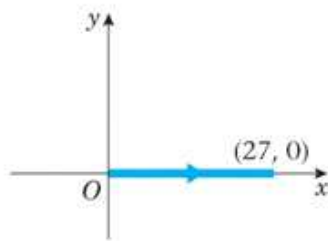
$$\therefore k = 0, z = \cos\frac{\pi}{6} + i \sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$k = 1, z = \cos\frac{5\pi}{6} + i \sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$k = -1, z = \cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right) = 0 - i$$

$$\text{Therefore, } z = \frac{\sqrt{3}}{2} + \frac{1}{2}i, -\frac{\sqrt{3}}{2} + \frac{1}{2}i, -i$$

c $z^3 = 27$



for 27, $r = 27$ and $\theta = 0$

$$\text{So } z^3 = 27(\cos 0 + i \sin 0)$$

$$z^3 = 27[\cos(0 + 2k\pi) + i \sin(0 + 2k\pi)] \quad k \in \mathbb{Z}$$

$$\text{Hence, } z = [27(\cos(2k\pi) + i \sin(2k\pi))]^{\frac{1}{3}}$$

de Moivre's Theorem.

$$z = 3 \left[\cos\left(\frac{2k\pi}{3}\right) + i \sin\left(\frac{2k\pi}{3}\right) \right]$$

$$k = 0; z = 3(\cos 0 + i \sin 0) = 3$$

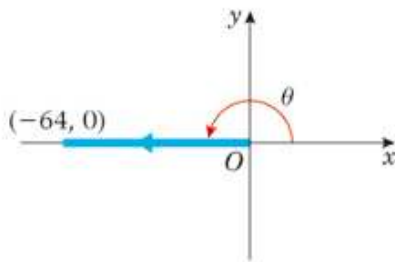
$$k = 1; z = 3 \left(\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3} \right) = 3 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) = -\frac{3}{2} + \frac{3\sqrt{3}}{2}i$$

$$k = -1; z = 3 \left(\cos\left(-\frac{2\pi}{3}\right) + i \sin\left(-\frac{2\pi}{3}\right) \right) = 3 \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i \right) = -\frac{3}{2} - \frac{3\sqrt{3}}{2}i$$

$$\text{Therefore, } z = 3, -\frac{3}{2} + \frac{3\sqrt{3}}{2}i, -\frac{3}{2} - \frac{3\sqrt{3}}{2}i$$

$$\mathbf{d} \quad z^4 + 64 = 0$$

$$z^4 = -64$$



for -64 , $r = 64$ and $\theta = \pi$

So $z^4 = 64(\cos \pi + i \sin \pi)$

$$z^4 = 64(\cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi)) \quad k \in \mathbb{Z}$$

Hence, $z = [64(\cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi))]^{\frac{1}{4}}$

$$z = 64^{\frac{1}{4}} \left(\cos\left(\frac{\pi + 2k\pi}{4}\right) + i \sin\left(\frac{\pi + 2k\pi}{4}\right) \right)$$

de Moivre's Theorem.

$$z = 2\sqrt{2} \left(\cos\left(\frac{\pi}{4} + \frac{k\pi}{2}\right) + i \sin\left(\frac{\pi}{4} + \frac{k\pi}{2}\right) \right)$$

$$k = 0; z = 2\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) = 2\sqrt{2} \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) = 2 + 2i$$

$$k = 1; z = 2\sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) = 2\sqrt{2} \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right) = -2 + 2i$$

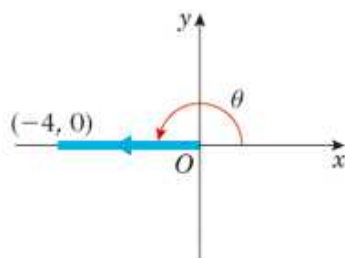
$$k = -1; z = 2\sqrt{2} \left(\cos \left(-\frac{\pi}{4}\right) + i \sin \left(-\frac{\pi}{4}\right) \right) = 2\sqrt{2} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right) = 2 - 2i$$

$$k = -2; z = 2\sqrt{2} \left(\cos \left(-\frac{3\pi}{4}\right) + i \sin \left(-\frac{3\pi}{4}\right) \right) = 2\sqrt{2} \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right) = -2 - 2i$$

Therefore, $z = 2 + 2i, -2 + 2i, 2 - 2i, -2 - 2i$

$$\mathbf{e} \quad z^4 + 4 = 0$$

$$z^4 = -4$$



for -4 , $r = 4$ and $\theta = \pi$

So $z^4 = 4(\cos \pi + i \sin \pi)$

$$z^4 = 4(\cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi)) \quad k \in \mathbb{Z}$$

Hence, $z = [4(\cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi))]^{\frac{1}{4}}$

$$z = 4^{\frac{1}{4}} \left(\cos\left(\frac{\pi + 2k\pi}{4}\right) + i \sin\left(\frac{\pi + 2k\pi}{4}\right) \right)$$

de Moivre's Theorem.

$$z = \sqrt{2} \left(\cos\left(\frac{\pi}{4} + \frac{k\pi}{2}\right) + i \sin\left(\frac{\pi}{4} + \frac{k\pi}{2}\right) \right)$$

$$k = 0; z = \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) = \sqrt{2} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \right) = 1 + i$$

$$k = 1; z = \sqrt{2} \left(\cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) \right) = \sqrt{2} \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \right) = -1 + i$$

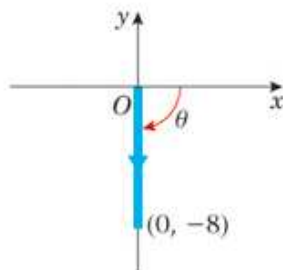
$$k = -1; z = \sqrt{2} \left(\cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right) \right) = \sqrt{2} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i \right) = 1 - i$$

$$k = -2; z = \sqrt{2} \left(\cos\left(-\frac{3\pi}{4}\right) + i \sin\left(-\frac{3\pi}{4}\right) \right) = \sqrt{2} \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i \right) = -1 - i$$

Therefore, $z = 1 + i, -1 + i, 1 - i, -1 - i$

f $z^3 + 8i = 0$

$$z^3 = -8i$$



for $-8i$, $r = 8$, $\theta = -\frac{\pi}{2}$

$$\text{So } z^3 = 8\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$$

$$z^3 = 8\left(\cos\left(-\frac{\pi}{2} + 2k\pi\right) + i\sin\left(-\frac{\pi}{2} + 2k\pi\right)\right) \quad k \in \mathbb{Z}$$

$$\text{Hence, } z = \left[8\left(\cos\left(-\frac{\pi}{2} + 2k\pi\right) + i\sin\left(-\frac{\pi}{2} + 2k\pi\right)\right)\right]^{\frac{1}{3}}$$

$$z = 8^{\frac{1}{3}}\left(\cos\left(\frac{-\frac{\pi}{2} + 2k\pi}{3}\right) + i\sin\left(\frac{-\frac{\pi}{2} + 2k\pi}{3}\right)\right)$$

de Moivre's Theorem.

$$z = 2\left(\cos\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right) + i\sin\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right)\right)$$

$$k = 0; z = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \sqrt{3} - i$$

$$k = 1; z = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2(0 + i) = 2i$$

$$k = -1; z = 2\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right) = 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -\sqrt{3} - i$$

Therefore, $z = \sqrt{3} - i, 2i, -\sqrt{3} - i$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise E, Question 2

Question:

Solve the following equations, expressing your answers for z in the form $r(\cos \theta + i \sin \theta)$, where $-\pi < \theta \leq \pi$.

a $z^7 = 1$

b $z^4 + 16i = 0$

c $z^5 + 32 = 0$

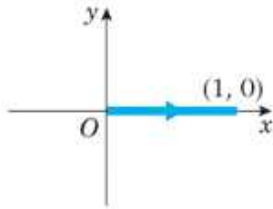
d $z^3 = 2 + 2i$

e $z^4 + 2\sqrt{3}i = 2$

f $z^3 + 32\sqrt{3} + 32i = 0$

Solution:

a $z^7 = 1$



for 1, $r = 1$ and $\theta = 0$

So $z^7 = 1 (\cos 0 + i \sin 0)$

$$z^7 = \cos(0 + 2k\pi) + i \sin(0 + 2k\pi) \quad k \in \mathbb{Z}$$

Hence, $z = (\cos(2k\pi) + i \sin(2k\pi))^{\frac{1}{7}}$

$$z = \cos\left(\frac{2k\pi}{7}\right) + i \sin\left(\frac{2k\pi}{7}\right)$$

de Moivre's Theorem.

$k = 0, z = \cos 0 + i \sin 0$

$k = 1, z = \cos\left(\frac{2\pi}{7}\right) + i \sin\left(\frac{2\pi}{7}\right)$

$k = 2, z = \cos\left(\frac{4\pi}{7}\right) + i \sin\left(\frac{4\pi}{7}\right)$

$k = 3, z = \cos\left(\frac{6\pi}{7}\right) + i \sin\left(\frac{6\pi}{7}\right)$

$k = -1, z = \cos\left(-\frac{2\pi}{7}\right) + i \sin\left(-\frac{2\pi}{7}\right)$

$k = -2, z = \cos\left(-\frac{4\pi}{7}\right) + i \sin\left(-\frac{4\pi}{7}\right)$

$k = -3, z = \cos\left(-\frac{6\pi}{7}\right) + i \sin\left(-\frac{6\pi}{7}\right)$

Therefore, $z = \cos 0 + i \sin 0, \cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7}$

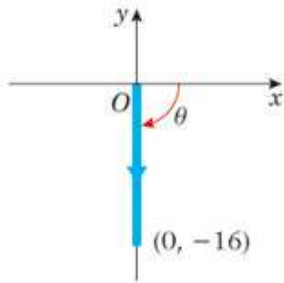
$$\cos \frac{4\pi}{7} + i \sin \frac{4\pi}{7}, \cos \frac{6\pi}{7} + i \sin \frac{6\pi}{7}$$

$$\cos\left(-\frac{2\pi}{7}\right) + i \sin\left(-\frac{2\pi}{7}\right), \cos\left(-\frac{4\pi}{7}\right) + i \sin\left(-\frac{4\pi}{7}\right)$$

$$\cos\left(-\frac{6\pi}{7}\right) + i \sin\left(-\frac{6\pi}{7}\right)$$

$$\mathbf{b} \quad z^4 + 16i = 0$$

$$z^4 = -16i$$



for $-16i$, $r = 16$ and $\theta = -\frac{\pi}{2}$

$$\text{So } z^4 = 16\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$$

$$z^4 = 16\left(\cos\left(-\frac{\pi}{2} + 2k\pi\right) + i\sin\left(-\frac{\pi}{2} + 2k\pi\right)\right) \quad k \in \mathbb{Z}$$

$$\text{Hence, } z = \left[16\left(\cos\left(-\frac{\pi}{2} + 2k\pi\right) + i\sin\left(-\frac{\pi}{2} + 2k\pi\right)\right)\right]^{\frac{1}{4}}$$

$$z = 16^{\frac{1}{4}}\left(\cos\left(\frac{-\frac{\pi}{2} + 2k\pi}{4}\right) + i\sin\left(\frac{-\frac{\pi}{2} + 2k\pi}{4}\right)\right)$$

de Moivre's Theorem.

$$z = \left(\cos\left(-\frac{\pi}{8} + \frac{k\pi}{2}\right) + i\sin\left(-\frac{\pi}{8} + \frac{k\pi}{2}\right)\right)$$

$$k = 0, z = 2\left(\cos\left(-\frac{\pi}{8}\right) + i\sin\left(-\frac{\pi}{8}\right)\right)$$

$$k = 1, z = 2\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)$$

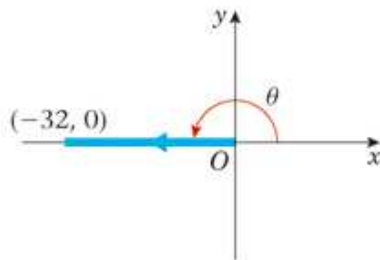
$$k = 2, z = 2\left(\cos\frac{7\pi}{8} + i\sin\frac{7\pi}{8}\right)$$

$$k = -1, z = 2\left(\cos\left(-\frac{5\pi}{8}\right) + i\sin\left(-\frac{5\pi}{8}\right)\right)$$

$$\text{Therefore, } z = 2\left(\cos\left(-\frac{\pi}{8}\right) + i\sin\left(-\frac{\pi}{8}\right)\right), 2\left(\cos\left(\frac{3\pi}{8}\right) + i\sin\left(\frac{3\pi}{8}\right)\right)$$

$$2\left(\cos\left(\frac{7\pi}{8}\right) + i\sin\left(\frac{7\pi}{8}\right)\right), 2\left(\cos\left(-\frac{5\pi}{8}\right) + i\sin\left(-\frac{5\pi}{8}\right)\right)$$

$$\begin{aligned} \text{c } z^5 + 32 &= 0 \\ z^5 &= -32 \end{aligned}$$



for -32 , $r = 32$ and $\theta = \pi$

So $z^5 = 32(\cos \pi + i \sin \pi)$

$$z^5 = 32(\cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi)) \quad k \in \mathbb{Z}$$

Hence, $z = [32(\cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi))]^{\frac{1}{5}}$

$$z = 32^{\frac{1}{5}} \left(\cos\left(\frac{\pi + 2k\pi}{5}\right) + i \sin\left(\frac{\pi + 2k\pi}{5}\right) \right)$$

$$z = 2 \left(\cos\left(\frac{\pi}{5} + \frac{2k\pi}{5}\right) + i \sin\left(\frac{\pi}{5} + \frac{2k\pi}{5}\right) \right)$$

de Moivre's Theorem.

$$k = 0, z = 2 \left(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5} \right)$$

$$k = 1, z = 2 \left(\cos \frac{3\pi}{5} + i \sin \frac{3\pi}{5} \right)$$

$$k = 1, z = 2(\cos \pi + i \sin \pi)$$

$$k = 2, z = 2 \left(\cos\left(-\frac{\pi}{5}\right) + i \sin\left(-\frac{\pi}{5}\right) \right)$$

$$k = -1, z = 2 \left(\cos\left(-\frac{5\pi}{8}\right) + i \sin\left(-\frac{5\pi}{8}\right) \right)$$

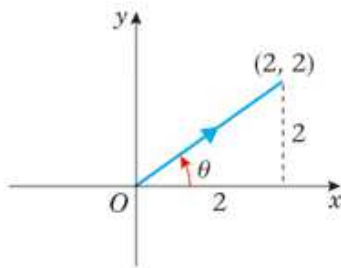
$$k = -2, z = 2 \left(\cos\left(-\frac{3\pi}{5}\right) + i \sin\left(-\frac{3\pi}{5}\right) \right)$$

Therefore, $z = 2 \left(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5} \right), 2 \left(\cos \frac{3\pi}{5} + i \sin \frac{3\pi}{5} \right),$

$$2(\cos \pi + i \sin \pi), 2 \left(\cos\left(-\frac{\pi}{5}\right) + i \sin\left(-\frac{\pi}{5}\right) \right),$$

$$2 \left(\cos\left(-\frac{3\pi}{5}\right) + i \sin\left(-\frac{3\pi}{5}\right) \right)$$

d $z^3 = 2 + 2i$



$$r = \sqrt{2^2 + 2^2} = \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$$

So $z^3 = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$$z^3 = 2\sqrt{2}\left(\cos\left(\frac{\pi}{4} + 2k\pi\right) + i\sin\left(\frac{\pi}{4} + 2k\pi\right)\right) \quad k \in \mathbb{Z}$$

Hence, $z = \left[2\sqrt{2}\left(\cos\left(\frac{\pi}{4} + 2k\pi\right) + i\sin\left(\frac{\pi}{4} + 2k\pi\right)\right)\right]^{\frac{1}{3}}$

$$z = (2\sqrt{2})^{\frac{1}{3}}\left(\cos\left(\frac{\frac{\pi}{4} + 2k\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{4} + 2k\pi}{3}\right)\right)$$

de Moivre's Theorem.

$$z = \sqrt{2}\left(\cos\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right)\right)$$

$k = 0, z = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

$k = 1, z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

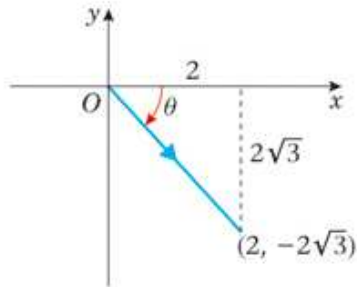
$k = -1, z = \sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right)$

Therefore, $z = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right), \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right),$

$$\sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right)$$

$$\mathbf{e} \quad z^4 + 2\sqrt{3}i = 2$$

$$z^4 = 2 - 2\sqrt{3}i$$



$$r = \sqrt{2^2 + (-2\sqrt{3})^2} = \sqrt{4 + 12} = \sqrt{16} = 4$$

$$\theta = -\tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

$$\text{So } z^4 = 4\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$$

$$z^4 = 4\left(\cos\left(-\frac{\pi}{3} + 2k\pi\right) + i\sin\left(-\frac{\pi}{3} + 2k\pi\right)\right) \quad k \in \mathbb{Z}$$

$$\text{Hence, } z = \left[4\left(\cos\left(-\frac{\pi}{3} + 2k\pi\right) + i\sin\left(-\frac{\pi}{3} + 2k\pi\right)\right)\right]^{\frac{1}{4}}$$

$$z = 4^{\frac{1}{4}}\left(\cos\left(\frac{-\frac{\pi}{3} + 2k\pi}{4}\right) + i\sin\left(\frac{-\frac{\pi}{3} + 2k\pi}{4}\right)\right)$$

de Moivre's Theorem.

$$z = \sqrt{2}\left(\cos\left(-\frac{\pi}{12} + \frac{k\pi}{2}\right) + i\sin\left(-\frac{\pi}{12} + \frac{k\pi}{2}\right)\right)$$

$$k = 0, z = \sqrt{2}\left(\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right)$$

$$k = 1, z = \sqrt{2}\left(\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right)$$

$$k = 1, z = \sqrt{2}\left(\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right)$$

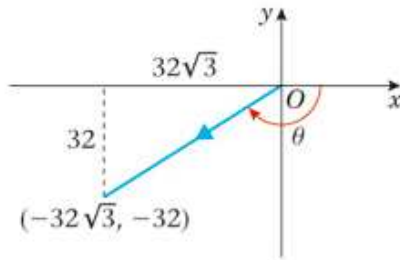
$$k = -1, z = \sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right)$$

$$\text{Therefore, } z = \sqrt{2}\left(\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right), \sqrt{2}\left(\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right),$$

$$\sqrt{2}\left(\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right), \sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right)$$

$$\mathbf{f} \quad z^3 + 32\sqrt{3} + 32i = 0$$

$$z^3 = -32\sqrt{3} - 32i$$



$$r = \sqrt{(-32\sqrt{3})^2 + (-32)^2} = \sqrt{3072 + 1024} = \sqrt{4096} = 64$$

$$\theta = -\pi + \tan^{-1}\left(\frac{32}{32\sqrt{3}}\right) = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$$

$$\text{So } z^3 = 64\left(\cos\left(-\frac{5\pi}{6}\right) + i \sin\left(-\frac{5\pi}{6}\right)\right)$$

$$z^3 = 64\left(\cos\left(-\frac{5\pi}{6} + 2k\pi\right) + i \sin\left(-\frac{5\pi}{6} + 2k\pi\right)\right) \quad k \in \mathbb{Z}$$

$$\text{Hence, } z = \left[64\left(\cos\left(-\frac{5\pi}{6} + 2k\pi\right) + i \sin\left(-\frac{5\pi}{6} + 2k\pi\right)\right)\right]^{\frac{1}{3}}$$

$$z = 64^{\frac{1}{3}}\left(\cos\left(\frac{-\frac{5\pi}{6} + 2k\pi}{3}\right) + i \sin\left(\frac{-\frac{5\pi}{6} + 2k\pi}{3}\right)\right)$$

de Moivre's Theorem.

$$z = 4\left(\cos\left(-\frac{5\pi}{18} + \frac{2k\pi}{3}\right) + i \sin\left(-\frac{5\pi}{18} + \frac{2k\pi}{3}\right)\right)$$

$$k = 0, z = 4\left(\cos\left(-\frac{5\pi}{18}\right) + i \sin\left(-\frac{5\pi}{18}\right)\right)$$

$$k = 1, z = 4\left(\cos\left(\frac{7\pi}{18}\right) + i \sin\left(\frac{7\pi}{18}\right)\right)$$

$$k = -1, z = 4\left(\cos\left(-\frac{17\pi}{18}\right) + i \sin\left(-\frac{17\pi}{18}\right)\right)$$

$$\text{Therefore, } z = 4\left(\cos\left(-\frac{5\pi}{18}\right) + i \sin\left(-\frac{5\pi}{18}\right)\right), 4\left(\cos\left(\frac{7\pi}{18}\right) + i \sin\left(\frac{7\pi}{18}\right)\right),$$

$$4\left(\cos\left(-\frac{17\pi}{18}\right) + i \sin\left(-\frac{17\pi}{18}\right)\right)$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise E, Question 3

Question:

Solve the following equations, expressing your answers for z in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. Give θ to 2 d.p.

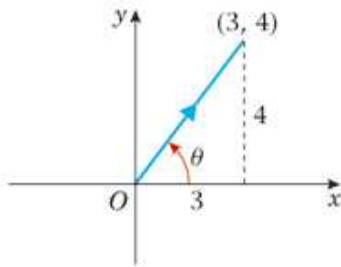
a $z^4 = 3 + 4i$

b $z^3 = \sqrt{11} - 4i$

c $z^4 = -\sqrt{7} + 3i$

Solution:

a $z^4 = 3 + 4i$



$$r = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right) = 0.927295\dots$$

So, $z^4 = 5e^{i(0.927295\dots)}$

$$z^4 = 5e^{i(0.927295\dots + 2k\pi)}, \quad k \in \mathbb{Z}$$

Hence, $z = [5e^{i(0.927295\dots + 2k\pi)}]^{\frac{1}{4}}$

$$= 5^{\frac{1}{4}}e^{i\left(\frac{0.927295\dots + 2k\pi}{4}\right)}$$

$$= 5^{\frac{1}{4}}e^{i\left(\frac{0.927295\dots}{4} + \frac{k\pi}{2}\right)}$$

de Moivre's Theorem.

$$k = 0, z = 5^{\frac{1}{4}}e^{i(0.2318\dots)}$$

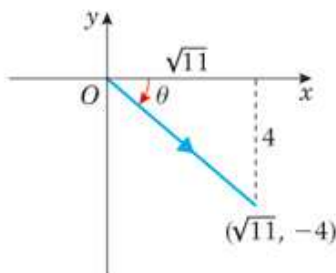
$$k = 1, z = 5^{\frac{1}{4}}e^{i(1.8026\dots)}$$

$$k = -1, z = 5^{\frac{1}{4}}e^{i(-1.3389\dots)}$$

$$k = -2, z = 5^{\frac{1}{4}}e^{i(-2.9097\dots)}$$

Therefore, $z = 5^{\frac{1}{4}}e^{0.231i}, 5^{\frac{1}{4}}e^{1.80i}, 5^{\frac{1}{4}}e^{-1.34i}, 5^{\frac{1}{4}}e^{-2.91i}$

b $z^3 = \sqrt{11} + 4i$



$$r = \sqrt{(\sqrt{11})^2 + (-4)^2} = \sqrt{11 + 16} = \sqrt{27}$$

$$\theta = -\tan^{-1}\left(\frac{4}{\sqrt{11}}\right) = 0.878528\dots$$

$$\text{So, } z^3 = \sqrt{27} e^{i(-0.878528\dots)}$$

$$z^3 = \sqrt{27} e^{i(-0.878528\dots + 2k\pi)}, \quad k \in \mathbb{Z}$$

$$\text{Hence, } z = [\sqrt{27} e^{i(-0.878528\dots + 2k\pi)}]^{1/3}$$

$$= (\sqrt{27})^{1/3} e^{i\left(\frac{-0.878528\dots + 2k\pi}{3}\right)}$$

de Moivre's Theorem.

$$= \sqrt{3} e^{i\left(-\frac{0.878528\dots}{3} + \frac{2k\pi}{3}\right)}$$

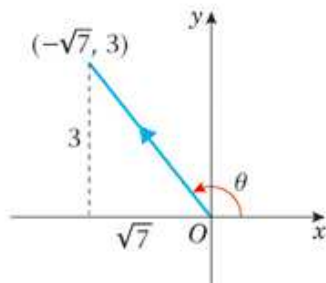
$$k = 0, z = \sqrt{3} e^{i(-0.2928\dots)}$$

$$k = 1, z = \sqrt{3} e^{i(1.8015\dots)}$$

$$k = -1, z = \sqrt{3} e^{i(-2.3872\dots)}$$

$$\text{Therefore, } z = \sqrt{3} e^{-0.29i}, \sqrt{3} e^{1.80i}, \sqrt{3} e^{-2.39i}$$

c $z^4 = -\sqrt{7} + 3i$



$$r = \sqrt{(-\sqrt{7})^2 + 3^2} = \sqrt{7 + 9} = \sqrt{16} = 4$$

$$\theta = \pi - \tan^{-1}\left(\frac{3}{\sqrt{7}}\right) = 2.293530\dots$$

$$\text{So, } z^4 = 4e^{i(2.293530\dots)}$$

$$z^4 = 4e^{i(2.293530\dots + 2k\pi)}, \quad k \in \mathbb{Z}$$

$$\text{Hence, } z = [4e^{i(2.293530\dots + 2k\pi)}]^{\frac{1}{4}}$$

$$= 4^{\frac{1}{4}} e^{i\left(\frac{2.293530\dots + 2k\pi}{4}\right)}$$

de Moivre's Theorem.

$$= \sqrt{2} e^{i\left(\frac{2.293530\dots}{4} + \frac{k\pi}{2}\right)}$$

$$k = 0, z = \sqrt{2} e^{i(0.5733\dots)}$$

$$k = 1, z = \sqrt{2} e^{i(2.1441\dots)}$$

$$k = -1, z = \sqrt{2} e^{i(-0.9974\dots)}$$

$$k = -2, z = \sqrt{2} e^{i(-2.5682\dots)}$$

$$\text{Therefore, } z = \sqrt{2} e^{0.57i}, z = \sqrt{2} e^{2.14i}, z = \sqrt{2} e^{-1.00i}, z = \sqrt{2} e^{-2.57i}$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

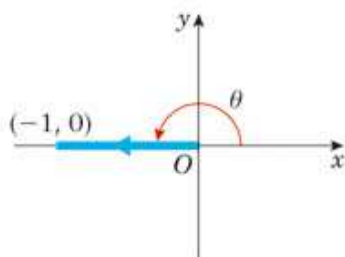
Exercise E, Question 4

Question:

- a** Find the three roots of the equation $(z + 1)^3 = -1$.
Give your answers in the form $x + iy$, where $x \in \mathbb{R}$ and $y \in \mathbb{R}$.
- b** Plot the points representing these three roots on an Argand diagram.
- c** Given that these three points lie on a circle, find its centre and radius.

Solution:

a $(z + 1)^3 = -1$



For -1 , $r = 1$ and $\theta = \pi$

So, $(z + 1)^3 = 1(\cos \pi + i \sin \pi)$

$$(z + 1)^3 = (\pi + 2k\pi) + i \sin(\pi + 2k\pi) \quad k \in \mathbb{Z}$$

Hence, $z + 1 = [\cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi)]^{\frac{1}{3}}$

$$z + 1 = \cos\left(\frac{\pi + 2k\pi}{3}\right) + i \sin\left(\frac{\pi + 2k\pi}{3}\right)$$

de Moivre's Theorem.

$$z + 1 = \cos\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right) + i \sin\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right)$$

$$k = 0, z + 1 = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$\Rightarrow z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

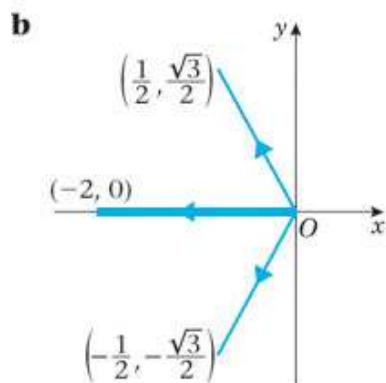
$$k = 1, z + 1 = \cos \pi + i \sin \pi = -1 + 0i$$

$$\Rightarrow z = -2$$

$$k = -1, z + 1 = \cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\Rightarrow z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

Therefore, $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -2, \frac{1}{2} - \frac{\sqrt{3}}{2}i$



c The solutions to $w^3 = -1$, lie on a circle centre $(0, 0)$, radius 1.

As $w = z + 1$, then the three solutions for z are the three solutions for w translated by $(-\frac{1}{0})$.

Hence the three points (the solutions for z), lie on a circle centre $(-1, 0)$, radius 1.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise E, Question 5

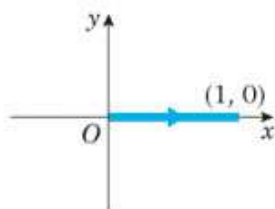
Question:

- a** Find the five roots of the equation $z^5 - 1 = 0$.
Give your answers in the form $r(\cos \theta + i \sin \theta)$, where $-\pi < \theta \leq \pi$.
- b** Given that the sum of all five roots of $z^5 - 1 = 0$ is zero, show that
- $$\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = -\frac{1}{2}.$$

Solution:

a $z^5 - 1 = 0$

$$z^5 = 1$$



For 1, $r = 1$ and $\theta = 0$

So, $z^5 = 1(\cos 0 + i \sin 0)$

$$z^5 = \cos(0 + 2k\pi) + i \sin(0 + 2k\pi) \quad k \in \mathbb{Z}$$

Hence, $z = [\cos(2k\pi) + i \sin(2k\pi)]^{\frac{1}{5}}$

$$z = \cos\left(\frac{2k\pi}{5}\right) + i \sin\left(\frac{2k\pi}{5}\right) \quad \boxed{\text{de Moivre's Theorem.}}$$

$k = 0, z_1 = \cos 0 + i \sin 0 = 1 + i(0) = 1$

$k = 1, z_2 = \cos\left(\frac{2\pi}{5}\right) + i \sin\left(\frac{2\pi}{5}\right)$

$k = 2, z_3 = \cos\left(\frac{4\pi}{5}\right) + i \sin\left(\frac{4\pi}{5}\right)$

$k = -1, z_4 = \cos\left(-\frac{2\pi}{5}\right) + i \sin\left(-\frac{2\pi}{5}\right)$

$k = -2, z_5 = \cos\left(-\frac{4\pi}{5}\right) + i \sin\left(-\frac{4\pi}{5}\right)$

Therefore $z = 1, \cos\left(\frac{2\pi}{5}\right) + i \sin\left(\frac{2\pi}{5}\right), \cos\left(\frac{4\pi}{5}\right) + i \sin\left(\frac{4\pi}{5}\right),$

$$\cos\left(-\frac{2\pi}{5}\right) + i \sin\left(-\frac{2\pi}{5}\right), \cos\left(-\frac{4\pi}{5}\right) + i \sin\left(-\frac{4\pi}{5}\right)$$

b So, $z_1 + z_2 + z_3 + z_4 + z_5 = 0$

$$1 + \cos\left(\frac{2\pi}{5}\right) + i \sin\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) + i \sin\left(\frac{4\pi}{5}\right) \\ + \cos\left(-\frac{2\pi}{5}\right) + i \sin\left(-\frac{2\pi}{5}\right) + \cos\left(-\frac{4\pi}{5}\right) + i \sin\left(-\frac{4\pi}{5}\right) = 0$$

$$\Rightarrow 1 + \cos\left(\frac{2\pi}{5}\right) + i \sin\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) + i \sin\left(\frac{4\pi}{5}\right) \\ + \cos\left(\frac{2\pi}{5}\right) - i \sin\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) - i \sin\left(\frac{4\pi}{5}\right) = 0$$

$$1 + 2 \cos\left(\frac{2\pi}{5}\right) + 2 \cos\left(\frac{4\pi}{5}\right) = 0$$

$$2 \cos\left(\frac{2\pi}{5}\right) + 2 \cos\left(\frac{4\pi}{5}\right) = -1$$

$$2 \left(\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) \right) = -1$$

$$\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = -\frac{1}{2} \text{ (as required)}$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

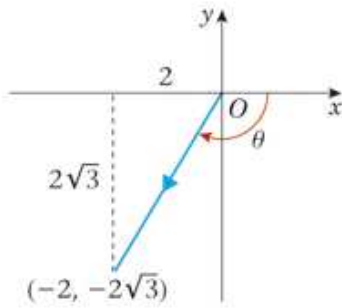
Exercise E, Question 6

Question:

- a** Find the modulus and argument of $-2 - 2\sqrt{3}i$.
- b** Hence find all the solutions of the equation $z^4 + 2 + 2\sqrt{3}i = 0$.
Give your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$.

Solution:

a $-2 - 2\sqrt{3}i$.



$$\text{modulus} = r = \sqrt{(-2)^2 + (-2\sqrt{3})^2} = \sqrt{4 + 12} = \sqrt{16} = 4$$

$$\text{argument} = \theta = 2\pi + \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$$

$$\text{Therefore, } r = 4, \theta = -\frac{2\pi}{3}$$

b $z^4 + 2 + 2\sqrt{3}i = 0$

$$z^4 = -2 - 2\sqrt{3}i$$

$$\text{and } r = 4, \theta = -\frac{2\pi}{3} \text{ for } -2 - 2\sqrt{3}i$$

$$\text{So } z^4 = 4e^{i(-\frac{2\pi}{3})}$$

$$z^4 = 4e^{i(-\frac{2\pi}{3} + 2k\pi)}, \quad k \in \mathbb{Z}$$

$$\text{Hence, } z = \left[4e^{i(-\frac{2\pi}{3} + 2k\pi)}\right]^{\frac{1}{4}}$$

$$= 4^{\frac{1}{4}} e^{i\left(\frac{-\frac{2\pi}{3} + 2k\pi}{4}\right)}$$

de Moivre's Theorem.

$$= \sqrt{2} e^{i\left(-\frac{\pi}{6} + \frac{k\pi}{2}\right)}$$

$$k = 0, z = \sqrt{2} e^{i\left(-\frac{\pi}{6}\right)}$$

$$k = 1, z = \sqrt{2} e^{i\left(\frac{\pi}{3}\right)}$$

$$k = 2, z = \sqrt{2} e^{i\left(\frac{5\pi}{6}\right)}$$

$$k = -1, z = \sqrt{2} e^{i\left(-\frac{2\pi}{3}\right)}$$

$$\text{Therefore, } z = \sqrt{2} e^{-\frac{\pi i}{6}}, \sqrt{2} e^{\frac{\pi i}{3}}, \sqrt{2} e^{\frac{5\pi i}{6}}, \sqrt{2} e^{-\frac{2\pi i}{3}}$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

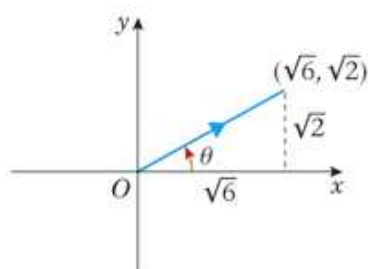
Exercise E, Question 7

Question:

- a** Find the modulus and argument of $\sqrt{6} + \sqrt{2}i$.
- b** Solve the equation $z^{\frac{1}{4}} = \sqrt{6} + \sqrt{2}i$.
Give your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$.

Solution:

a $\sqrt{6} + \sqrt{2}i$.



$$\text{modulus} = r = \sqrt{(\sqrt{6})^2 + (\sqrt{2})^2} = \sqrt{6 + 2} = \sqrt{8}$$

$$\text{argument} = \theta = \tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{6}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$\text{Therefore, } r = \sqrt{8}, \theta = \frac{\pi}{6}$$

b $z^{\frac{1}{3}} = \sqrt{6} + \sqrt{2}i$

$$\text{For } \sqrt{6} + \sqrt{2}i, r = \sqrt{8}, \theta = \frac{\pi}{6}$$

$$\text{So, } z^{\frac{1}{3}} = \sqrt{8} e^{i\left(\frac{\pi}{6}\right)}$$

$$z^3 = \left[\sqrt{8} e^{i\left(\frac{\pi}{6}\right)}\right]^4$$

$$z^3 = (\sqrt{8})^4 e^{i\left(\frac{4\pi}{6}\right)}$$

$$z^3 = 64 e^{i\left(\frac{2\pi}{3}\right)}$$

$$z^3 = 64 e^{i\left(\frac{2\pi}{3} + 2k\pi\right)}, \quad k \in \mathbb{Z}$$

$$\text{Hence, } z = \left[64 e^{i\left(\frac{2\pi}{3} + 2k\pi\right)}\right]^{\frac{1}{3}}$$

$$= (64)^{\frac{1}{3}} e^{i\left(\frac{\frac{2\pi}{3} + 2k\pi}{3}\right)}$$

$$= 4 e^{i\left(\frac{2\pi}{9} + \frac{2k\pi}{3}\right)}$$

$$k = 0, z = 4 e^{i\left(\frac{2\pi}{9}\right)}$$

$$k = 1, z = 4 e^{i\left(\frac{8\pi}{9}\right)}$$

$$k = -1, z = 4 e^{i\left(-\frac{4\pi}{9}\right)}$$

$$\text{Therefore, } z = 4 e^{\frac{2\pi i}{9}}, z = 4 e^{\frac{8\pi i}{9}}, z = 4 e^{-\frac{4\pi i}{9}}$$

de Moivre's Theorem.

$$\begin{aligned} (\sqrt{8})^4 &= (8^{\frac{1}{2}})^4 \\ &= 8^2 = 64 \end{aligned}$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise F, Question 1

Question:

Sketch the locus of z and give the Cartesian equation of the locus of z when:

a $|z| = 6$

b $|z| = 10$

c $|z - 3| = 2$

d $|z + 3i| = 3$

e $|z - 4i| = 5$

f $|z + 1| = 1$

g $|z - 1 - i| = 5$

h $|z + 3 + 4i| = 4$

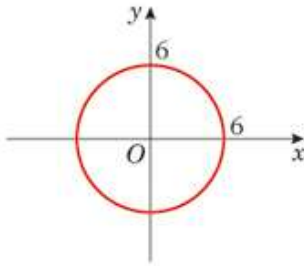
i $|z - 5 + 6i| = 5$

j $|2z + 6 - 4i| = 6$

k $|3z - 9 - 6i| = 12$

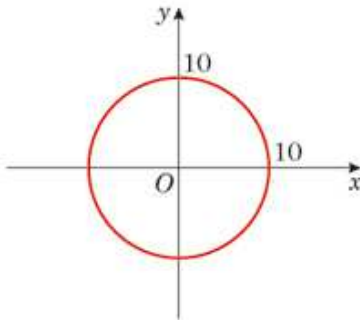
Solution:

a $|z| = 6$



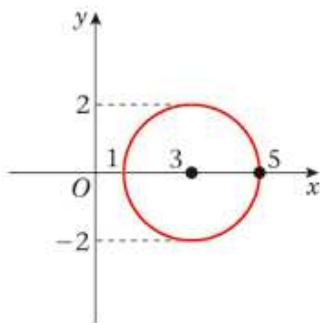
circle centre $(0, 0)$, radius 6
 equation: $x^2 + y^2 = 6^2$
 $x^2 + y^2 = 36$

b $|z| = 10$



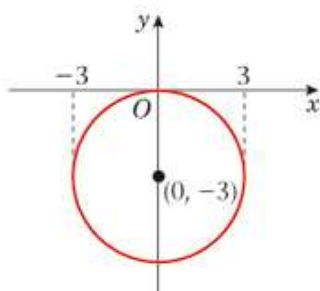
circle centre $(0, 0)$, radius 10
 equation: $x^2 + y^2 = 10^2$
 $x^2 + y^2 = 100$

c $|z - 3| = 2$



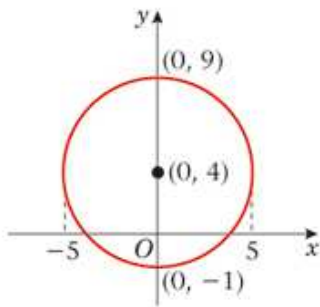
circle centre $(3, 0)$, radius 2
 equation: $(x - 3)^2 + y^2 = 2^2$
 $(x - 3)^2 + y^2 = 4$

d $|z + 3i| = 3 \Rightarrow |z - (-3i)| = 3$



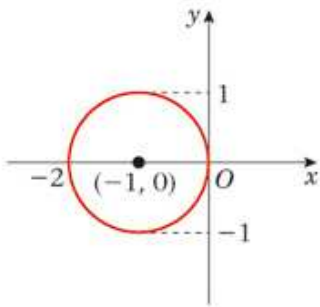
circle centre $(0, -3)$, radius 3
 equation: $x^2 + (y + 3)^2 = 3^2$
 $x^2 + (y + 3)^2 = 9$

e $|z - 4i| = 5$



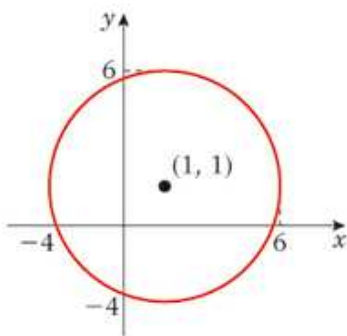
circle centre $(0, 4)$, radius 5
 equation: $x^2 + (y - 4)^2 = 5^2$
 $x^2 + (y - 4)^2 = 25$

f $|z + 1| = 1 \Rightarrow |z - (-1)| = 1$



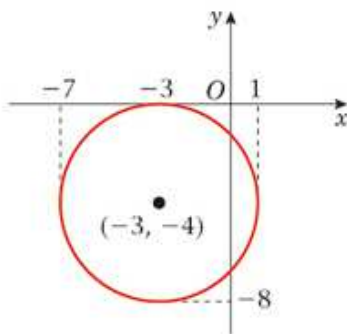
circle centre $(-1, 0)$, radius 1
 equation: $(x + 1)^2 + y^2 = 1^2$
 $(x + 1)^2 + y^2 = 1$

g $|z - 1 - i| = 5 \Rightarrow |z - (1 + i)| = 5$



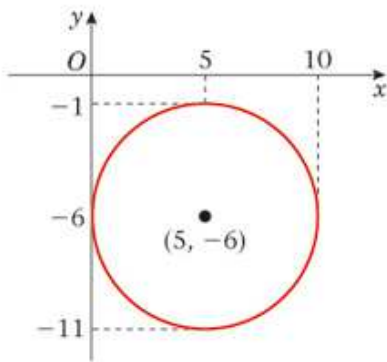
circle centre $(1, 1)$, radius 5
 equation: $(x - 1)^2 + (y - 1)^2 = 5^2$
 $(x - 1)^2 + (y - 1)^2 = 25$

h $|z + 3 + 4i| = 4 \Rightarrow |z - (-3 - 4i)| = 4$



circle centre $(-3, -4)$, radius 4
 equation: $(x + 3)^2 + (y + 4)^2 = 4^2$
 $(x + 3)^2 + (y + 4)^2 = 16$

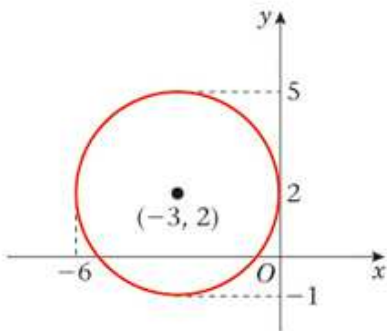
i $|z - 5 + 6i| = 5 \Rightarrow |z - (5 - 6i)| = 4$



circle centre $(5, -6)$, radius 5

equation: $(x - 5)^2 + (y + 6)^2 = 5^2$
 $(x - 5)^2 + (y + 6)^2 = 25$

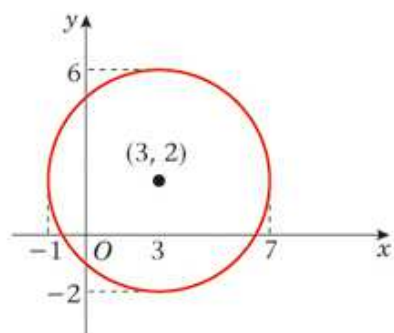
j $|2z + 6 - 4i| = 6$
 $\Rightarrow |2(z + 3 - 2i)| = 6$
 $\Rightarrow |2||z + 3 - 2i| = 6$
 $\Rightarrow 2|z + 3 - 2i| = 6$
 $\Rightarrow |z + 3 - 2i| = 3$
 $\Rightarrow |z - (-3 + 2i)| = 3$



circle centre $(-3, 2)$, radius 3

equation: $(x + 3)^2 + (y - 2)^2 = 3^2$
 $(x + 3)^2 + (y - 2)^2 = 9$

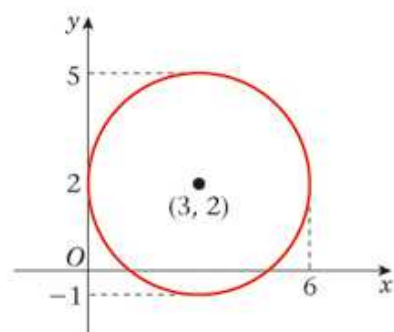
k $|3z - 9 - 6i| = 12$
 $\Rightarrow |3(z - 3 - 2i)| = 12$
 $\Rightarrow |3||z - 3 - 2i| = 12$
 $\Rightarrow 3|z - (3 + 2i)| = 12$
 $\Rightarrow |z - (3 + 2i)| = 4$



circle centre $(3, 2)$, radius 4

equation: $(x - 3)^2 + (y - 2)^2 = 4^2$
 $(x - 3)^2 + (y - 2)^2 = 16$

I $|3z - 9 - 6i| = 9$
 $\Rightarrow |3(z - 3 - 2i)| = 9$
 $\Rightarrow |3||z - 3 - 2i| = 9$
 $\Rightarrow 3|z - 3 - 2i| = 9$
 $\Rightarrow |z - 3 - 2i| = 3$
 $\Rightarrow |z - (3 + 2i)| = 3$



circle centre $(3, 2)$, radius 3

equation: $(x - 3)^2 + (y - 2)^2 = 3^2$
 $(x - 3)^2 + (y - 2)^2 = 9$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise F, Question 2

Question:

Sketch the locus of z when:

a $\arg z = \frac{\pi}{3}$

b $\arg(z + 3) = \frac{\pi}{4}$

c $\arg(z - 2) = \frac{\pi}{2}$

d $\arg(z + 2 + 2i) = -\frac{\pi}{4}$

e $\arg(z - 1 - i) = \frac{3\pi}{4}$

f $\arg(z + 3i) = \pi$

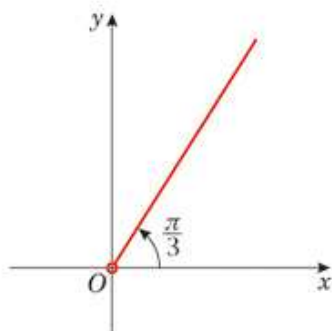
g $\arg(z - 1 + 3i) = \frac{2\pi}{3}$

h $\arg(z - 3 + 4i) = -\frac{\pi}{2}$

i $\arg(z - 4i) = -\frac{3\pi}{4}$

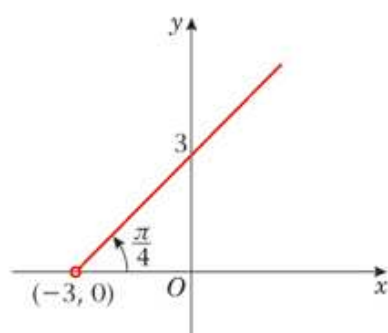
Solution:

a $\arg z = \frac{\pi}{3}$

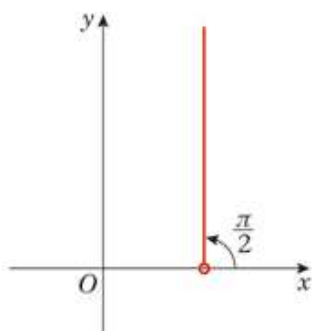


b $\arg(z + 3) = \frac{\pi}{4}$

$$\Rightarrow \arg(z - (-3)) = \frac{\pi}{4}$$

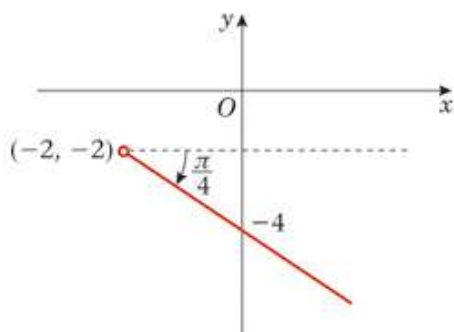


c $\arg(z - 2) = \frac{\pi}{2}$



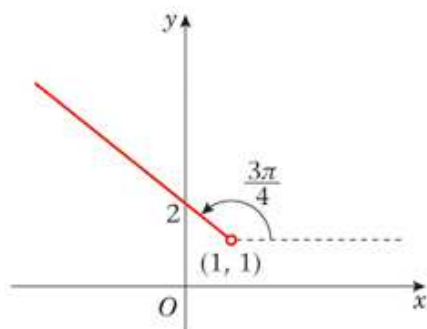
$$\mathbf{d} \quad \arg(z + 2 + 2i) = -\frac{\pi}{4}$$

$$\Rightarrow \arg(z - (-2 - 2i)) = -\frac{\pi}{4}$$



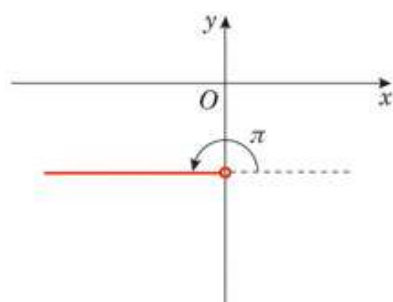
$$\mathbf{e} \quad \arg(z - 1 - i) = \frac{3\pi}{4}$$

$$\Rightarrow \arg(z - (1 + i)) = \frac{3\pi}{4}$$



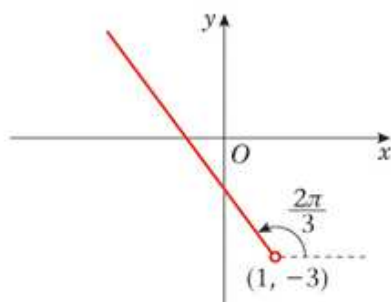
$$\mathbf{f} \quad \arg(z + 3i) = \pi$$

$$\Rightarrow \arg(z - (-3i)) = \pi$$



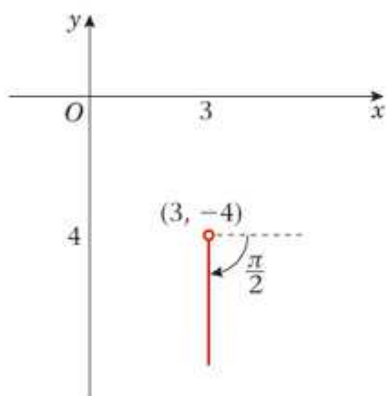
$$\mathbf{g} \quad \arg(z - 1 + 3i) = \frac{2\pi}{3}$$

$$\Rightarrow \arg(z - (1 - 3i)) = \frac{2\pi}{3}$$

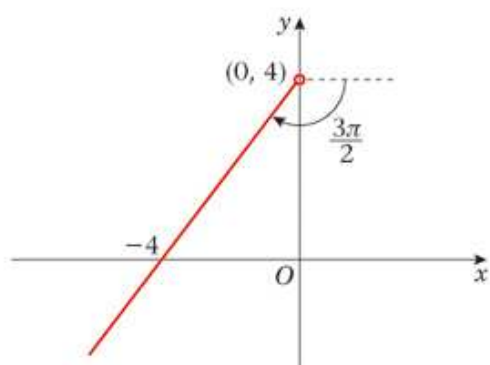


$$\mathbf{h} \quad \arg(z - 3 + 4i) = -\frac{\pi}{2}$$

$$\Rightarrow \arg(z - (3 - 4i)) = -\frac{\pi}{2}$$



$$\mathbf{i} \quad \arg(z - 4i) = -\frac{3\pi}{4}$$



Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise F, Question 3

Question:

Sketch the locus of z and give the Cartesian equation of the locus of z when:

a $|z - 6| = |z - 2|$

b $|z + 8| = |z - 4|$

c $|z| = |z + 6i|$

d $|z + 3i| = |z - 8i|$

e $|z - 2 - 2i| = |z + 2 + 2i|$

f $|z + 4 + i| = |z + 4 + 6i|$

g $|z + 3 - 5i| = |z - 7 - 5i|$

h $|z + 4 - 2i| = |z - 8 + 2i|$

i $\frac{|z + 3i|}{|z - 6i|} = 1$

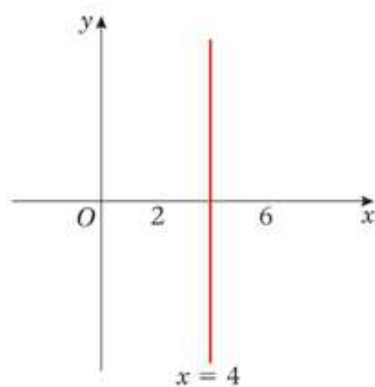
j $|z + 7 + 2i| = |z - 4 - 3i|$

k $|z + 1 - 6i| = |2 + 3i - z|$

Solution:

a $|z - 6| = |z - 2|$

perpendicular bisector of the line joining $(6, 0)$ and $(2, 0)$.

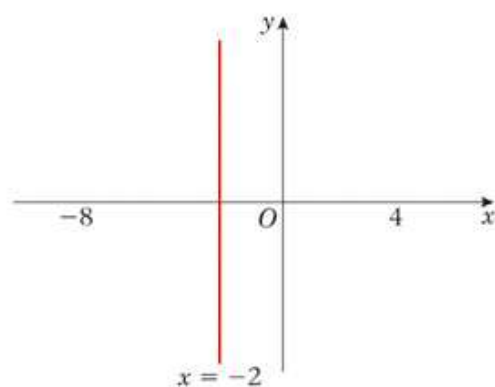


Equation: $x = 4$

b $|z + 8| = |z - 4|$

$\Rightarrow |z - (-8)| = |z - 4|$

perpendicular bisector of the line joining $(-8, 0)$ and $(4, 0)$.

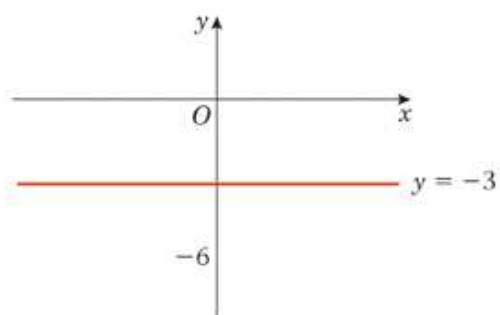


Equation: $x = -2$

c $|z| = |z + 6i|$

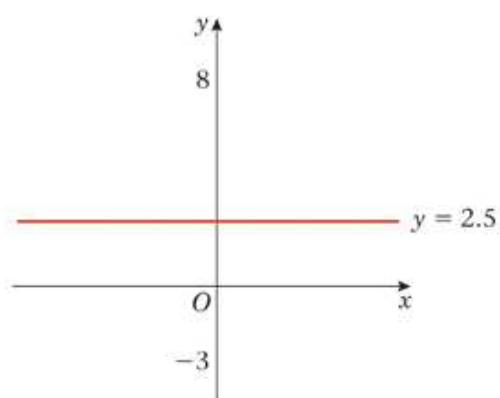
$\Rightarrow |z| = |z - (-6i)|$

perpendicular bisector of the line joining $(0, 0)$ to $(0, -6)$.



Equation: $y = -3$

- d** $|z + 3i| = |z - 8i|$
 $\Rightarrow |z - (-3i)| = |z - 8i|$
perpendicular bisector of the line joining $(0, -3)$ to $(0, 8)$.

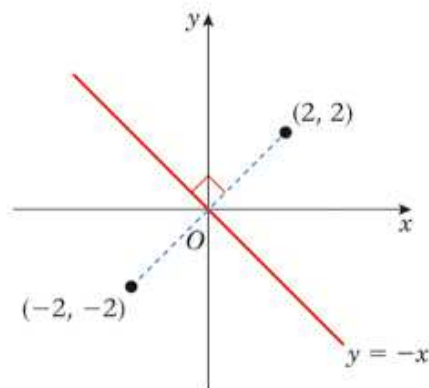


Equation: $y = 2.5$

Equation: $y = 2.5$

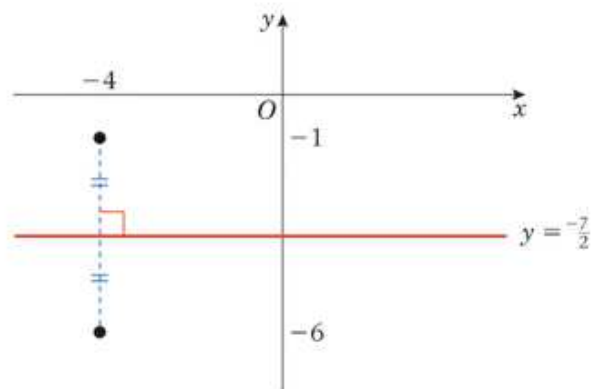
- e** $|z - 2 - 2i| = |z + 2 + 2i|$
 $\Rightarrow |z - (2 + 2i)| = |z - (-2 - 2i)|$
 perpendicular bisector of the line joining $(2, 2)$ to $(-2, -2)$.

So, $|x + iy - 2 - 2i| = |x + iy + 2 + 2i|$
 $\Rightarrow |(x - 2) + i(y - 2)| = |(x + 2) + i(y + 2)|$
 $\Rightarrow (x - 2)^2 + (y - 2)^2 = (x + 2)^2 + (y + 2)^2$
 $\Rightarrow x^2 - 4x + 4 + y^2 - 4y + 4 = x^2 + 4x + 4 + y^2 + 4y + 4$
 $\Rightarrow -4x - 4y^2 + 8 = 4x + 4y + 8$
 $\Rightarrow 0 = 8x + 8y$
 $\Rightarrow -8x = 8y$
 $\Rightarrow y = -x$



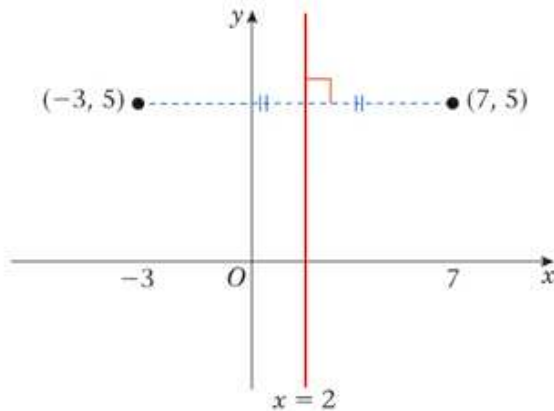
Equation: $y = -x$

- f** $|z + 4 + i| = |z + 4 + 6i|$
 $\Rightarrow |z - (-4 - i)| = |z + (-4 - 6i)|$
 perpendicular bisector of the line joining $(-4, -1)$ to $(-4, -6)$.



$$\text{Equation: } y = -\frac{7}{2}$$

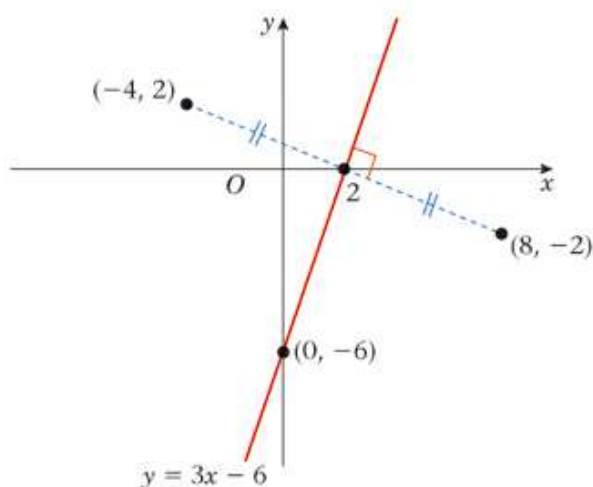
- g** $|z + 3 - 5i| = |z - 7 - 5i|$
 $\Rightarrow |z - (-3 + 5i)| = |z - (7 + 5i)|$
 perpendicular bisector of the line joining $(-3, 5)$ to $(7, 5)$.



$$\text{Equation: } x = 2$$

- h** $|z + 4 - 2i| = |z - 8 + 2i|$
 $\Rightarrow |z - (-4 + 2i)| = |z - (8 - 2i)|$
 perpendicular bisector of the line joining $(-4, 2)$ to $(8, -2)$.

$$\begin{aligned} \text{So, } |x + iy + 4 - 2i| &= |x + iy - 8 + 2i| \\ \Rightarrow |(x + 4) + i(y - 2)| &= |(x - 8) + i(y + 2)| \\ \Rightarrow (x + 4)^2 + (y - 2)^2 &= (x - 8)^2 + (y + 2)^2 \\ \Rightarrow x^2 + 8x + 16 + y^2 - 4y + 4 &= x^2 - 16x + 64 + y^2 + 4y + 4 \\ \Rightarrow 8x - 4y + 20 &= -16x + 4y + 68 \\ \Rightarrow 0 &= -24x + 8y + 48 \\ \Rightarrow 0 &= -3x + y + 6 \\ \Rightarrow 3x - 6 &= y \end{aligned}$$



$$\begin{aligned} y &= 0 \\ \Rightarrow 3x - 6 &= 0 \\ \Rightarrow 3x &= 6 \\ \Rightarrow x &= 2 \end{aligned}$$

$$x = 0, y = -6$$

$$\text{Equation: } y = 3x - 6$$

$$\mathbf{i} \quad \frac{|z + 3|}{|z - 6i|} = 1$$

$$\Rightarrow |z + 3| = |z - 6i|$$

$$\Rightarrow |z - (-3)| = |z - 6i|$$

perpendicular bisector of the line joining $(-3, 0)$ to $(0, 6)$.

$$\text{So, } |x + iy + 3| = |x + iy - 6i|$$

$$\Rightarrow |(x + 3) + iy| = |x + i(y - 6)|$$

$$\Rightarrow (x + 3)^2 + y^2 = x^2 + (y - 6)^2$$

$$\Rightarrow x^2 + 6x + 9 + y^2 = x^2 + y^2 - 12y + 36$$

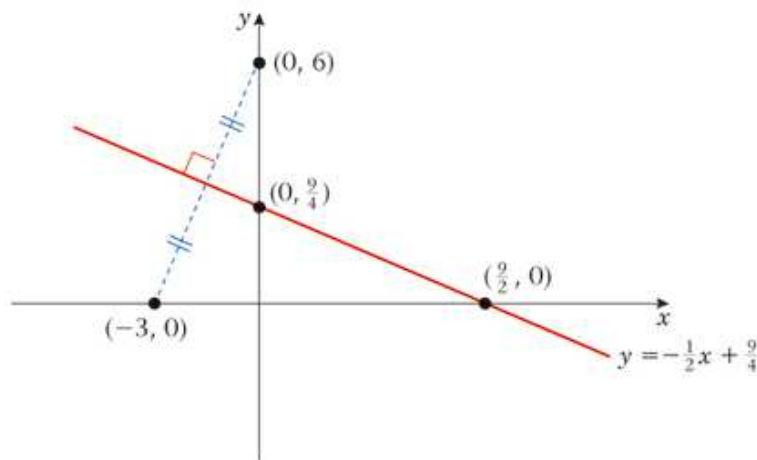
$$\Rightarrow 6x + 12y = 36 - 9$$

$$\Rightarrow 6x + 12y = 27$$

$$\Rightarrow 2x + 4y = 9$$

$$\Rightarrow 4y = 9 - 2x$$

$$\Rightarrow y = -\frac{1}{2}x + \frac{9}{4}$$



$$\begin{aligned} y &= 0 \\ \Rightarrow 0 &= 9 - 2x \\ \Rightarrow 2x &= 9 \\ \Rightarrow x &= \frac{9}{2} \end{aligned}$$

$$x = 0, y = \frac{9}{4}$$

$$\text{Equation: } y = -\frac{1}{2}x + \frac{9}{4}$$

$$\mathbf{j} \quad \frac{|z + 6 - i|}{|z - 10 - 5i|} = 1$$

$$\Rightarrow |z + 6 - i| = |z - 10 - 5i|$$

$$\Rightarrow |z - (-6 + i)| = |z - (10 + 5i)|$$

perpendicular bisector of the line joining $(-6, 1)$ to $(10, 5)$.

$$\text{So, } |x + iy + 6 - i| = |x + iy - 10 - 5i|$$

$$\Rightarrow |(x + 6) + i(y - 1)| = |(x - 10) + i(y - 5)|$$

$$\Rightarrow (x + 6)^2 + (y - 1)^2 = (x - 10)^2 + (y - 5)^2$$

$$\Rightarrow x^2 + 12x + 36 + y^2 - 2y + 1 = x^2 - 20x + 100 + y^2 - 10y + 25$$

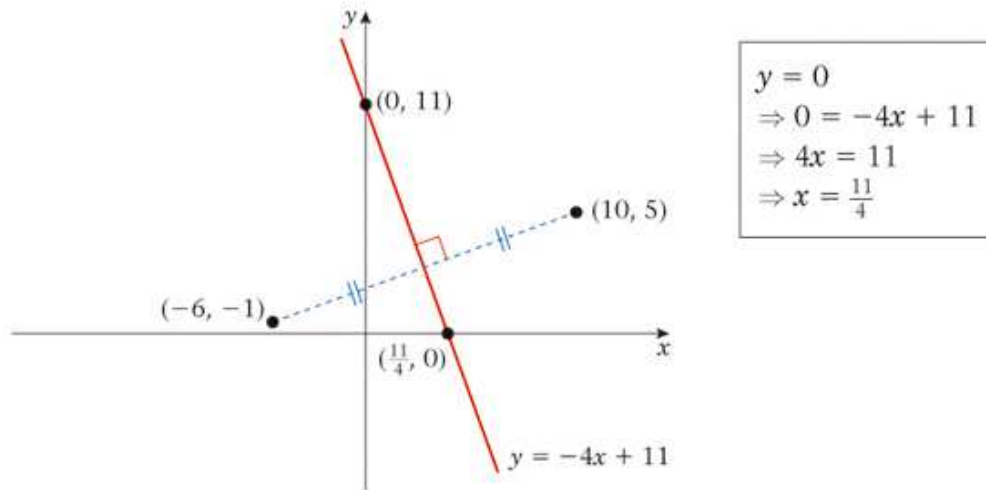
$$\Rightarrow 12x - 2y + 37 = -20x - 10y + 125$$

$$\Rightarrow 32x + 8y + 37 - 125 = 0$$

$$\Rightarrow 32x + 8y - 88 = 0$$

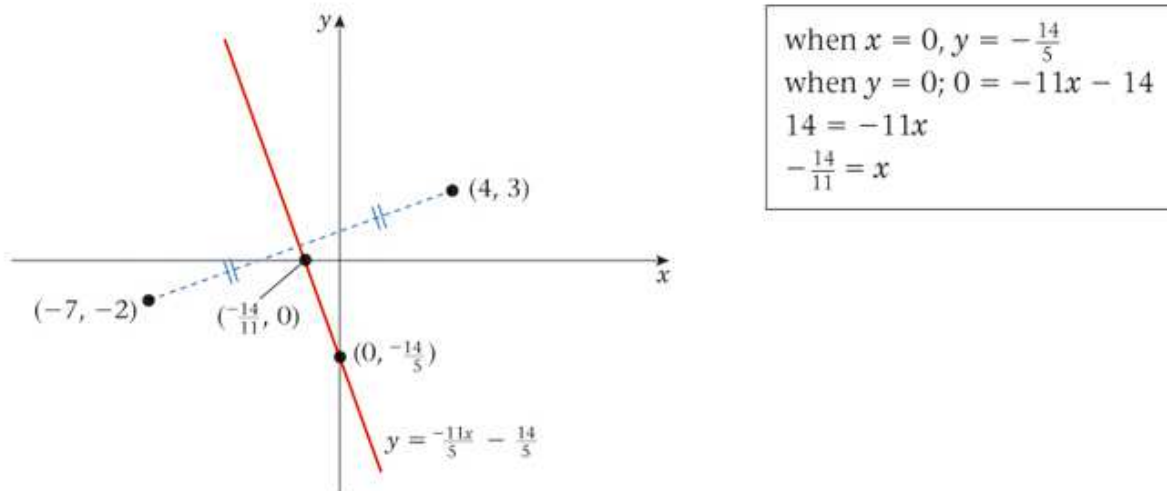
$$\Rightarrow 4x + y - 11 = 0$$

$$\Rightarrow y = -4x + 11$$



Equation: $y = -4x + 11$

k $|z + 7 + 2i| = |z - 4 - 3i|$
 $\Rightarrow |z - (-7 - 2i)| = |z - (4 + 3i)|$
 perpendicular bisector of the line joining $(-7, -2)$ to $(4, 3)$.
 So, $|x + iy + 7 + 2i| = |x + iy - 4 - 3i|$
 $\Rightarrow |(x + 7) + i(y + 2)| = |(x - 4) + i(y - 3)|$
 $\Rightarrow (x + 7)^2 + (y + 2)^2 = (x - 4)^2 + (y - 3)^2$
 $\Rightarrow x^2 + 14x + 49 + y^2 + 4y + 4 = x^2 - 8x + 16 + y^2 - 6y + 9$
 $\Rightarrow 14x + 4y + 53 = -8x - 6y + 25$
 $\Rightarrow 22x + 10y + 28 = 0$
 $\Rightarrow 11x + 5y + 14 = 0$
 $\Rightarrow 5x = -11x - 14$
 $\Rightarrow y = -\frac{11x}{5} - \frac{14}{5}$



Equation: $y = -\frac{11x}{5} - \frac{14}{5}$

$$\mathbf{I} \quad |z + 1 - 6i| = |2 + 3i - z|$$

$$\Rightarrow |z + 1 - 6i| = |(-1)(z - 2 - 3i)|$$

$$\Rightarrow |z + 1 - 6i| = |(-1)||z - 2 - 3i|$$

$$\Rightarrow |z - (-1 + 6i)| = |z - (2 + 3i)|$$

perpendicular bisector of the line joining $(-1, 6)$ to $(2, 3)$.

$$\text{So, } |x + iy + 1 - 6i| = |x + iy - 2 - 3i|$$

$$\Rightarrow |(x + 1) + i(y - 6)| = |(x - 2) + i(y - 3)|$$

$$\Rightarrow (x + 1)^2 + (y - 6)^2 = (x - 2)^2 + (y - 3)^2$$

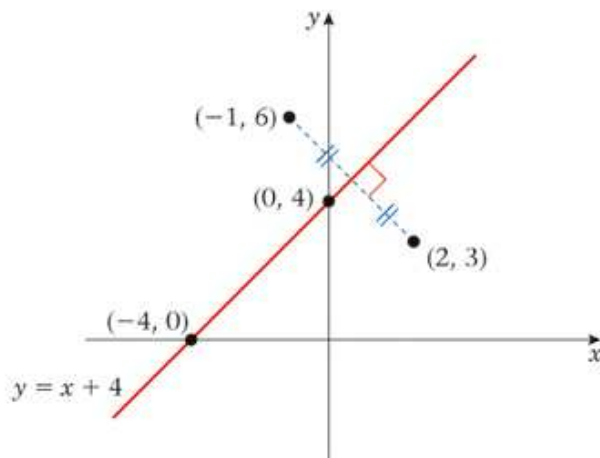
$$\Rightarrow x^2 + 2x + 1 + y^2 - 12y + 36 = x^2 - 4x + 4 + y^2 - 6y + 9$$

$$\Rightarrow 2x - 12y + 37 = -4x - 6y + 13$$

$$\Rightarrow 6x - 6y + 24 = 0$$

$$\Rightarrow x - y + 4 = 0$$

$$\Rightarrow y = x + 4$$



$$\begin{array}{l} y = 0 \\ x = -4 \end{array}$$

$$x = 0, y = 4$$

Equation: $y = x + 4$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise F, Question 4

Question:

Find the Cartesian equation of the locus of z when:

a $z - z^* = 0$

b $z + z^* = 0$

Solution:

a $z - z^* = 0$

$$\Rightarrow (x + iy) - (x - iy) = 0$$

$$\Rightarrow 2iy = 0 \quad (\times i)$$

$$\Rightarrow -2y = 0$$

$$\Rightarrow y = 0$$

The Cartesian equation of the locus of $z - z^* = 0$ is $y = 0$.

$\begin{aligned} z &= x + iy \\ z^* &= x - iy \end{aligned}$
--

b $z + z^* = 0$

$$\Rightarrow (x + iy) + (x - iy) = 0$$

$$\Rightarrow 2x = 0$$

$$x = 0$$

The Cartesian equation of the locus of $z + z^* = 0$ is $x = 0$.

$\begin{aligned} z &= x + iy \\ z^* &= x - iy \end{aligned}$
--

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise F, Question 5

Question:

Sketch the locus of z and give the Cartesian equation of the locus of z when:

a $|2 - z| = 3$

b $|5i - z| = 4$

c $|3 - 2i - z| = 3$

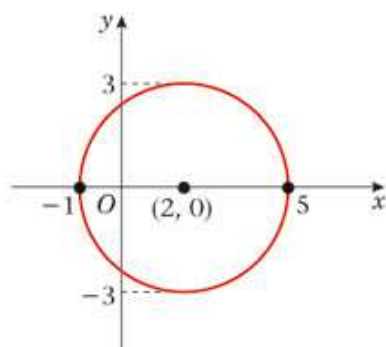
Solution:

a $|2 - z| = 3$

$$\Rightarrow |(-1)(z - 2)| = 3$$

$$\Rightarrow |(-1)||z - 2| = 3 \quad \leftarrow \boxed{|-1| = 1}$$

$$\Rightarrow |z - 2| = 3$$



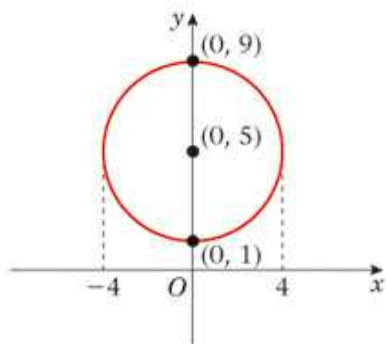
circle centre $(2, 0)$, radius 3
equation: $(x - 2)^2 + y^2 = 3^2$
 $(x - 2)^2 + y^2 = 9$

b $|5i - z| = 4$

$$\Rightarrow |(-1)(z - 5i)| = 4$$

$$\Rightarrow |(-1)||z - 5i| = 4$$

$$\Rightarrow |z - 5i| = 4 \quad \leftarrow \boxed{|-1| = 1}$$



circle centre $(0, 5)$, radius 4
equation: $x^2 + (y - 5)^2 = 4^2$
 $x^2 + (y - 5)^2 = 16$

$$\mathbf{c} \quad |3 - 2i - z| = 3$$

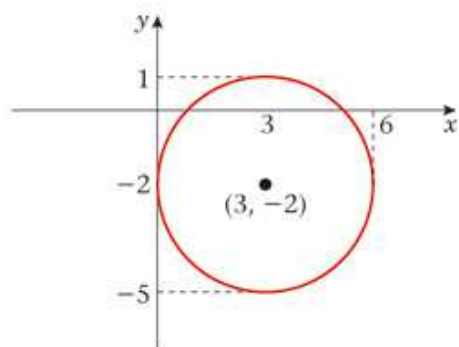
$$\Rightarrow |(-1)(z - 3 + 2i)| = 3$$

$$\Rightarrow |(-1)||z - 3 + 2i| = 3$$

$$\Rightarrow |z - 3 + 2i| = 3$$

$$|-1| = 1$$

$$\Rightarrow |z - (3 - 2i)| = 3$$



circle centre $(3, -2)$, radius 3

$$\text{equation: } (x - 3)^2 + (y + 2)^2 = 3^2$$

$$(x - 3)^2 + (y + 2)^2 = 9$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise F, Question 6

Question:

Sketch the locus of z and give the Cartesian equation of the locus of z when:

a $|z + 3| = 3|z - 5|$

b $|z - 3| = 4|z + 1|$

c $|z - i| = 2|z + i|$

d $|z + 2 - 7i| = 2|z - 10 + 2i|$

e $|z + 4 - 2i| = 2|z - 2 - 5i|$

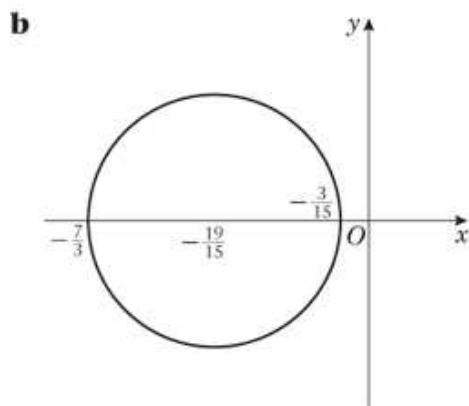
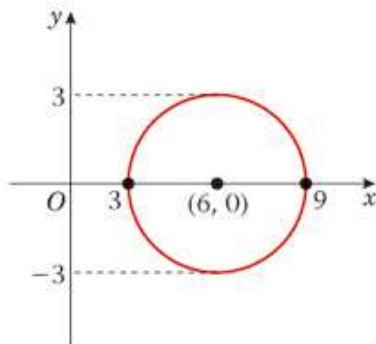
f $|z| = 2|2 - z|$

Solution:

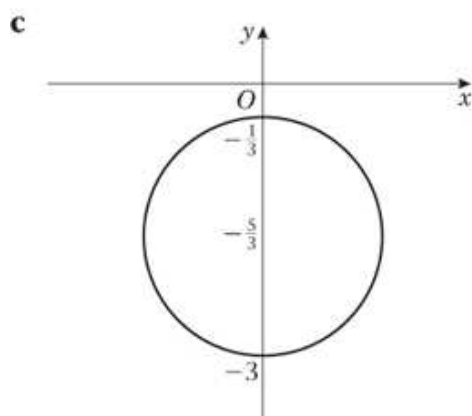
$$\begin{aligned}
 \mathbf{a} \quad |z + 3| &= 3|z - 5| \\
 \Rightarrow |x + iy + 3| &= 3|x + iy - 5| \\
 \Rightarrow |(x + 3) + iy| &= 3|(x - 5) + iy| \\
 \Rightarrow |(x + 3) + iy|^2 &= 3^2|(x - 5) + iy|^2 \\
 \Rightarrow (x + 3)^2 + y^2 &= 9[(x - 5)^2 + y^2] \\
 \Rightarrow x^2 + 6x + 9 + y^2 &= 9[x^2 - 10x + 25 + y^2] \\
 \Rightarrow x^2 + 6x + 9 + y^2 &= 9x^2 - 90x + 225 + 9y^2 \\
 \Rightarrow 0 &= 8x^2 - 96x + 8y^2 + 216 \quad (\div 8) \\
 \Rightarrow x^2 - 12x + y^2 + 27 &= 0 \\
 \Rightarrow (x - 6)^2 - 36 + y^2 + 27 &= 0 \\
 \Rightarrow (x - 6)^2 + y^2 - 9 &= 0 \\
 \Rightarrow (x - 6)^2 + y^2 &= 9
 \end{aligned}$$

The Cartesian equation of the locus of z is $(x - 6)^2 + y^2 = 9$

This is a circle centre $(6, 0)$, radius = 3



$$\begin{aligned}
 |z - 3| &= 4|z + 1| \\
 |x + iy - 3| &= 4|x + iy + 1| \\
 |x - 3 + iy|^2 &= 16|x + 1 + iy|^2 \\
 (x - 3)^2 + y^2 &= 16((x + 1)^2 + y^2) \\
 x^2 - 6x + 9 + y^2 &= 16(x^2 + 2x + 1 + y^2) \\
 &= 16x^2 + 32x + 16 + 16y^2 \\
 15x^2 + 38x + 15y^2 + 7 &= 0 \\
 x^2 + \frac{38}{15}x + y^2 + \frac{7}{15} &= 0 \\
 \left(x + \frac{19}{15}\right)^2 - \frac{19^2}{15^2} + y^2 + \frac{7}{15} &= 0 \\
 \left(x + \frac{19}{15}\right)^2 + y^2 &= \frac{256}{225} \\
 \text{Circle centre } \left(-\frac{19}{15}, 0\right) \text{ radius } \frac{16}{15}
 \end{aligned}$$



$$|z - i| = 2|z + i|$$

$$|x + iy - i| = 2|x + iy + i|$$

$$|x + i(y - 1)|^2 = 4|x + i(y + 1)|^2$$

$$x^2 + (y - 1)^2 = 4[x^2 + (y + 1)^2]$$

$$x^2 + y^2 - 2y + 1 = 4(x^2 + y^2 + 2y)$$

$$= 4x^2 + 4y^2 + 8y$$

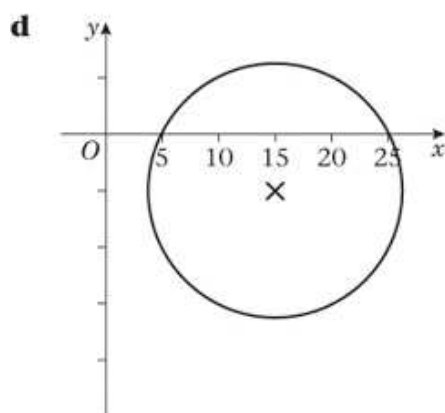
$$3x^2 + 3y^2 + 10y + 3 = 0$$

$$x^2 + y^2 + \frac{10}{3}y + 1 = 0$$

$$x^2 + \left(y + \frac{5}{3}\right)^2 - \frac{25}{9} + 1 = 0$$

$$x^2 + \left(y + \frac{5}{3}\right)^2 = \frac{16}{9}$$

Circle centre $\left(0, -\frac{5}{3}\right)$ radius $\frac{4}{3}$



$$|z + 2 - 7i| = 2|z - 10 + 2i|$$

$$|x + iy + 2 - 7i| = 2|x + iy - 10 + 2i|$$

$$|(x + 2) + i(y - 7)|^2 = 4|(x - 10) + i(y + 2)|^2$$

$$(x + 2)^2 + (y - 7)^2 = 4[(x - 10)^2 + (y + 2)^2]$$

$$x^2 + 4x^2 + 4 + y^2 - 14y + 49 = [x^2 - 20x + 100$$

$$+ y^2 + 4y + 4]$$

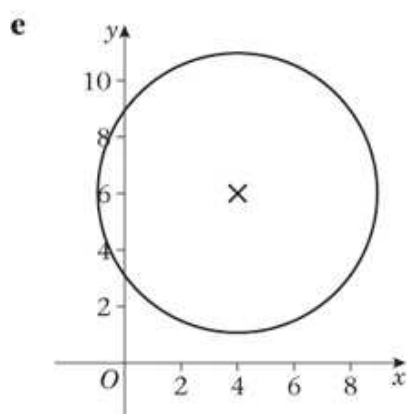
$$3x^2 - 84x + 3y^2 + 30y + 363 = 0$$

$$x^2 - 28x + y^2 + 10y + 121 = 0$$

$$(x - 14)^2 - 14^2 + (y + 5)^2 - 5^2 + 121 = 0$$

$$(x - 14)^2 + (y + 5)^2 = 100$$

Circle centre $(14, -5)$ radius 10



$$|z + 4 - 2i| = 2|z - 2 - 5i|$$

$$|x + iy + 4 - 2i| = 2|x + iy - 2 - 5i|$$

$$|(x + 4) + i(y - 2)|^2 = 4|(x - 2) + i(y - 5)|^2$$

$$(x + 4)^2 + (y - 2)^2 = 4[(x - 2)^2 + (y - 5)^2]$$

$$x^2 + 8x^2 + 16 + y^2 - 4y + 4 = [x^2 - 4x + 4$$

$$+ y^2 + 10y + 25]$$

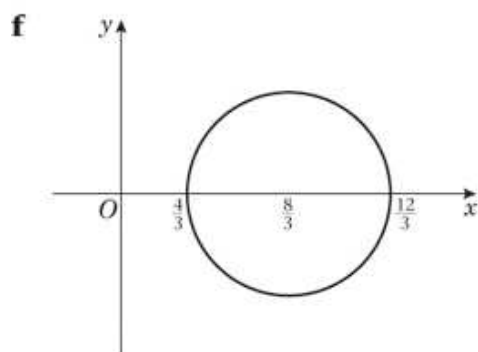
$$3x^2 - 24x + 3y^2 - 36y + 96 = 0$$

$$x^2 - 8x + y^2 - 12y + 32 = 0$$

$$(x - 4)^2 - 16 + (y - 6)^2 - 36 + 32 = 0$$

$$(x - 4)^2 + (y - 6)^2 = 20$$

Circle centre $(4, 6)$ radius $\sqrt{20} = 2\sqrt{5}$



$$\begin{aligned}
 |z| &= 2|2 - z| \\
 &= 2|-1||z - 2| \\
 |x + iy| &= 2 \times 1 \times |x + iy - 2| \\
 x^2 + y^2 &= 4((x - 2)^2 + y^2) \\
 x^2 + y^2 &= 4(x^2 - 4x + 4 + y^2) \\
 3x^2 - 16x + 3y^2 + 16 &= 0 \\
 x^2 - \frac{16}{3}x + y^2 + \frac{16}{3} &= 0 \\
 \left(x - \frac{8}{3}\right)^2 - \frac{64}{9} + y^2 + \frac{16}{3} &= 0 \\
 \left(x - \frac{8}{3}\right)^2 + y^2 &= \frac{16}{9} \\
 \text{Circle centre } \left(\frac{8}{3}, 0\right) \text{ radius } \frac{4}{3}
 \end{aligned}$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise F, Question 7

Question:

Sketch the locus of z when:

a $\arg\left(\frac{z}{z+3}\right) = \frac{\pi}{4}$

c $\arg\left(\frac{z}{z-2}\right) = \frac{\pi}{3}$

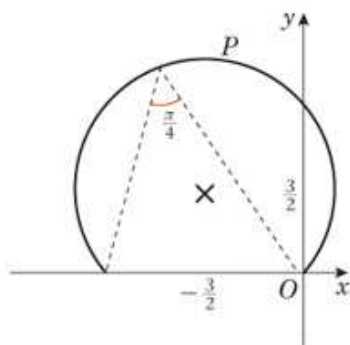
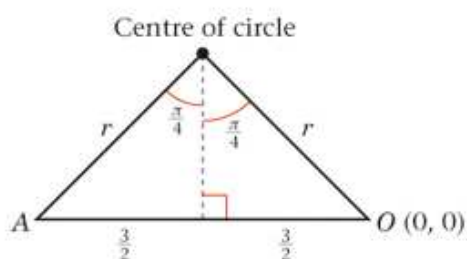
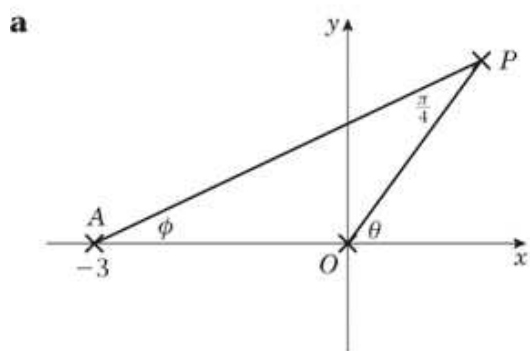
e $\arg z - \arg(z-2+3i) = \frac{\pi}{3}$

b $\arg\left(\frac{z-3i}{z+4}\right) = \frac{\pi}{6}$

d $\arg\left(\frac{z-3i}{z-5}\right) = \frac{\pi}{4}$

f $\arg\left(\frac{z-4i}{z+4}\right) = \frac{\pi}{2}$

Solution:



$$\arg\left(\frac{z}{z+3}\right) = \frac{\pi}{4}$$

$$\arg z - \arg(z+3) = \frac{\pi}{4}$$

$$\arg z - \arg(z - (-3)) = \frac{\pi}{4}$$

$$\arg z = \theta$$

$$\arg(z - (-3)) = \phi$$

$$\theta - \phi = \frac{\pi}{4}$$

$$\theta = \phi + \frac{\pi}{4}$$

P lies on an arc of a circle cut off at

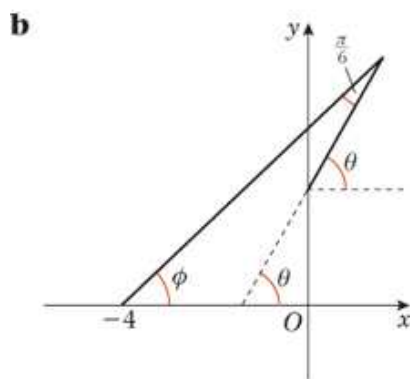
$A(-3, 0)$ and $O(0, 0)$

Angle at the centre is twice the angle at the

circumference $\therefore \frac{\pi}{2}$

It follows that the centre is at $\left(-\frac{3}{2}, \frac{3}{2}\right)$

and the radius is $\frac{3\sqrt{2}}{2}$



$$\arg\left(\frac{z-3i}{z+4}\right) = \frac{\pi}{6}$$

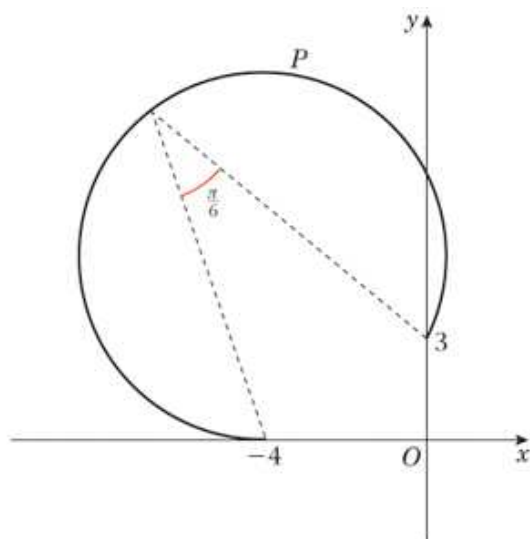
$$\arg(z-3i) - \arg(z - (-4)) = \frac{\pi}{6}$$

$$\arg(z-3i) = \theta.$$

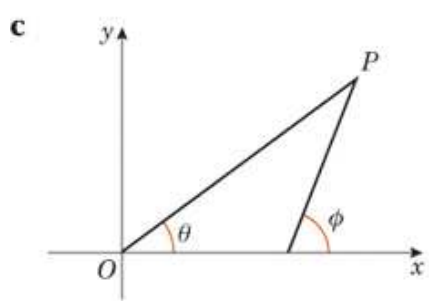
$$\arg(z - (-4)) = \phi$$

$$\theta - \phi = \frac{\pi}{6}$$

Arc of a circle from $(-4, 0)$ to $(0, 3)$



(The centre is at $(-\frac{4 + 3\sqrt{3}}{2}, \frac{3 + 4\sqrt{3}}{2})$ you do not need to calculate this for a sketch!)



$$\arg\left(\frac{z}{z-2}\right) = \frac{\pi}{3}$$

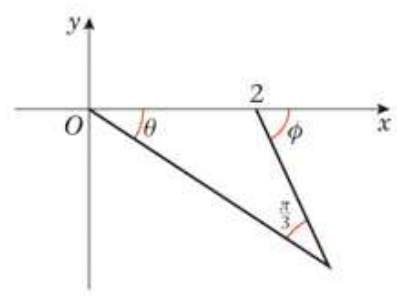
$$\arg z = \theta$$

$$\arg(z-2) = \phi$$

$$\theta - \phi = \frac{\pi}{3}$$

As our diagram has $\phi > \theta$, we have P on the wrong side of the line joining O or ϕ .
We want the arc below the x -axis.

Redrawing:



$$\arg z = -\theta$$

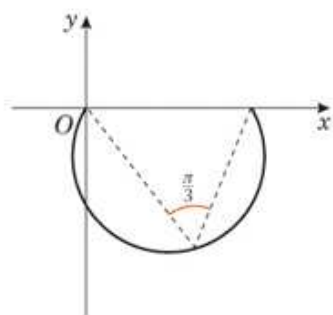
$$\arg(z-2) = -\phi$$

$$\text{Hence } \arg z - \arg(z-2) = \frac{\pi}{3}$$

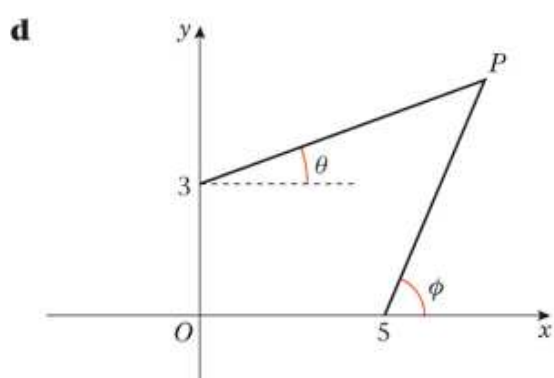
$$\text{becomes } -\theta - (-\phi) = \frac{\pi}{3}$$

$$\phi = \theta + \frac{\pi}{3}$$

Arc of a circle, ends 0 and 2, subtending angle $\frac{\pi}{3}$



[The centre is at $(1, -\frac{1}{\sqrt{3}})$ radius $\frac{2\sqrt{3}}{3}$ not needed to be calculated for a sketch]



$$\arg\left(\frac{z - 3i}{z - 5}\right) = \frac{\pi}{4}$$

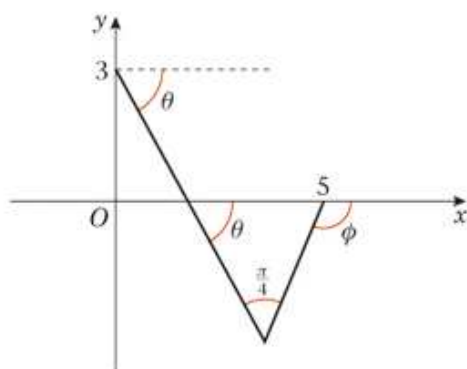
$$\arg(z - 3i) - \arg(z - 5) = \frac{\pi}{4}$$

$$\arg(z - 3i) = \theta$$

$$\arg(z - 5) = \phi$$

$$\theta - \phi = \frac{\pi}{4}$$

But $\phi > \theta$, we have P on the wrong side of the line joining $3i$ and 5 .

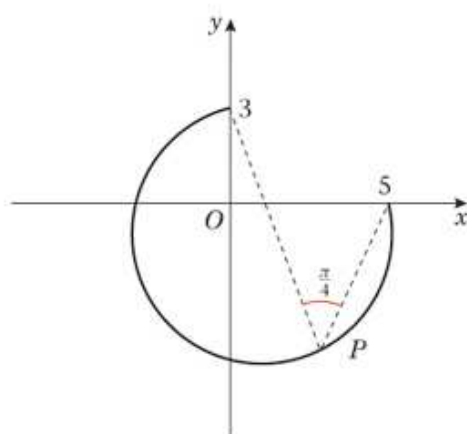


$$\arg(z - 3i) = -\theta$$

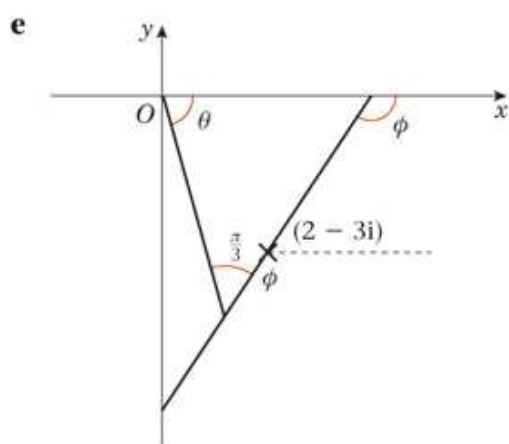
$$\arg(z - 5) = -\phi$$

$$-\theta - (-\phi) = \frac{\pi}{4}$$

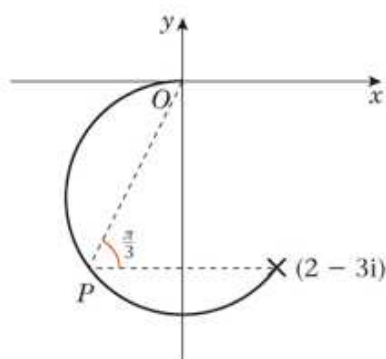
$$\phi = \theta + \frac{\pi}{4}$$



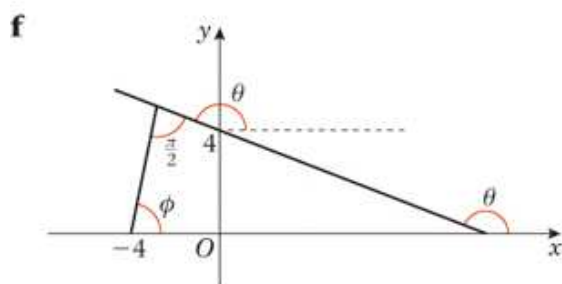
(Arc of Circle centre $(1, -1)$ radius $\sqrt{17}$ not needed for sketch)



$$\begin{aligned} \arg z - \arg(z - 2 + 3i) &= \frac{\pi}{3} \\ \arg z - \arg(z - (2 - 3i)) &= \frac{\pi}{3} \\ \arg z &= -\theta \\ \arg(z - (2 - 3i)) &= -\phi \\ -\theta - (-\phi) &= \frac{\pi}{3} \\ \phi &= \theta + \frac{\pi}{3} \end{aligned}$$

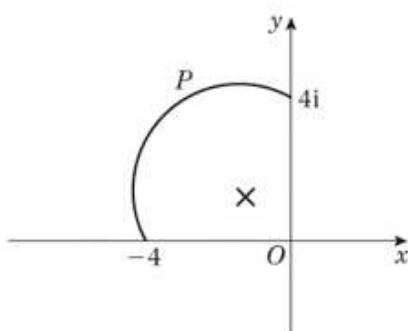


Arc of circle, centre at $\left(\frac{2 - \sqrt{3}}{2}, -\frac{9 + 2\sqrt{3}}{6}\right)$, this need not be calculated for your sketch.



$$\begin{aligned} \arg\left(\frac{z - 4i}{z + 4}\right) &= \frac{\pi}{2} \\ \arg(z - 4i) - \arg(z + 4) &= \frac{\pi}{2} \\ \arg(z - 4i) &= \theta \\ \arg(z + 4) &= \phi = \arg(z - (-4i)) \\ \theta - \phi &= \frac{\pi}{2} \\ \theta &= \phi + \frac{\pi}{2} \end{aligned}$$

The locus is an arc of a circle, ends at -4 and $4i$, angle subtended being $\frac{\pi}{2}$.
 \therefore It is a semi-circle.



(Circle arc has centre $(-2, 2)$, radius $2\sqrt{2}$)

© Pearson Education Ltd 2009

Solutionbank FP2

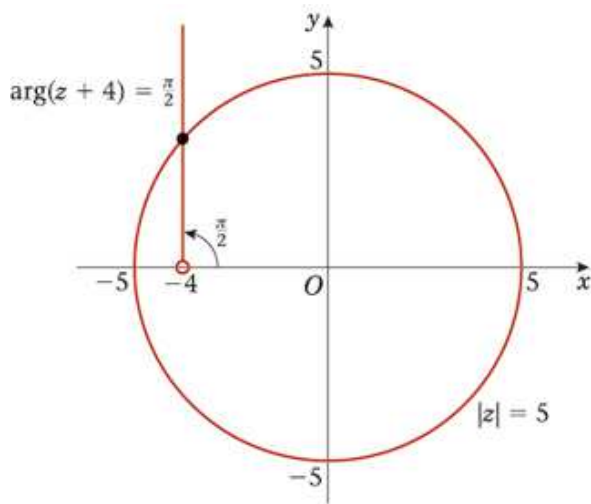
Edexcel AS and A Level Modular Mathematics

Exercise F, Question 8

Question:

Use the Argand diagram to find the value of z that satisfies the equations
 $|z| = 5$ and $\arg(z + 4) = \frac{\pi}{2}$.

Solution:



$|z| = 5$
 is a circle centre $(0, 0)$,
 radius 5

$\arg(z + 4) = \frac{\pi}{2}$ is a
 half-line from $(-4, 0)$
 making an angle of
 $\frac{\pi}{2}$ with the positive
 x -axis.

$$|z| = 5 \Rightarrow x^2 + y^2 = 25 \quad \text{①}$$

$$\arg(z + 4) = \frac{\pi}{2} \Rightarrow x = -4 \text{ and } y > 0 \quad \text{②}$$

Substituting ② into ① gives

$$\begin{aligned} (-4)^2 + y^2 &= 25 \\ 16 + y^2 &= 25 \\ y^2 &= 9 \\ y &= 3 \end{aligned}$$

Therefore, $z = -4 + 3i$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise F, Question 9

Question:

Given that the complex number z satisfies $|z - 2 - 2i| = 2$,

a sketch, on an Argand diagram, the locus of z .

Given further that $\arg(z - 2 - 2i) = \frac{\pi}{6}$,

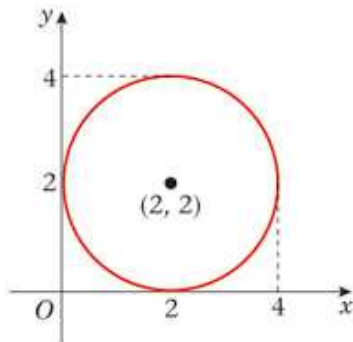
b find the value of z in the form $a + ib$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

Solution:

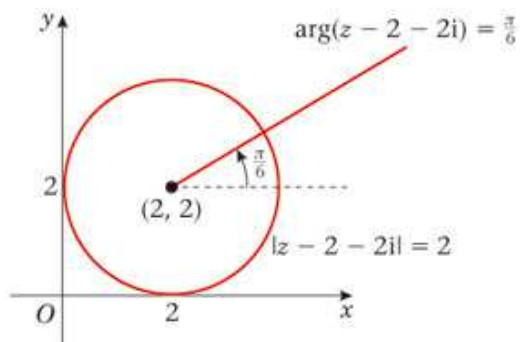
a $|z - 2 - 2i| = 2$

$$\Rightarrow |z - (2 + 2i)| = 2$$

The locus of z is a circle centre $(2, 2)$, radius 2.



b $\arg(z - 2 - 2i) = \frac{\pi}{6}$, is a half-line from $(2, 2)$, as shown below.



$$|z - 2 - 2i| = 2 \Rightarrow (x - 2)^2 + (y - 2)^2 = 4 \quad \textcircled{1}$$

$$\arg(z - 2 - 2i) = \frac{\pi}{6} \Rightarrow \arg(x + iy - 2 - 2i) = \frac{\pi}{6}$$

$$\Rightarrow \arg((x - 2) + i(y - 2)) = \frac{\pi}{6}$$

$$\Rightarrow \frac{y - 2}{x - 2} = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

$$\Rightarrow y - 2 = \frac{1}{\sqrt{3}}(x - 2)$$

$$\Rightarrow (y - 2)^2 = \left[\frac{1}{\sqrt{3}}(x - 2)\right]^2$$

$$\Rightarrow (y - 2)^2 = \frac{1}{3}(x - 2)^2 \quad \textcircled{2}$$

$$\begin{aligned}\text{Substituting ② into ① gives } (x - 2)^2 + \frac{1}{3}(x - 2)^2 &= 4 \\ \Rightarrow \frac{4}{3}(x - 2)^2 &= 4 \\ \Rightarrow 4(x - 2)^2 &= 12 \\ \Rightarrow (x - 2)^2 &= 3 \\ \Rightarrow x - 2 &= \pm\sqrt{3} \\ \Rightarrow x &= 2 \pm \sqrt{3}\end{aligned}$$

From the Argand diagram, $x > 2$.

$$\text{So } x = 2 + \sqrt{3} \quad \text{③}$$

$$\text{As } y - 2 = \frac{1}{\sqrt{3}}(x - 2) \quad \text{④}$$

$$\begin{aligned}\text{Substituting ③ into ④ gives } y - 2 &= \frac{1}{\sqrt{3}}(2 + \sqrt{3} - 2) \\ \Rightarrow y - 2 &= \frac{1}{\sqrt{3}}(\sqrt{3}) \\ \Rightarrow y - 2 &= 1 \\ \Rightarrow y &= 3\end{aligned}$$

Therefore, $z = (2 + \sqrt{3}) + 3i$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise F, Question 10

Question:

Sketch on the same Argand diagram the locus of points satisfying

a $|z - 2i| = |z - 8i|$, **b** $\arg(z - 2 - i) = \frac{\pi}{4}$.

The complex number z satisfies both $|z - 2i| = |z - 8i|$ and $\arg(z - 2 - i) = \frac{\pi}{4}$.

c Use your answers to parts **a** and **b** to find the value of z .

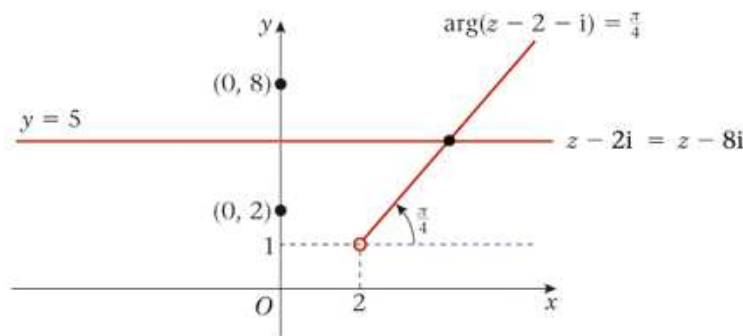
Solution:

a $|z - 2i| = |z - 8i|$

perpendicular bisector of the line joining $(0, 2)$ to $(0, 8)$, having equation $y = 5$.

b $\arg(z - 2 - i) = \frac{\pi}{4}$

is a half-line from $(1, 1)$, as shown below.



c $|z - 2i| = |z - 8i| \Rightarrow y = 5$ ①

$$\arg(z - 2 - i) = \frac{\pi}{4} \Rightarrow \arg(x + iy - 2 - i) = \frac{\pi}{4}$$

$$\Rightarrow \arg((x - 2) + i(y - 1)) = \frac{\pi}{4}$$

$$\Rightarrow \frac{y - 1}{x - 2} = \tan \frac{\pi}{4}$$

$$\Rightarrow \frac{y - 1}{x - 2} = 1$$

$$\Rightarrow y - 1 = x - 2$$

$$\Rightarrow y - x - 1 = 0 \quad \text{②}$$

Substituting ① into ② gives $5 = x - 1$

$$\Rightarrow 6 = x$$

Therefore, $z = 6 + 5i$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise F, Question 11

Question:

Sketch on the same Argand diagram the locus of points satisfying

a $|z - 3 + 2i| = 4$

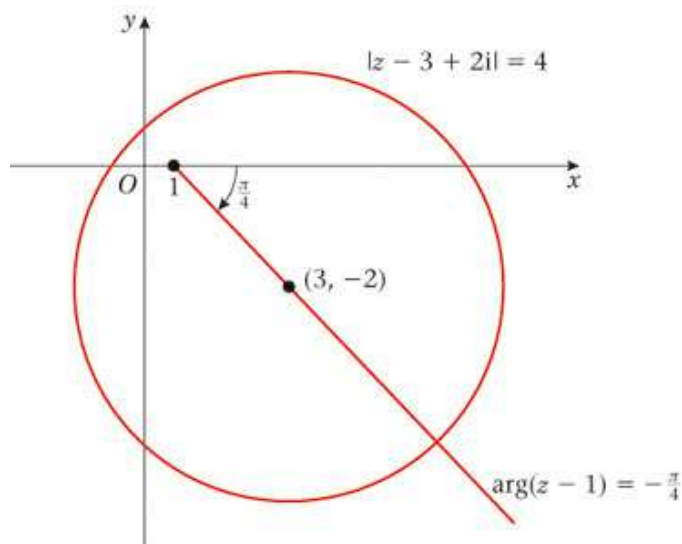
b $\arg(z - 1) = -\frac{\pi}{4}$.

The complex number z satisfies both $|z - 3 + 2i| = 4$ and $\arg(z - 1) = -\frac{\pi}{4}$.

Given that $z = a + ib$ where $a \in \mathbb{R}$ and $b \in \mathbb{R}$,

c find the exact value of a and the exact value of b .

Solution:



a $|z - 3 + 2i| = 4$
is a circle centre $(3, -2)$
radius 4.

b $\arg(z - 1) = -\frac{\pi}{4}$
is a half-line from $(1, 0)$
making an angle of $-\frac{\pi}{4}$
with the positive x -axis.

c $|z - 3 + 2i| = 4 \Rightarrow (x - 3)^2 + (y + 2)^2 = 16$ ①

$$\arg(z - 1) = -\frac{\pi}{4} \Rightarrow \arg(x + iy - 1) = -\frac{\pi}{4}$$

$$\Rightarrow \arg((x - 1) + iy) = -\frac{\pi}{4}$$

$$\Rightarrow \frac{y}{x - 1} = \tan\left(-\frac{\pi}{4}\right)$$

$$\Rightarrow \frac{y}{x - 1} = -1$$

$$\Rightarrow y = -1(x - 1)$$

$$\Rightarrow y = -x + 1$$

② for $x > 1, y < 0$

$$\begin{aligned}
\text{Substituting ② into ① gives } & (x - 3)^2 + (-x + 1 + 2)^2 = 16 \\
& \Rightarrow (x - 3)^2 + (-x + 3)^2 = 16 \\
& \Rightarrow x^2 - 6x + 9 + x^2 - 6x + 9 = 16 \\
& \Rightarrow 2x^2 - 12x + 18 = 16 \\
& \Rightarrow 2x^2 - 12x + 2 = 0 \\
& \Rightarrow x^2 - 6x + 1 = 0 \\
& \Rightarrow x = \frac{6 \pm \sqrt{36 - 4(1)(1)}}{2} \\
& \Rightarrow x = \frac{6 \pm \sqrt{32}}{2} \\
& \Rightarrow x = \frac{6 \pm \sqrt{16}\sqrt{2}}{2} \\
& \Rightarrow x = \frac{6 \pm 4\sqrt{2}}{2} \\
& \Rightarrow x = 3 \pm 2\sqrt{2}
\end{aligned}$$

as $x > 1$ then $x = 3 + 2\sqrt{2}$

$$\text{②} \Rightarrow y = -(3 + 2\sqrt{2}) + 1$$

$$\Rightarrow y = -3 - 2\sqrt{2} + 1$$

$$\Rightarrow y = -2 - 2\sqrt{2}$$

Therefore, $z = (3 + 2\sqrt{2}) + (-2 - 2\sqrt{2})i$

Note: $z = a + ib$

So $a = 3 + 2\sqrt{2}$, $b = -2 - 2\sqrt{2}$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise F, Question 12

Question:

On an Argand diagram the point P represents the complex number z .
Given that $|z - 4 - 3i| = 8$,

- find the Cartesian equation for the locus of P ,
- sketch the locus of P ,
- find the maximum and minimum values of $|z|$ for points on this locus.

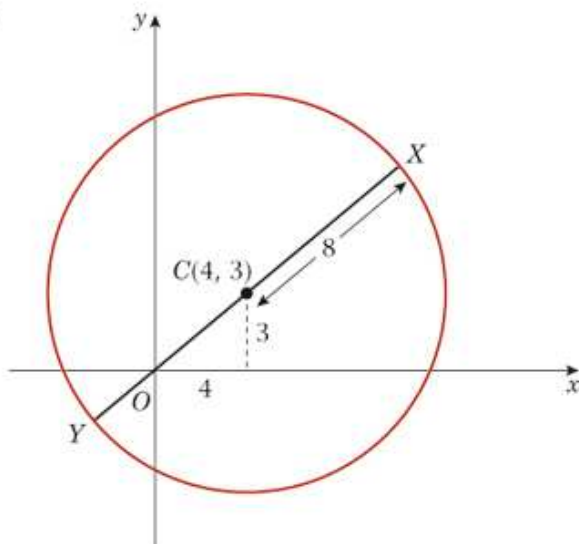
Solution:

a $|z - 4 - 3i| = 8 \Rightarrow |z - (4 + 3i)| = 8$

circle centre $(4, 3)$, radius 8

Hence the Cartesian equation of the locus of P is $(x - 4)^2 + (y - 3)^2 = 64$

b



c $|z|$ is the distance from $(0, 0)$ to the locus of points.

$|z|_{\max}$ is the distance OX .

$|z|_{\min}$ is the distance OY .

Note radius = $CY = CX = 8$

and $OC = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$

$|z|_{\max} = OC + CX = 5 + 8 = 13$

$|z|_{\min} = CY - OC = 8 - 5 = 3$

The maximum value of $|z|$ is 13 and the minimum value of $|z|$ is 3.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

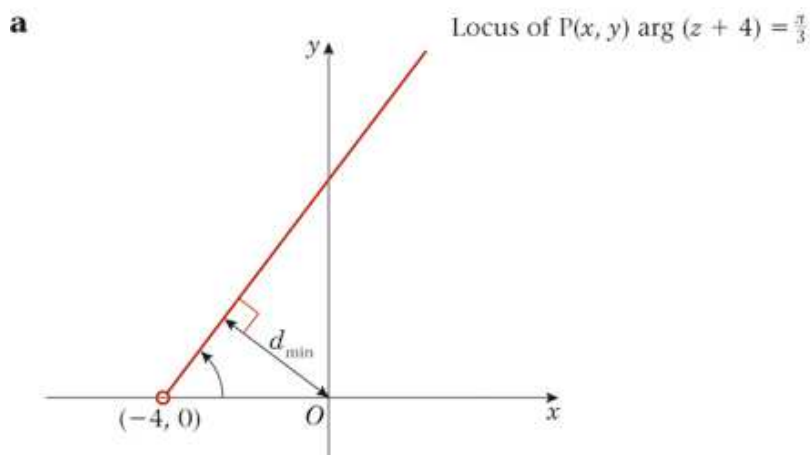
Exercise F, Question 13

Question:

On an Argand diagram the point P represents the complex number z .
Given that $|z - 4 - 3i| = 8$,

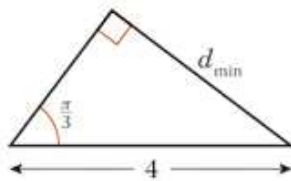
- find the Cartesian equation for the locus of P ,
- sketch the locus of P ,
- find the maximum and minimum values of $|z|$ for points on this locus.

Solution:



- b** $|z|$ is the distance from $(0, 0)$ to the locus of points.
Marked as d_{\min} on the Argand diagram is the minimum value of $|z|$.

Hence,



$$\frac{d_{\min}}{4} = \sin\left(\frac{\pi}{3}\right)$$

$$d_{\min} = 4 \sin\left(\frac{\pi}{3}\right)$$

$$d_{\min} = \frac{4\sqrt{3}}{2} = 2\sqrt{3}$$

Hence the minimum value of $|z|$ is $|z|_{\min} = 2\sqrt{3}$.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise F, Question 14

Question:

The complex number $z = x + iy$ satisfies the equation $|z + 1 + i| = 2|z + 4 - 2i|$.

The complex number z is represented by the point P on the Argand diagram.

- a** Show that the locus of P is a circle with centre $(-5, 3)$.
- b** Find the exact radius of this circle.

Solution:

$$\begin{aligned}
 \mathbf{a} \quad & |z + 1 + i| = 2|z + 4 - 2i| \\
 & \Rightarrow |x + iy + 1 + i| = 2|x + iy + 4 - 2i| \\
 & \Rightarrow |(x + 1) + i(y + 1)| = 2|(x + 4) + i(y - 2)| \\
 & \Rightarrow |(x + 1) + i(y + 1)|^2 = 2^2|(x + 4) + i(y - 2)|^2 \\
 & \Rightarrow (x + 1)^2 + (y + 1)^2 = 4[(x + 4)^2 + (y - 2)^2] \\
 & \Rightarrow x^2 + 2x + 1 + y^2 + 2y + 1 = 4[x^2 + 8x + 16 + y^2 - 4y + 4] \\
 & \Rightarrow x^2 + 2x + 1 + y^2 + 2y + 1 = 4x^2 + 32x + 64 + 4y^2 - 16y + 16 \\
 & \Rightarrow 0 = 3x^2 + 30x + 3y^2 - 18y + 64 + 16 - 1 - 1 \\
 & \Rightarrow 3x^2 + 30x + 3y^2 - 18y + 78 = 0 \\
 & \Rightarrow x^2 + 10x + y^2 - 6y + 26 = 0 \\
 & \Rightarrow (x + 5)^2 - 25 + (y - 3)^2 - 9 + 26 = 0 \\
 & \Rightarrow (x + 5)^2 + (y - 3)^2 = 25 + 9 - 26 \\
 & \Rightarrow (x + 5)^2 + (y - 3)^2 = 8
 \end{aligned}$$

Therefore the locus of P is a circle centre $(-5, 3)$. (as required)

$$\begin{aligned}
 \mathbf{b} \quad & \text{radius} = \sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2} \\
 & \text{The exact radius is } 2\sqrt{2}.
 \end{aligned}$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise F, Question 15

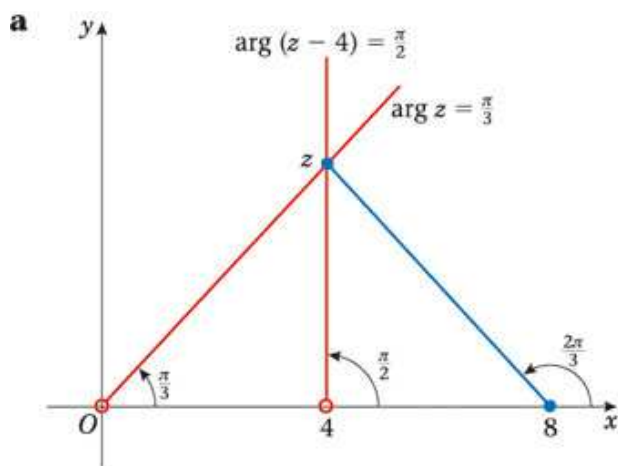
Question:

If the complex number z satisfies both $\arg z = \frac{\pi}{3}$ and $\arg(z - 4) = \frac{\pi}{2}$,

a find the value of z in the form $a + ib$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

b Hence, find $\arg(z - 8)$.

Solution:



From part **b** $\arg(z - 8) = \frac{2\pi}{3}$.

$$\arg z = \frac{\pi}{3} \Rightarrow \arg(x + iy) = \frac{\pi}{3}$$

$$\Rightarrow \frac{y}{x} = \tan \frac{\pi}{3}$$

$$\Rightarrow \frac{y}{x} = \sqrt{3}$$

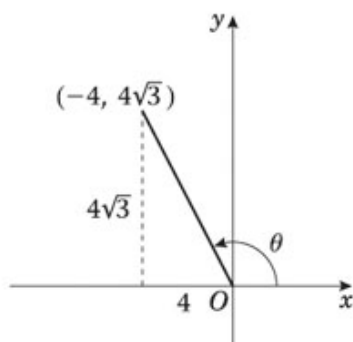
$$\Rightarrow y = \sqrt{3}x \text{ (for } x > 0, y > 0) \quad \textcircled{1}$$

$$\arg(z - 4) = \frac{\pi}{2} \Rightarrow x = 4 \text{ (for } y > 0) \quad \textcircled{2}$$

Substituting $\textcircled{2}$ and $\textcircled{1}$ gives $y = \sqrt{3}(4) = 4\sqrt{3}$

The value of z satisfying both equations is $z = 4 + 4\sqrt{3}i$.

$$\begin{aligned}\mathbf{b} \quad \arg(z - 8i) &= \arg(4 + 4\sqrt{3}i - 8) \\ &= \arg(-4 + 4\sqrt{3}i) = \theta\end{aligned}$$



$$\therefore \theta = \pi - \tan^{-1}\left(\frac{4\sqrt{3}}{4}\right) = \pi - \frac{\pi}{3}$$

$$\theta = \frac{2\pi}{3}$$

$$\text{Therefore, } \arg(z - 8) = \frac{2\pi}{3}.$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise F, Question 16

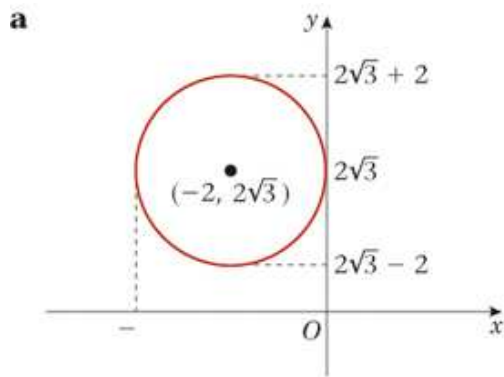
Question:

The point P represents a complex number z in an Argand diagram.

Given that $|z + 2 - 2\sqrt{3}i| = 2$,

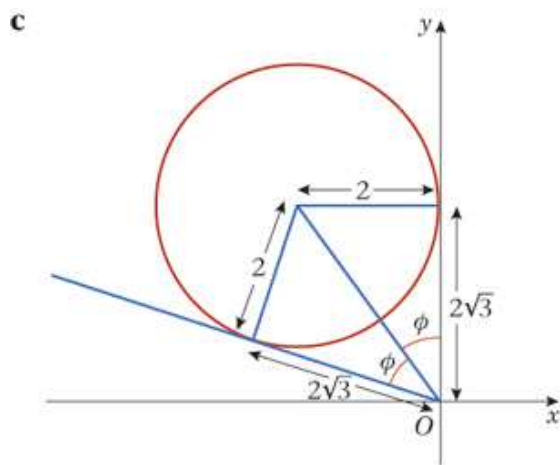
- a** sketch the locus of P on an Argand diagram.
- b** Write down the minimum value of $\arg z$.
- c** Find the maximum value of $\arg z$.

Solution:



$|z + 2 - 2\sqrt{3}i| = 2$ is a circle centre $(-2, 2\sqrt{3})$, radius 2.

b From the diagram, the minimum value of $\arg(z)$ is $\frac{\pi}{2}$.



The maximum value of $\arg z$ is $\frac{\pi}{2} + \phi + \phi = \frac{\pi}{2} + 2\phi$.

$$\tan \phi = \frac{2}{2\sqrt{3}}$$

$$\Rightarrow \tan \phi = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \phi = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$\arg(z)_{\max} = \frac{\pi}{2} + 2\left(\frac{\pi}{6}\right) = \frac{5\pi}{6}$$

The maximum value of $\arg(z)$ is $\frac{5\pi}{6}$.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise F, Question 17

Question:

The point P represents a complex number z in an Argand diagram.

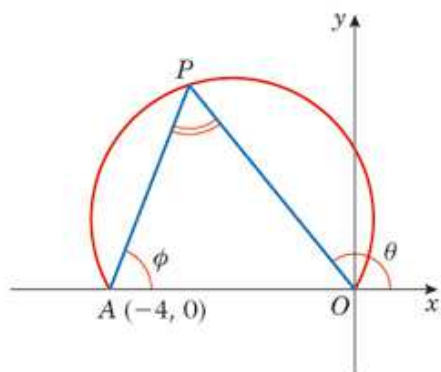
Given that $\arg z - \arg(z + 4) = \frac{\pi}{4}$ is a locus of points P lying on an arc of a circle C ,

- a** sketch the locus of points P ,
- b** find the coordinates of the centre of C ,
- c** find the radius of C ,
- d** find a Cartesian equation for the circle C ,
- e** find the finite area bounded by the locus of P and the x -axis.

Solution:

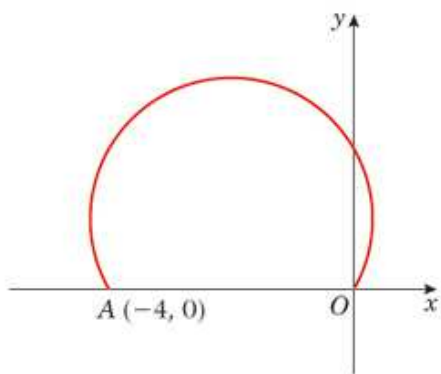
a $\arg(z) - \arg(z + 4) = \frac{\pi}{4}$

$\Rightarrow \theta - \phi = \frac{\pi}{4}$, where $\arg(z) = \theta$ and $\arg(z + 4) = \phi$

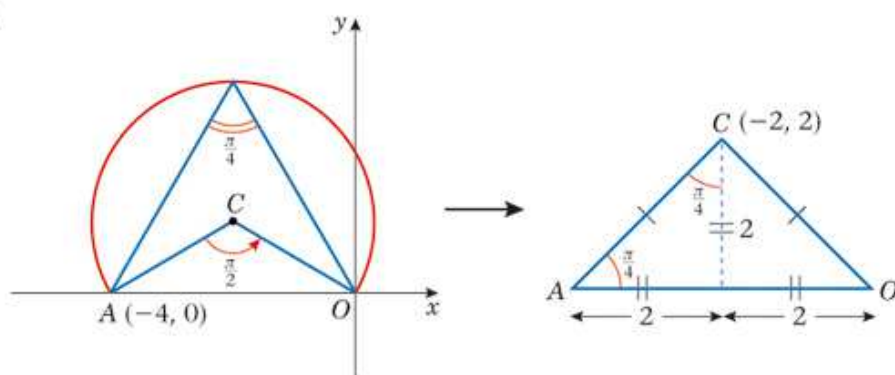


from $\triangle AOP$,
 $\widehat{APO} + \phi = \theta$
 $\Rightarrow \widehat{APO} = \theta - \phi$
 $\Rightarrow \widehat{APO} = \frac{\pi}{4}$

The locus of points P is an arc of a circle cut off at $(-4, 0)$ and $(0, 0)$, as shown below.



b



Therefore the centre of the circle has coordinates $(-2, 2)$.

c $r = \sqrt{2^2 + 2^2} = \sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$

Therefore, the radius of C is $2\sqrt{2}$.

d The Cartesian equation of C is $(x + 2)^2 + (y - 2)^2 = 8$.

e Finite area = Area of major sector ACO + Area $\triangle ACO$

$$\begin{aligned} &= \frac{1}{2}(\sqrt{8})^2\left(2\pi - \frac{\pi}{2}\right) + \frac{1}{2}(4)(2) \\ &= \frac{1}{2}(8)\left(2\pi - \frac{\pi}{2}\right) + 4 \\ &= 4\left(\frac{3\pi}{2}\right) + 4 \\ &= 6\pi + 4 \end{aligned}$$

Finite area bounded by the locus of P and the x -axis is $6\pi + 4$.

b, c, d Method ②:

$$\begin{aligned} \arg z - \arg(z + 4) &= \arg\left(\frac{z}{z + 4}\right) \\ &= \arg\left(\frac{x + iy}{x + iy + 4}\right) \\ &= \arg\left[\frac{x + iy}{(x + 4) + iy}\right] \\ &= \arg\left[\frac{(x + iy)}{(x + 4) + iy} \times \frac{(x + 4) - iy}{(x + 4) - iy}\right] \\ &= \arg\left[\frac{x(x + 4) - iyx + iy(x + 4) + y^2}{(x + 4)^2 + y^2}\right] \\ &= \arg\left[\left(\frac{x(x + 4) + y^2}{(x + 4)^2 + y^2}\right) + i\left(\frac{y(x + 4) - yx}{(x + 4)^2 + y^2}\right)\right] \\ &= \arg\left[\left(\frac{x^2 + 4x + y^2}{(x + 4)^2 + y^2}\right) + i\left(\frac{xy + 4y - xy}{(x + 4)^2 + y^2}\right)\right] \\ &= \arg\left[\left(\frac{x^2 + 4x + y^2}{(x + 4)^2 + y^2}\right) + i\left(\frac{4y}{(x + 4)^2 + y^2}\right)\right] \end{aligned}$$

$$\text{Applying } \arg\left(\frac{z}{z + 4}\right) = \frac{\pi}{4} \Rightarrow \frac{\left(\frac{4y}{(x + 4)^2 + y^2}\right)}{\left(\frac{x^2 + 4x + y^2}{(x + 4)^2 + y^2}\right)} = \tan\left(\frac{\pi}{4}\right) = 1$$

$$\Rightarrow \frac{4y}{x^2 + 4x + y^2} = 1$$

$$\Rightarrow 4y = x^2 + 4x + y^2$$

$$\Rightarrow 0 = x^2 + 4x + y^2 - 4y$$

$$\Rightarrow (x + 2)^2 - 4 + (y - 2)^2 - 4 = 0$$

$$\Rightarrow (x + 2)^2 + (y - 2)^2 = 8$$

$$\Rightarrow (x + 2)^2 + (y - 2)^2 = (2\sqrt{2})^2$$

C is a circle with centre $(-2, 2)$, radius $2\sqrt{2}$ and has Cartesian equation $(x + 2)^2 + (y - 2)^2 = 8$.

© Pearson Education Ltd 2009

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

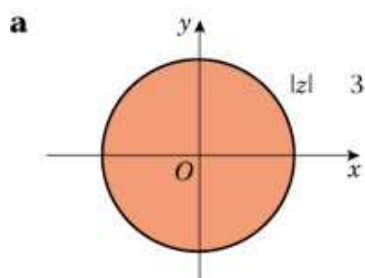
Exercise G, Question 1

Question:

On an Argand diagram shade in the regions represented by the following inequalities:

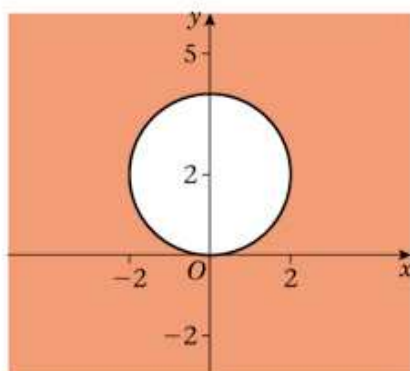
- a** $|z| < 3$ **b** $|z - 2i| > 2$ **c** $|z + 7| \geq |z - 1|$ **d** $|z + 6| > |z + 2 + 8i|$
e $2 \leq |z| \leq 3$ **f** $1 \leq |z + 4i| \leq 4$ **g** $3 \leq |z - 3 + 5i| \leq 5$ **h** $2|z| \leq |z - 3|$

Solution:

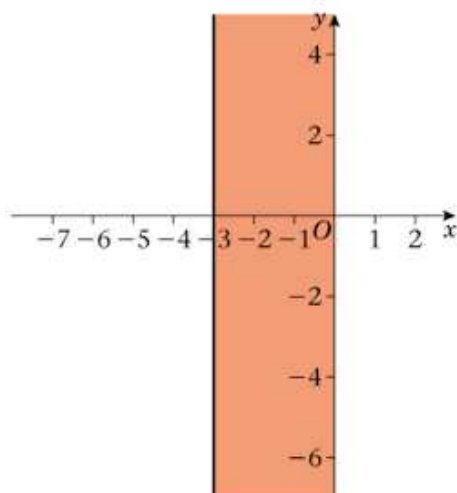


$|z| = 3$ represents a circle centre $(0, 0)$, radius 3

- b** $|z - (2i)| > 2$

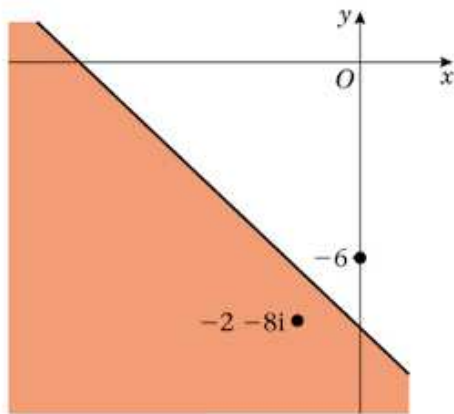


- c** $|z + 7| \leq |z - 1|$



$|z + 7| = |z - 1|$ represents a perpendicular bisector of the line joining $(-7, 0)$ to $(1, 0)$ which has equation $x = -3$.

d $|z + 6| > |z + 2 + 8i|$



$|z + 6| = |z + 2 + 8i|$ represents a perpendicular bisector of the line joining $(-6, 0)$ to $(-2, -8)$.

$$|x + iy + 6| = |x + iy + 2 + 8i|$$

$$\Rightarrow |x + 6 + iy| = |(x + 2) + i(y + 8)|$$

$$\Rightarrow |(x + 6) + iy|^2 = |(x + 2) + i(y + 8)|^2$$

$$\Rightarrow (x + 6)^2 + y^2 = (x + 2)^2 + (y + 8)^2$$

$$\Rightarrow x^2 + 12x + 36 + y^2 = x^2 + 4x + 4 + y^2 + 16y + 64$$

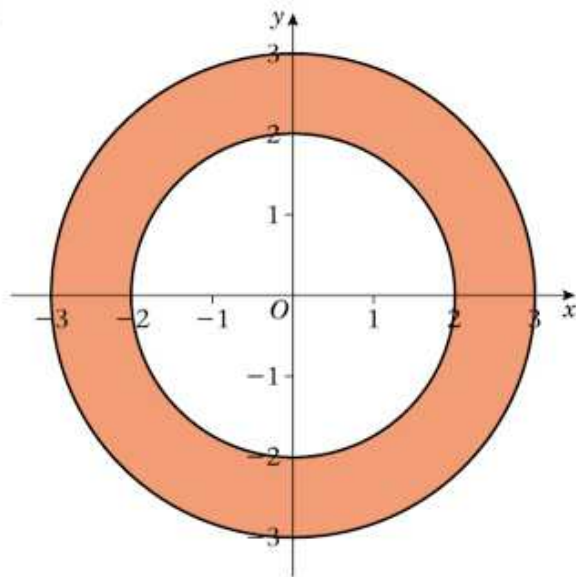
$$\Rightarrow (2x + 36 = 4x + 16y + 68$$

$$\Rightarrow 8x + 36 - 68 = 16y$$

$$\Rightarrow 8x - 32 = 16y$$

$$\Rightarrow y = \frac{1}{2}x - 2$$

e

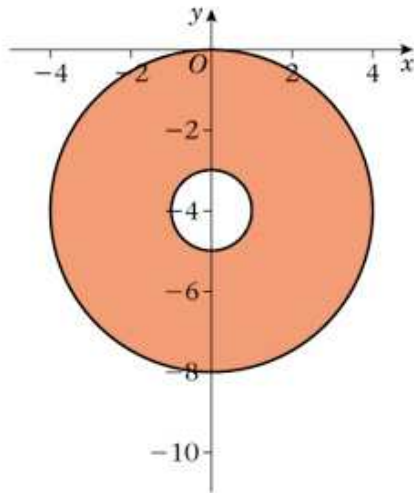


$$2 \leq |z| \leq 3$$

$|z| = 2$ represents a circle centre $(0, 0)$, radius 2

$|z| = 3$ represents a circle centre $(0, 0)$, radius 3

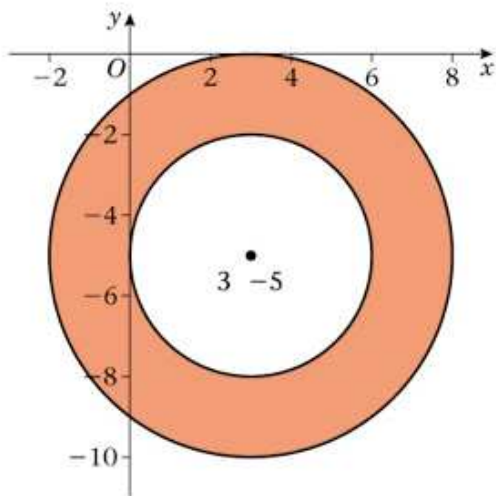
f $1 \leq |z + 4i|$



$|z + 4i| = 1$ represents a circle centre $(0, 4)$, radius 1.

$|z + 4i| = 4$ represents a circle centre $(0, -4)$, radius 4.

g $3 \leq |z - 3 + 5i| \leq 5$



h $2|z| \geq |z - 3|$

Consider $2|z| = |z - 3|$ let $z = x + iy$

$$2|x + iy| = |x + iy - 3|$$

$$4(x^2 + y^2) = (x - 3)^2 + y^2$$

$$4x^2 + 4y^2 = x^2 - 6x + 9 + y^2$$

$$3x^2 + 6x + 3y^2 - 9 = 0$$

$$(x + 1)^2 - 1 + y^2 - 3 = 0$$

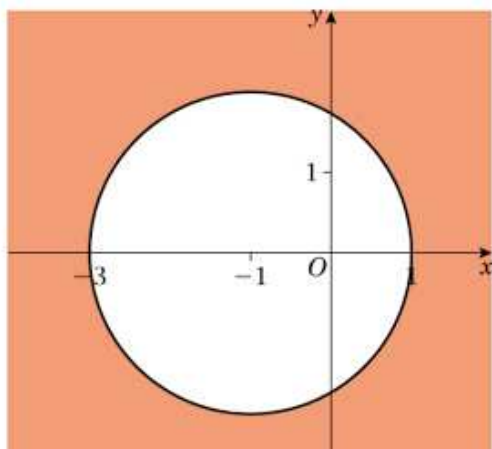
$$(x + 1)^2 + y^2 = 4$$

Circle centre $(-1, 0)$ radius 2.

Consider $z = 0$ in $2|z| \geq |z - 3|$

$$2 \times 0 \geq 3$$

So $z = 0$ is not in the region.



Solutionbank FP2

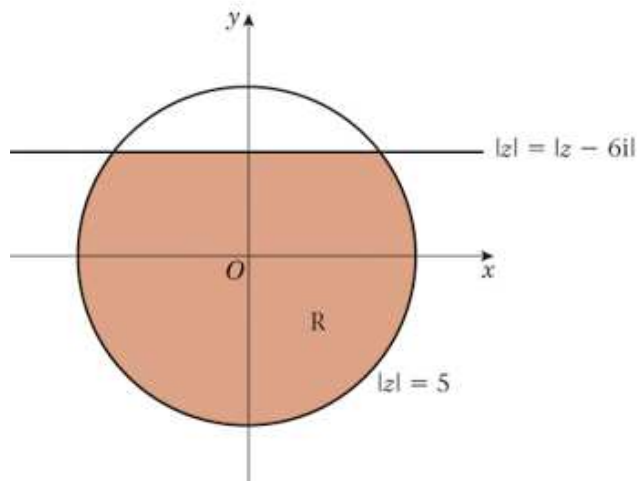
Edexcel AS and A Level Modular Mathematics

Exercise G, Question 2

Question:

The region R in an Argand diagram is satisfied by the inequalities $|z| \leq 5$ and $|z| \leq |z - 6i|$. Draw an Argand diagram and shade in the region R .

Solution:



$$|z| \leq 5$$

$$|z| \leq |z - 6i|$$

$|z| = 5$ represents a circle centre $(0, 0)$, radius 5

$|z| = |z - 6i|$ represents a perpendicular bisector of the line joining $(0, 0)$, to $(0, 6)$ and has the equation $y = 3$.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise G, Question 3

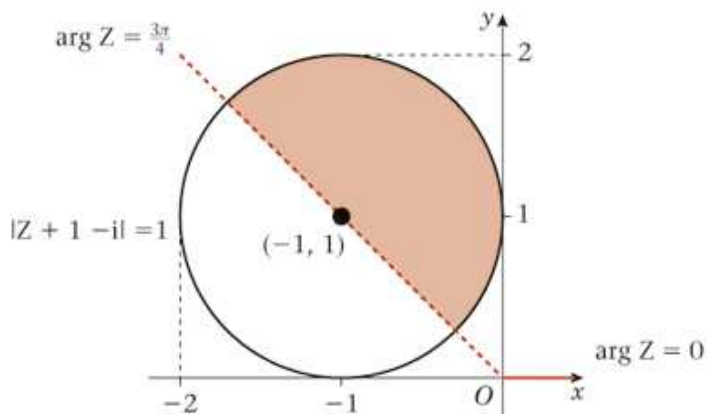
Question:

Shade in on an Argand diagram the region satisfied by the set of points $P(x, y)$, where $|z + 1 - i| \leq 1$ and $0 \leq \arg z < \frac{3\pi}{4}$.

Solution:

$$|z + 1 - i| \leq 1$$

$$|z - (-1 - i)| \leq 1$$



Inside of a circle centre $(-1, 1)$ radius 1

$\arg z = \frac{3\pi}{4}$ is a half-line with equation $y = -x$, which goes through the centre of the circle, $(-1, 1)$.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise G, Question 4

Question:

Shade in on an Argand diagram the region satisfied by the set of points $P(x, y)$, where $|z| \leq 3$ and $\frac{\pi}{4} \leq \arg(z + 3) \leq \pi$.

Solution:

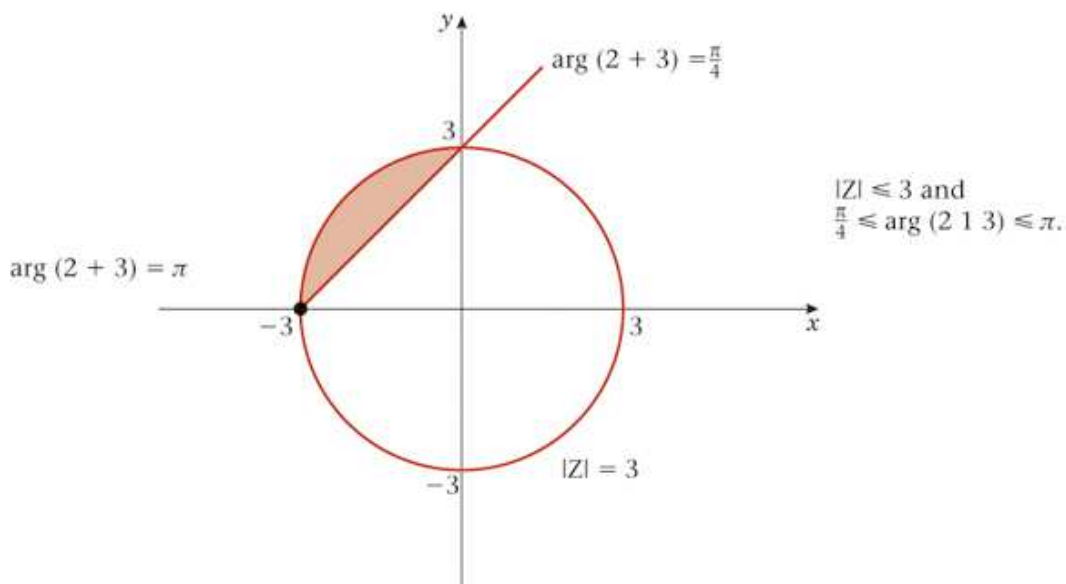
$$|z| \leq 3 \text{ and } \frac{\pi}{4} \leq \arg(z + 3) \leq \pi$$

$|z| = 3$ represents a circle centre $(0, 0)$ radius 3.

$\arg(z + 3) = \frac{\pi}{4}$ is a half-line with equation $y - 0 = 1(x + 3) \Rightarrow y = x + 3, x > -3$.

Note it passes through the points $(-3, 0)$ and $(0, 3)$.

$\arg(z + 3) = \pi$ is a half-line with equation $y = 0, x < -3$.



© Pearson Education Ltd 2009

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise G, Question 5

Question:

a Sketch on the same Argand diagram:

i the locus of points representing $|z - 2| = |z - 6 - 8i|$,

ii the locus of points representing $\arg(z - 4 - 2i) = 0$,

iii the locus of points representing $\arg(z - 4 - 2i) = \frac{\pi}{2}$.

The region R is defined by the inequalities $|z - 2| \leq |z - 6 - 8i|$ and $0 \leq \arg(z - 4 - 2i) \leq \frac{\pi}{2}$.

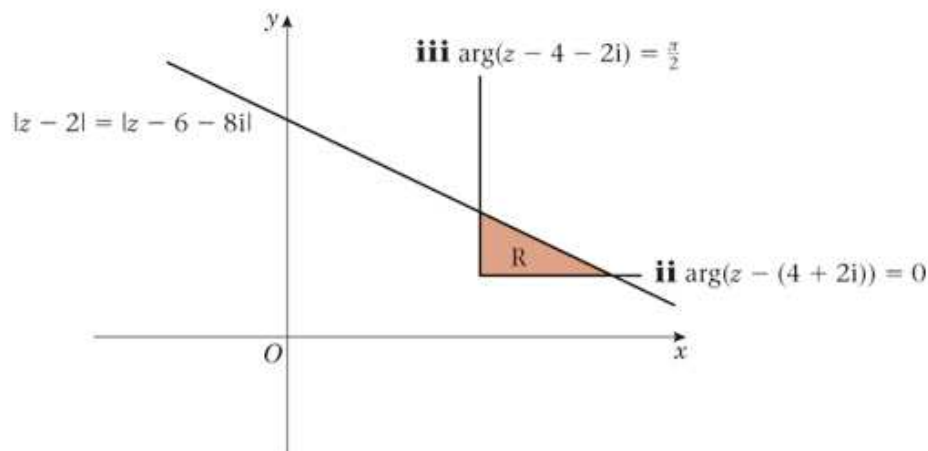
b On your sketch in part **a**, identify, by shading, the region R .

Solution:

a $|z - 2| = |z - 6 - 8i|$ represents a perpendicular bisector of the line joining $(2, 0)$ to $(6, 8)$.

$$\begin{aligned} |x + iy - 2| &= |x + iy - 6 - 8i| \\ \Rightarrow |(x - 2) + iy| &= |(x - 6) + i(y - 8)| \\ \Rightarrow |(x - 2) + iy|^2 &= |(x - 6) + i(y - 8)|^2 \\ \Rightarrow (x - 2)^2 + y^2 &= (x - 6)^2 + (y - 8)^2 \\ \Rightarrow x^2 - 4x + 4 + y^2 &= x^2 - 12x + 36 + y^2 - 16y + 64 \\ \Rightarrow -4x + 4 &= -12x - 16y + 100 \\ \Rightarrow 8x + 16y - 96 &= 0 \quad (\div 8) \\ \Rightarrow x + 2y - 12 &= 0 \\ \Rightarrow 2y &= -x + 12 \\ \Rightarrow y &= \frac{-1}{2}x + 6 \end{aligned}$$

i $|z - 2| = |z - (6 - 8i)|$



Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise G, Question 6

Question:

a Find the Cartesian equations of:

i the locus of points representing $|z + 10| = |z - 6 - 4\sqrt{2}i|$,

ii the locus of points representing $|z + 1| = 3$.

b Find the two values of z that satisfy both $|z + 10| = |z - 6 - 4\sqrt{2}i|$ and $|z + 1| = 3$.

c Hence shade in the region R on an Argand diagram which satisfies both

$|z + 10| \leq |z - 6 - 4\sqrt{2}i|$ and $|z + 1| \leq 3$.

Solution:

a i $|x + iy + 10| = |x + iy - 6 - 4\sqrt{2}i|$

so $(x + 10)^2 + y^2 = (x - 6)^2 + (y - 4\sqrt{2})^2$

$$x^2 + 20x + 100 + y^2 = x^2 + 12x + 36 + y^2 - 8\sqrt{2}y + 32$$

$$32x = -8\sqrt{2}y - 32$$

$$8\sqrt{2}y + (x + 1)32 = 0$$

$$y + (x + 1)2\sqrt{2} = 0$$

$$y = -2\sqrt{2}(x + 1)$$

ii $(x + 1)^2 + y^2 = 9$

$$(x^2 + 2x + y^2 = 8)$$

b Substitute $y = -2\sqrt{2}(x + 1)$ into $(x + 1)^2 + y^2 = 9$

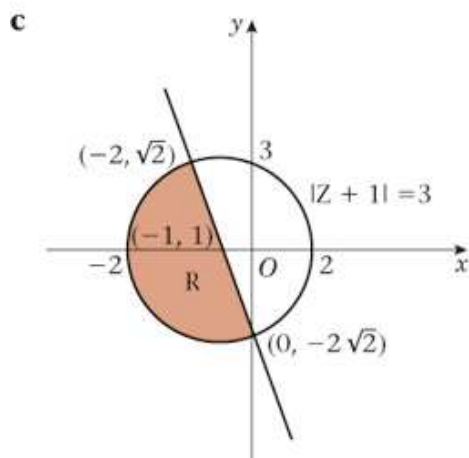
$$(x + 1)^2 + 8(x + 1)^2 = 9$$

$$9(x + 1)^2 = 9$$

$$x + 1 = \pm 1$$

$$x = 0, -2 \quad (0, -2\sqrt{2}) \text{ and } (-2, 2\sqrt{2})$$

$$z = -2\sqrt{2}i \text{ and } z = -2 + 2\sqrt{2}i$$



Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise H, Question 1

Question:

For the transformation $w = z + 4 + 3i$, sketch on separate Argand diagrams the locus of w when

a z lies on the circle $|z| = 1$,

b z lies on the half-line $\arg z = \frac{\pi}{2}$,

c z lies on the line $y = x$.

Solution:

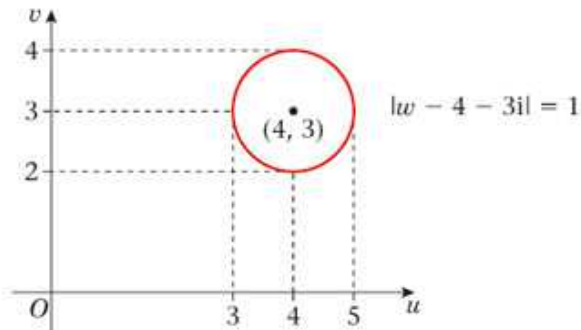
$$w = z + 4 + 3i$$

a $|z| = 1$ is a circle, centre $(0, 0)$, radius 1

METHOD ① $|z|$ is translated by a translation vector $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ to give a circle, centre $(4, 3)$, radius 1, in the w plane.

METHOD ② $w = z + 4 + 3i$
 $\Rightarrow w - 4 - 3i = z$
 $\Rightarrow |w - 4 - 3i| = |z|$
 $\Rightarrow |w - 4 - 3i| = 1$

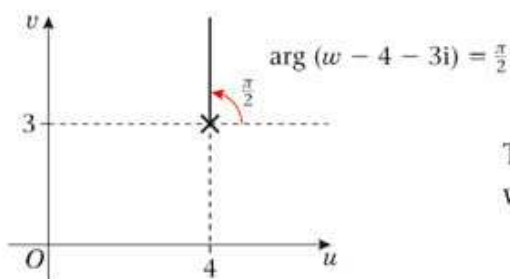
The locus of w is a circle centre $(4, 3)$, radius 1.



b $\arg z = \frac{\pi}{2}$

METHOD ① $\arg z = \frac{\pi}{2}$ is translated by a translation vector $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ to give a half-line from $(4, 3)$ at $\frac{\pi}{2}$ with the positive real axis.

METHOD ② $w = z + 4 + 3i$
 $\Rightarrow w - 4 - 3i = z$
 So $\arg z = \frac{\pi}{2} \Rightarrow \arg(w - 4 - 3i) = \frac{\pi}{2}$



The locus of w is the half-line with equation $u = 4, v > 3$.

c $y = x$

$$w = z + 4 + 3i$$

$$\Rightarrow z = w - 4 - 3i$$

$$\Rightarrow x + iy = u + iv - 4 - 3i$$

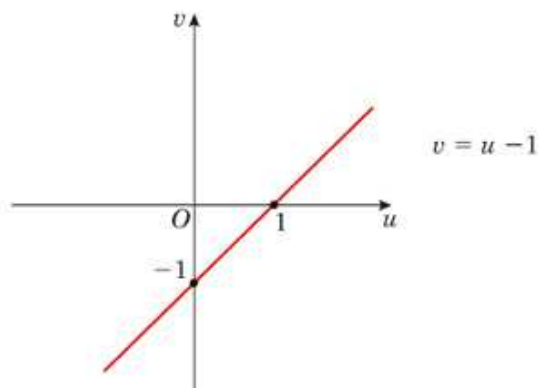
$$\Rightarrow x + iy = (u - 4) + i(v - 3)$$

$$y = x \Rightarrow v - 3 = u - 4$$

$$\Rightarrow v = u - 4 + 3$$

$$\Rightarrow v = u - 1$$

The locus of w is a line with equation $v = u - 1$.



© Pearson Education Ltd 2009

Solutionbank FP2

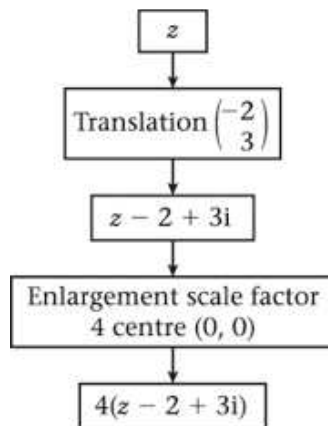
Edexcel AS and A Level Modular Mathematics

Exercise H, Question 2

Question:

A transformation T from the z -plane to the w -plane is a translation with translation vector $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ followed by an enlargement scale factor 4, centre O . Write down the transformation T in the form $w = a z + b$, where $a, b \in \mathbb{C}$.

Solution:



$$\begin{aligned} \text{Hence } T: w &= 4(z - 2 + 3i) \\ &= 4z - 8 + 12i \end{aligned}$$

The transformation T is $w = 4z - 8 + 12i$

Note: $a = 4, b = -8 + 12i$.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise H, Question 3

Question:

For the transformation $w = 3z + 2 - 5i$, find the equation of the locus of w when z lies on a circle centre O , radius 2.

Solution:

$$w = 3z + 2 - 5i$$

METHOD ① z lies on a circle, centre O , radius 2.

$$\Rightarrow |z| = 2$$

$$w = 3z + 2 - 5i$$

$$\Rightarrow w - 2 + 5i = 3z$$

$$\Rightarrow |w - 2 + 5i| = |3z|$$

$$\Rightarrow |w - 2 - 5i| = |3||z|$$

$$\Rightarrow |w - 2 - 5i| = 3|z|$$

$$\Rightarrow |w - 2 - 5i| = 3(2)$$

$$\Rightarrow |w - 2 - 5i| = 6$$

$$\Rightarrow |w - (2 - 5i)| = 6$$

So the locus of w is a circle centre $(2, -5)$, radius 6 with equation $(u - 2)^2 + (v + 5)^2 = 36$.

METHOD ② z lies on a circle, centre O , radius 2.



enlargement scale factor 3, centre O .

$3z$ lies on a circle, centre O , radius 6.



translation by a translation vector $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$.

$3z + 2 - 5i$ lies on a circle centre $(2, -5)$, radius 6.

So the locus of w is a circle, centre $(2, -5)$, radius 6 with equation $(u - 2)^2 + (v + 5)^2 = 36$.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise H, Question 4

Question:

For the transformation $w = 2z - 5 + 3i$, find the equation of the locus of w as z moves on the circle $|z - 2| = 4$.

Solution:

z moves on a circle $|z - 2| = 4$

METHOD ①

$$\begin{aligned} w &= 2z - 5 + 3i \\ \Rightarrow w + 5 - 3i &= 2z \\ \Rightarrow \frac{w + 5 - 3i}{2} &= z \\ \Rightarrow \frac{w + 5 - 3i}{2} - 2 &= z - 2 \\ \Rightarrow \frac{w + 5 - 3i - 4}{2} &= z - 2 \\ \Rightarrow \frac{w + 1 - 3i}{2} &= z - 2 \\ \Rightarrow \left| \frac{w + 1 - 3i}{2} \right| &= |z - 2| \\ \Rightarrow \frac{|w + 1 - 3i|}{|2|} &= |z - 2| \\ \Rightarrow |w + 1 - 3i| &= 2|z - 2| \\ \Rightarrow |w + 1 - 3i| &= 2(4) \\ \Rightarrow |w + 1 - 3i| &= 8 \\ \Rightarrow |w - (-1 + 3i)| &= 8 \end{aligned}$$

So the locus of w is a circle centre $(-1, 3)$, radius 8 with equation $(u + 1)^2 + (v - 3)^2 = 8$.

METHOD ②

$$|z - 2| = 4$$

z lies on a circle, centre $(2, 0)$, radius 4

↓ enlargement scale factor 2, centre 0.

$2z$ lies on a circle, centre $(4, 0)$, radius 8.

↓ translation by a translation vector $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$.

$w = 2z - 5 + 3i$ lies on a circle centre $(-1, 3)$, radius 8.

So the locus of w is a circle, centre $(-1, 3)$, radius 8 with equation $(u - 1)^2 + (v - 3)^2 = 8$.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise H, Question 5

Question:

For the transformation $w = z - 1 + 2i$ sketch on separate Argand diagrams the locus of w when:

a z lies on the circle $|z - 1| = 3$,

b z lies on the half-line $\arg(z - 1 + i) = \frac{\pi}{4}$,

c z lies on the line $y = 2x$.

Solution:

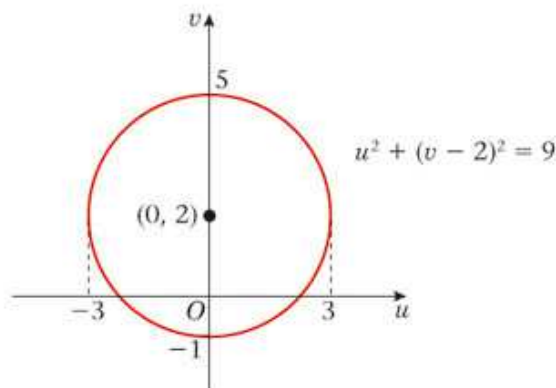
$$w = z - 1 + 2i$$

a $|z - 1| = 3$ circle centre (1, 0) radius 3.

METHOD ① $|z - 1| = 3$ is translated by a translation vector $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ to give a circle, centre (0, 2), radius 3, in the w -plane.

METHOD ② $w = z - 1 + 2i$
 $\Rightarrow w - 2i = z - 1$
 $\Rightarrow |w - 2i| = |z - 1|$
 $\Rightarrow |w - 2i| = 3$

The locus of w is a circle, centre (0, 2), radius 3.

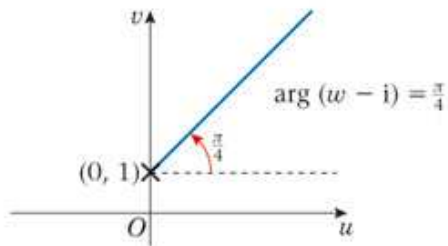


b $\arg(z - 1 + i) = \frac{\pi}{4}$ half-line from (1, -1) at $\frac{\pi}{4}$ with the positive real axis.

METHOD ① $\arg(z - 1 + i) = \frac{\pi}{4}$ is translated by a translation vector $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ to give a half-line from (0, 1) at $\frac{\pi}{4}$ with the positive real axis.

METHOD ② $w = z - 1 + 2i$
 $\Rightarrow w + 1 - 2i = z$
 So $\arg(z - 1 + i) = \frac{\pi}{4}$
 becomes $\arg(w + 1 - 2i - 1 + i) = \frac{\pi}{4}$
 $\Rightarrow \arg(w - i) = \frac{\pi}{4}$

Therefore, the locus of w is a half-line from $(0, 1)$ at $\frac{\pi}{4}$ with the positive real axis.



c $y = 2x$

$$w = z - 1 + 2i$$

$$\Rightarrow z = w + 1 - 2i$$

$$\Rightarrow x + iy = u + iv + 1 - 2i$$

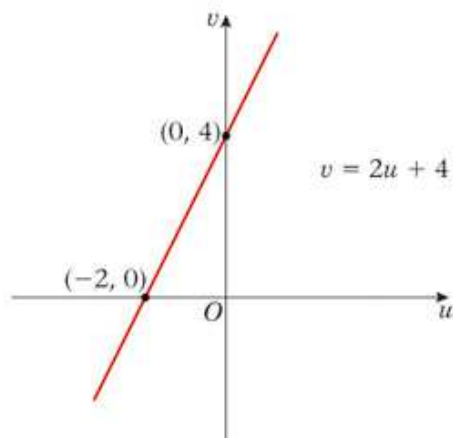
$$\Rightarrow x + iy = u + 1 + i(v - 2)$$

$$\text{So } y = 2x \Rightarrow v - 2 = 2(u + 1)$$

$$\Rightarrow v - 2 = 2u + 2$$

$$\Rightarrow v = 2u + 4$$

The locus of w is a line with equation $v = 2u + 4$.



Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise H, Question 6

Question:

For the transformation $w = \frac{1}{z}$, $z \neq 0$, find the locus of w when:

- a** z lies on the circle $|z| = 2$,
- b** z lies on the half-line with equation $\arg z = \frac{\pi}{4}$,
- c** z lies on the line with equation $y = 2x + 1$.

Solution:

$$w = \frac{1}{z}, z \neq 0$$

- a** z lies on a circle, $|z| = 2$

$$w = \frac{1}{z}$$

$$\Rightarrow |w| = \left| \frac{1}{z} \right|$$

$$\Rightarrow |w| = \frac{|1|}{|z|}$$

$$\Rightarrow |w| = \frac{1}{2} \quad \leftarrow \text{apply } |z| = 2$$

Therefore the locus of w is a circle, centre $(0, 0)$, radius $\frac{1}{2}$, with equation $u^2 + v^2 = \frac{1}{4}$.

- b** z lies on the half-line, $\arg z = \frac{\pi}{4}$

$$w = \frac{1}{z} \Rightarrow wz = 1 \Rightarrow z = \frac{1}{w}$$

$$\text{So } \arg z = \frac{\pi}{4}, \text{ becomes } \arg\left(\frac{1}{w}\right) = \frac{\pi}{4}$$

$$\Rightarrow \arg(1) - \arg(w) = \frac{\pi}{4}$$

$$\Rightarrow -\arg w = \frac{\pi}{4} \quad \leftarrow \text{arg } 1 = 0$$

$$\Rightarrow \arg w = -\frac{\pi}{4}$$

Therefore the locus of w is a half-line from $(0, 0)$ at $-\frac{\pi}{4}$ with the positive x -axis.
The locus of w has equation, $v = -u$, $u > 0$, $v < 0$.

c z lies on the line $y = 2x + 1$

$$w = \frac{1}{z} \Rightarrow wz = 1 \Rightarrow z = \frac{1}{w}.$$

$$\Rightarrow x + iy = \frac{1}{u + iv}$$

$$\Rightarrow x + iy = \frac{1}{(u + iv)(u - iv)}(u - iv)$$

$$\Rightarrow x + iy = \frac{u - iv}{u^2 + v^2}$$

$$\Rightarrow x + iy = \frac{u}{u^2 + v^2} + i\left(\frac{-v}{u^2 + v^2}\right)$$

$$\text{So } x = \frac{u}{u^2 + v^2} \text{ and } y = \frac{-v}{u^2 + v^2}$$

$$\text{Hence } y = 2x + 1 \text{ becomes } \frac{-v}{u^2 + v^2} = \frac{2u}{u^2 + v^2} + 1 \quad \times (u^2 + v^2)$$

$$\Rightarrow -v = 2u + u^2 + v^2$$

$$\Rightarrow 0 = u^2 + 2u + v^2 + v$$

$$\Rightarrow (u + 1)^2 - 1 + \left(v + \frac{1}{2}\right)^2 - \frac{1}{4} = 0$$

$$\Rightarrow (u + 1)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{5}{4}$$

$$\Rightarrow (u + 1)^2 + \left(v + \frac{1}{2}\right)^2 = \left(\frac{\sqrt{5}}{2}\right)^2$$

Therefore, the locus of w is a circle, centre $\left(-1, -\frac{1}{2}\right)$, radius $\frac{\sqrt{5}}{2}$, with equation

$$(u + 1)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{5}{4}.$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise H, Question 7

Question:

For the transformation $w = z^2$,

- a** show that as z moves once round a circle centre $(0, 0)$, radius 3, w moves twice round a circle centre $(0, 0)$, radius 9,
- b** find the locus of w when z lies on the real axis, with equation $y = 0$,
- c** find the locus of w when z lies on the imaginary axis.

Solution:

$$w = z^2$$

- a** z moves once round a circle, centre $(0, 0)$, radius 3.

The equation of the circle, $|z| = 3$ is also $r = 3$.

The equation of the circle can be written as $z = 3e^{i\theta}$

$$\text{or } z = 3(\cos \theta + i \sin \theta)$$

$$\begin{aligned} \Rightarrow w = z^2 &= (3(\cos \theta + i \sin \theta))^2 \\ &= 3^2(\cos 2\theta + i \sin 2\theta) \\ &= 9(\cos 2\theta + i \sin 2\theta) \end{aligned}$$

de Moivre's Theorem.

So, $w = 9(\cos 2\theta + i \sin 2\theta)$ can be written as $|w| = 9$

Hence, as $|w| = 9$ and $\arg w = 2\theta$ then w moves twice round a circle, centre $(0, 0)$, radius 9.

- b** z lies on the real-axis $\Rightarrow y = 0$

So $z = x + iy$ becomes $z = x$ (as $y = 0$)

$$\Rightarrow w = z^2 = x^2$$

$$\Rightarrow u + iv = x^2 + i(0)$$

$$\Rightarrow u = x^2 \text{ and } v = 0$$

As $v = 0$ and $u = x^2 \geq 0$ then w lies on the positive real-axis including the origin, 0.

- c** z lies on the imaginary axis $\Rightarrow x = 0$

So $z = x + iy$ becomes $z = iy$ (as $x = 0$)

$$\Rightarrow w = z^2 = (iy)^2 = -y^2$$

$$\Rightarrow u + iv = -y^2 + i(0)$$

$$\Rightarrow u = -y^2 \text{ and } v = 0$$

As $v = 0$ and $u = -y^2 \leq 0$ then w lies on the negative real-axis including the origin, 0.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise H, Question 8

Question:

If z is any point in the region R for which $|z + 2i| < 2$,

a shade in on an Argand diagram the region R .

Sketch on separate Argand diagrams the corresponding regions for \mathbb{R} where:

b $w = z - 2 + 5i$,

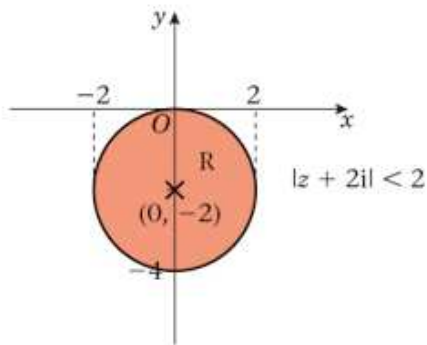
c $w = 4z + 2 + 4i$,

d $|zw + 2iw| = 1$.

Solution:

$$|z + 2i| < 2$$

a $|z + 2i| = 2$ is a circle, centre $(0, -2)$, radius 2.



b $w = z - 2 + 5i$

$$\Rightarrow w + 2 - 5i = z$$

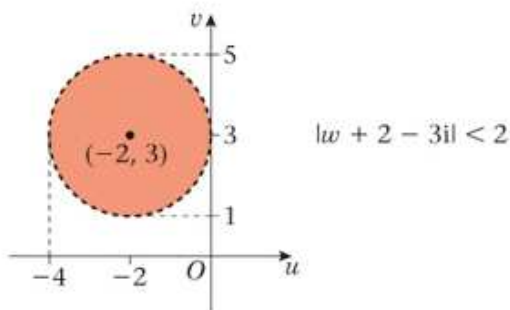
$$\Rightarrow z + 2i = w + 2 - 5i + 2i$$

$$\Rightarrow z + 2i = w + 2 - 3i$$

$$\Rightarrow |z + 2i| = |w + 2 - 3i|$$

As $|z + 2i| < 2$, then $|z + 2i| = |w + 2 - 3i| < 2$

Note that $|w + 2 - 3i| = 2$ is a circle, centre $(-2, 3)$, radius 2.



c $w = 4z + 2 + 4i$

$$\Rightarrow w - 2 - 4i = 4z$$

$$\Rightarrow \frac{w - 2 - 4i}{4} = z$$

$$\Rightarrow z + 2i = \frac{w - 2 - 4i}{4} + 2i$$

$$\Rightarrow z + 2i = \frac{w - 2 - 4i + 8i}{4}$$

$$\Rightarrow z + 2i = \frac{w - 2 + 4i}{4}$$

$$\Rightarrow |z + 2i| = \left| \frac{w - 2 + 4i}{4} \right|$$

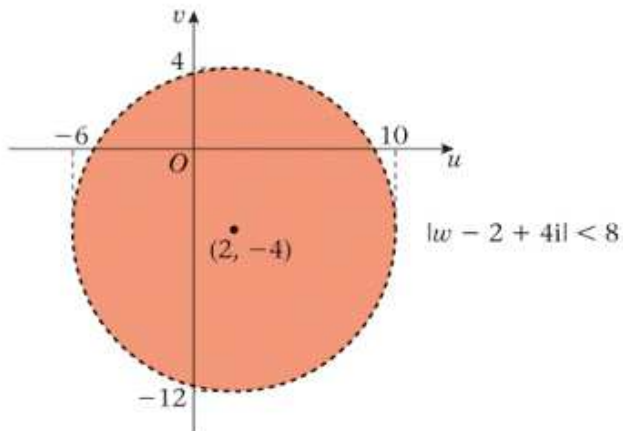
$$\Rightarrow |z + 2i| = \frac{|w - 2 + 4i|}{|4|}$$

$$\Rightarrow |z + 2i| = \frac{|w - 2 + 4i|}{4}$$

$$\text{As } |z + 2i| < 2, \text{ then } |z + 2i| = \frac{|w - 2 + 4i|}{4} < 2$$

$$\Rightarrow |w - 2 + 4i| < 8$$

Note that $|w - 2 + 4i| = 8$ is a circle, centre $(2, -4)$, radius 8.



d $|zw + 2iw| = 1$

$$\Rightarrow |w(z + 2i)| = 1$$

$$\Rightarrow |w| |z + 2i| = 1$$

$$\Rightarrow |z + 2i| = \frac{1}{|w|}$$

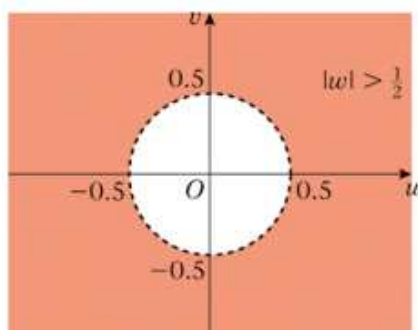
$$\text{As } |z + 2i| < 2, \text{ then } |z + 2i| = \frac{1}{|w|} < 2$$

$$\Rightarrow 1 < 2|w|$$

$$\Rightarrow \frac{1}{2} < |w|$$

$$\Rightarrow |w| > \frac{1}{2}$$

Note that $|w| = \frac{1}{2}$ is a circle, centre $(0, 0)$ radius $\frac{1}{2}$.



Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise H, Question 9

Question:

For the transformation $w = \frac{1}{2-z}$, $z \neq 2$, show that the image, under T , of the circle centre O , radius 2 in the z -plane is a line l in the w -plane. Sketch l on an Argand diagram.

Solution:

Circle, centre 0, radius 2 in the z -plane $\Rightarrow |z| = 2$

$$T: w = \frac{1}{2-z}$$

$$\Rightarrow w(2-z) = 1$$

$$\Rightarrow 2w - wz = 1$$

$$\Rightarrow 2w - 1 = wz$$

$$\Rightarrow \frac{2w-1}{w} = z$$

$$\Rightarrow \left| \frac{2w-1}{w} \right| = |z|$$

$$\Rightarrow \frac{|2w-1|}{|w|} = |z|$$

Applying $|z| = 2$ gives $\frac{|2w-1|}{|w|} = 2$

$$\Rightarrow |2w-1| = 2|w|$$

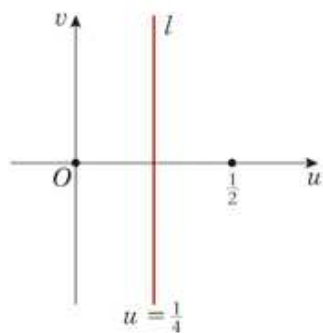
$$\Rightarrow |2(w - \frac{1}{2})| = 2|w|$$

$$\Rightarrow |2|(w - \frac{1}{2})| = 2|w|$$

$$\Rightarrow 2|w - \frac{1}{2}| = 2|w|$$

$$\Rightarrow |w - \frac{1}{2}| = |w|$$

The image under T of $|z| = 2$ is the perpendicular bisector of the line segment joining $(0, 0)$ and $(\frac{1}{2}, 0)$. Therefore the line l has equation $u = \frac{1}{4}$.

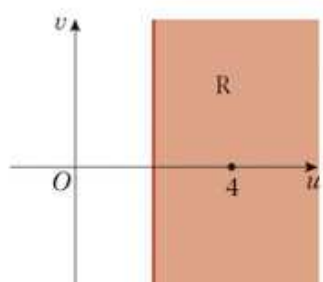


$$|z-4| < 4 \text{ gives } \frac{|16-4w|}{|w|} < 4$$

$$\Rightarrow |16-4w| < 4|w|$$

which leads to $|w-4| < |w|$

$$\Rightarrow |w| > |w-4|$$



© Pearson Education Ltd 2009

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise H, Question 10

Question:

The transformation T from the z -plane, where $z = x + iy$, to the w -plane where $w = u + iv$, is given by $w = \frac{16}{z}$, $z \neq 0$.

- a** The transformation T maps the points on the circle $|z - 4| = 4$, in the z -plane, to points on a line l in the w -plane. Find the equation of l .
- b** Hence, or otherwise, shade and label on an Argand diagram the region R which is the image of $|z - 4| < 4$ under T .

Solution:

$$T: w = \frac{16}{z}$$

$$|z - 4| = 4$$

$$w = \frac{16}{z}$$

$$\Rightarrow wz = 16$$

$$\Rightarrow z = \frac{16}{w}$$

$$\Rightarrow z - 4 = \frac{16}{w} - 4$$

$$\Rightarrow z - 4 = \frac{16 - 4w}{w}$$

$$\Rightarrow |z - 4| = \left| \frac{16 - 4w}{w} \right|$$

$$\Rightarrow |z - 4| = \frac{|16 - 4w|}{|w|}$$

$$\text{Applying } |z - 4| = 4 \text{ gives } \frac{|16 - 4w|}{|w|} = 4$$

$$\Rightarrow \frac{|-4(w - 4)|}{|w|} = 4|w|$$

$$\Rightarrow |-4||w - 4| = 4|w|$$

$$\Rightarrow 4|w - 4| = 4|w|$$

$$\Rightarrow |w - 4| = |w|$$

The image under T of $|z - 4| = 4$ is the perpendicular bisector of the line segment joining $(0, 0)$ to $(4, 0)$. Therefore the line l has equation $u = 2$.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise H, Question 11

Question:

The transformation T from the z -plane, where $z = x + iy$, to the w -plane where $w = u + iv$, is given by $w = \frac{3}{2-z}$, $z \neq 2$.

Show that under T the straight line with equation $2y = x$ is transformed to a circle in the w -plane with centre $(\frac{3}{4}, \frac{3}{2})$, radius $\frac{3}{4}\sqrt{5}$.

Solution:

$$T: w = \frac{3}{2-z}, z \neq 2$$

$$\Rightarrow w(2-z) = 3$$

$$\Rightarrow 2w - wz = 3$$

$$\Rightarrow 2w = 3 + wz$$

$$\Rightarrow 2w - 3 = wz$$

$$\Rightarrow \frac{2w-3}{w} = z$$

$$\Rightarrow z = \frac{2w-3}{w}$$

$$\Rightarrow z = \frac{2(u+iv)-3}{u+iv}$$

$$\Rightarrow z = \frac{(2u-3)+2iv}{u+iv}$$

$$\Rightarrow z = \frac{[(2u-3)+2iv]}{[u+iv]} \times \frac{[u-iv]}{[u-iv]}$$

$$\Rightarrow z = \frac{(2u-3)u - iv(2u-3) + 2iuv + 2v^2}{u^2 + v^2}$$

$$\Rightarrow z = \frac{2u^2 - 3u - 2uv + 3iv + 2uv + 2v^2}{u^2 + v^2}$$

$$\Rightarrow z = \frac{2u^2 - 3u + 2v^2}{u^2 + v^2} + i \left[\frac{3v}{u^2 + v^2} \right]$$

$$\text{So, } x + iy = \frac{2u^2 - 3u + 2v^2}{u^2 + v^2} + i \left[\frac{3v}{u^2 + v^2} \right]$$

$$\Rightarrow x = \frac{2u^2 - 3u + 2v^2}{u^2 + v^2}$$

$$\text{and } y = \frac{3v}{u^2 + v^2}$$

$$\begin{aligned}
\text{As, } 2y = x &\Rightarrow 2\left(\frac{3v}{u^2 + v^2}\right) = \frac{2u^2 - 3u + 2v^2}{u^2 + v^2} \\
&\Rightarrow \frac{6v}{u^2 + v^2} = \frac{2u^2 - 3u + 2v^2}{u^2 + v^2} \\
&\Rightarrow 6v = 2u^2 - 3u + 2v^2 \\
&\Rightarrow 0 = 2u^2 - 3u + 2v^2 - 6v \\
&\Rightarrow 2u^2 - 3u + 2v^2 - 6v = 0 \quad (\div 2) \\
&\Rightarrow u^2 - \frac{3}{2}u + v^2 - 3v = 0 \\
&\Rightarrow \left(u - \frac{3}{4}\right)^2 - \frac{9}{16} + \left(v - \frac{3}{2}\right)^2 - \frac{9}{4} = 0 \\
&\Rightarrow \left(u - \frac{3}{4}\right)^2 + \left(v - \frac{3}{2}\right)^2 = \frac{9}{16} + \frac{9}{4} \\
&\Rightarrow \left(u - \frac{3}{4}\right)^2 + \left(v - \frac{3}{2}\right)^2 = \frac{45}{16} \\
&\Rightarrow \left(u - \frac{3}{4}\right)^2 + \left(v - \frac{3}{2}\right)^2 = \left(\frac{3\sqrt{5}}{4}\right)^2
\end{aligned}$$

The image under T of $2y = x$ is a circle centre $\left(\frac{3}{4}, \frac{3}{2}\right)$, radius $\frac{3}{4}\sqrt{5}$, as required.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise H, Question 12

Question:

The transformation T from the z -plane, where $z = x + iy$, to the w -plane where $w = u + iv$, is given by $w = \frac{-iz + i}{z + 1}$, $z \neq -1$.

- a** The transformation T maps the points on the circle with equation $x^2 + y^2 = 1$ in the z -plane, to points on a line l in the w -plane. Find the equation of l .
- b** Hence, or otherwise, shade and label on an Argand diagram the region R of the w -plane which is the image of $|z| \leq 1$ under T .
- c** Show that the image, under T , of the circle with equation $x^2 + y^2 = 4$ in the z -plane is a circle C in the w -plane. Find the equation of C .

Solution:

$$T: w = \frac{-iz + i}{z + 1}, z \neq -1$$

a Circle with equation $x^2 + y^2 = 1 \Rightarrow |z| = 1$

$$w = \frac{-iz + i}{z + 1}$$

$$\Rightarrow w(z + 1) = -iz + i$$

$$\Rightarrow wz + w = -iz + i$$

$$\Rightarrow wz + iz = -i - w$$

$$\Rightarrow z(w + i) = i - w$$

$$\Rightarrow z = \frac{i - w}{w + i}$$

$$\Rightarrow |z| = \left| \frac{i - w}{w + i} \right|$$

$$\Rightarrow |z| = \frac{|i - w|}{|w + i|}$$

$$\text{Applying } |z| = 1 \Rightarrow 1 = \frac{|i - w|}{|w + i|}$$

$$\Rightarrow |w + i| = |i - w|$$

$$\Rightarrow |w + i| = |(-1)(w - i)|$$

$$\Rightarrow |w + i| = |(-1)||w - i|$$

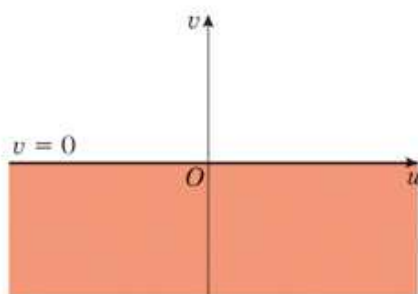
$$\Rightarrow |w + i| = |w - i|$$

The image under T of $x^2 + y^2 = 1$ is the perpendicular bisector of the line segment joining $(0, -1)$ to $(0, 1)$. Therefore the line l , has equation $v = 0$. (i.e. the u -axis.)

b $|z| \leq 1 \Rightarrow 1 \leq \frac{|i - w|}{|w + i|}$

$$\Rightarrow |w + i| \leq |i - w|$$

$$\Rightarrow |w + i| \leq |w - i|$$



c Circle with equation $x^2 + y^2 = 4 \Rightarrow |z| = 2$

from part a $w = \frac{-iz + i}{z + 1}$

$$\Rightarrow z = \frac{i - w}{w + i}$$

$$\Rightarrow |z| = \frac{|i - w|}{|w + i|}$$

Applying $|z| = 2 \Rightarrow 2 = \frac{|i - w|}{|w + i|}$

$$\Rightarrow 2|w + i| = |i - w|$$

$$\Rightarrow 2|w + i| = |(-1)(w - i)|$$

$$\Rightarrow 2|w + i| = |(-1)||w - i|$$

$$\Rightarrow 2|w + i| = |w - i|$$

$$\Rightarrow 2|u + iv + i| = |u + iv - i|$$

$$\Rightarrow 2|u + i(v + 1)| = |u + i(v - 1)|$$

$$\Rightarrow 2^2|u + i(v + 1)|^2 = |u + i(v - 1)|^2$$

$$\Rightarrow 4[u^2 + (v + 1)^2] = u^2 + (v - 1)^2$$

$$\Rightarrow 4[u^2 + v^2 + 2v + 1] = u^2 + v^2 - 2v + 1$$

$$\Rightarrow 4u^2 + 4v^2 + 8v + 4 = u^2 + v^2 - 2v + 1$$

$$\Rightarrow 3u^2 + 3v^2 + 10v + 3 = 0$$

$$\Rightarrow u^2 + v^2 + \frac{10}{3}v + 1 = 0$$

$$\Rightarrow u^2 + \left(v + \frac{5}{3}\right)^2 - \frac{25}{9} + 1 = 0$$

$$\Rightarrow u^2 + \left(v + \frac{5}{3}\right)^2 = \frac{25}{9} - 1$$

$$\Rightarrow u^2 + \left(v + \frac{5}{3}\right)^2 = \frac{16}{9}$$

$$\Rightarrow u^2 + \left(v + \frac{5}{3}\right)^2 = \left(\frac{4}{3}\right)^2$$

The image under T of $x^2 + y^2 = 4$ is a circle C with centre $\left(0, -\frac{5}{3}\right)$, radius $\frac{4}{3}$.

Therefore, the equation of C is $u^2 + \left(v + \frac{5}{3}\right)^2 = \frac{16}{9}$.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise H, Question 13

Question:

The transformation T from the z -plane, where $z = x + iy$, to the w -plane where $w = u + iv$, is given by $w = \frac{4z - 3i}{z - 1}$, $z \neq 1$.

Show that the circle $|z| = 1$ is mapped by T onto a circle C .
Find the centre and radius of C .

Solution:

$$T: w = \frac{4z - 3i}{z - 1}, z \neq 1$$

Circle with equation $|z| = 1$

$$w = \frac{4z - 3i}{z - 1},$$

$$\Rightarrow w(z - 1) = 4z - 3i$$

$$\Rightarrow wz - w = 4z - 3i$$

$$\Rightarrow wz + 4z = w - 3i$$

$$\Rightarrow z(w + 4) = w - 3i$$

$$\Rightarrow z = \frac{w - 3i}{w + 4}$$

$$\Rightarrow |z| = \left| \frac{w - 3i}{w + 4} \right|$$

$$\Rightarrow |z| = \frac{|w - 3i|}{|w + 4|}$$

$$\text{Applying } |z| = 1 \Rightarrow 1 = \frac{|w - 3i|}{|w + 4|}$$

$$\begin{aligned}
&\Rightarrow 3|w - 4| = |w - 3i| \\
&\Rightarrow 3|u + iv - 4| = |u + iv - 3i| \\
&\Rightarrow 3|(u - 4) + iv| = |u + i(v - 3)| \\
&\Rightarrow 3^2|(u - 4) + iv|^2 = |u + i(v - 3)|^2 \\
&\Rightarrow 9[(u - 4)^2 + v^2] = u^2 + (v - 3)^2 \\
&\Rightarrow 9[u^2 - 8u + 16 + v^2] = u^2 + v^2 - 6v + 9 \\
&\Rightarrow 9u^2 - 72u + 144 + 9v^2 = u^2 + v^2 - 6v + 9 \\
&\Rightarrow 8u^2 - 72u + 8v^2 + 6v + 144 - 9 = 0 \\
&\Rightarrow 8u^2 - 72u + 8v^2 + 6v + 135 = 0 \quad (\div 8) \\
&\Rightarrow u^2 - 9u + v^2 + \frac{3}{4}v + \frac{135}{8} = 0 \\
&\Rightarrow \left(u - \frac{9}{2}\right)^2 - \frac{81}{4} + \left(v + \frac{3}{8}\right)^2 - \frac{9}{64} + \frac{135}{8} = 0 \\
&\Rightarrow \left(u - \frac{9}{2}\right)^2 + \left(v + \frac{3}{8}\right)^2 = \frac{81}{4} + \frac{9}{64} - \frac{135}{8} \\
&\Rightarrow \left(u - \frac{9}{2}\right)^2 + \left(v + \frac{3}{8}\right)^2 = \frac{225}{64} \\
&\Rightarrow \left(u - \frac{9}{2}\right)^2 + \left(v + \frac{3}{8}\right)^2 = \left(\frac{15}{8}\right)^2
\end{aligned}$$

Therefore, the circle with equation $|z| = 1$ is mapped onto a circle C with centre $\left(\frac{9}{2} - \frac{3}{8}\right)$, radius $\frac{15}{8}$.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise H, Question 14

Question:

The transformation T from the z -plane, where $z = x + iy$, to the w -plane where $w = u + iv$, is given by $w = \frac{1}{z + i}$, $z \neq -i$.

- a** Show that the image, under T , of the real axis in the z -plane is a circle C_1 in the w -plane. Find the equation of C_1 .
- b** Show that the image, under T , of the line $x = 4$ in the z -plane is a circle C_2 in the w -plane. Find the equation of C_2 .

Solution:

$$T: w = \frac{1}{z + i}, z \neq -i$$

- a** Real axis in the z -plane $\Rightarrow y = 0$

$$w = \frac{1}{z - i}$$

$$\Rightarrow w(z + i) = 1$$

$$\Rightarrow wz + iw = 1$$

$$\Rightarrow wz = 1 - iw$$

$$\Rightarrow z = \frac{1 - iw}{w}$$

$$\Rightarrow z = \frac{1 - i(u + iv)}{u + iv}$$

$$\Rightarrow z = \frac{1 - iu + v}{u + iv}$$

$$\Rightarrow z = \frac{((1 + v) - iu)}{(u + iv)} \times \frac{(u - iv)}{(u - iv)}$$

$$\Rightarrow z = \frac{(1 + v)u - iv(1 + v) - iu^2 - uv}{u^2 + v^2}$$

$$\Rightarrow z = \frac{(1 + v)u - uv}{u^2 + v^2} + \frac{i(-v(1 + v) - u^2)}{u^2 + v^2}$$

$$\Rightarrow z = \frac{u + uv - uv}{u^2 + v^2} + \frac{i(-v - v^2 - u^2)}{u^2 + v^2}$$

$$\Rightarrow z = \frac{u}{u^2 + v^2} + \frac{i(-v - v^2 - u^2)}{u^2 + v^2}$$

$$\text{So } x + iy = \frac{u}{u^2 + v^2} + \frac{i(-v - v^2 - u^2)}{u^2 + v^2}$$

$$\Rightarrow x = \frac{u}{u^2 + v^2} \quad \text{and} \quad y = \frac{-v - v^2 - u^2}{u^2 + v^2}$$

$$\text{As } y = 0, \frac{-v - v^2 - u^2}{u^2 + v^2} = 0$$

$$\Rightarrow -v - v^2 - u^2 = 0$$

$$\Rightarrow u^2 + v^2 + v = 0$$

$$\Rightarrow u^2 + \left(v + \frac{1}{2}\right)^2 - \frac{1}{4} = 0$$

$$\Rightarrow u^2 + \left(v + \frac{1}{2}\right)^2 = \frac{1}{4}$$

$$\Rightarrow u^2 + \left(v + \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

Therefore, the image under T of the real axis in the z -plane is a circle C_1 with centre $\left(0, -\frac{1}{2}\right)$, radius $\frac{1}{2}$. The equation of C_1 is $u^2 + \left(v + \frac{1}{2}\right)^2 = \frac{1}{4}$.

b As $x = 4$, $\frac{u}{u^2 + v^2} = 2$

$$\Rightarrow u = 2(u^2 + v^2)$$

$$\Rightarrow u = 2u^2 + 2v^2$$

$$\Rightarrow 0 = 2u^2 - u + 2v^2 \quad (\div 2)$$

$$\Rightarrow 0 = u^2 - \frac{1}{2}u + v^2$$

$$\Rightarrow 0 = \left(u - \frac{1}{4}\right)^2 - \frac{1}{16} + v^2$$

$$\Rightarrow \left(u - \frac{1}{4}\right)^2 + v^2 = \frac{1}{16}$$

$$\Rightarrow \left(u - \frac{1}{4}\right)^2 + v^2 = \left(\frac{1}{4}\right)^2$$

Therefore, the image under T of the line $x = 2$ is a circle C_2 with centre $\left(\frac{1}{4}, 0\right)$, radius $\frac{1}{4}$.

The equation of C_2 is $\left(u - \frac{1}{4}\right)^2 + v^2 = \frac{1}{16}$.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise H, Question 15

Question:

The transformation T from the z -plane, where $z = x + iy$, to the w -plane where $w = u + iv$, is given by $w = z + \frac{4}{z}$, $z \neq 0$.

Show that the transformation T maps the points on a circle $|z| = 2$ to points in the interval $[-k, k]$ on the real axis. State the value of the constant k .

Solution:

$$T: w = z + \frac{4}{z}, z \neq 0$$

Circle with equation $|z| = 2 \Rightarrow x^2 + y^2 = 4$

$$w = z + \frac{4}{z}$$

$$\Rightarrow w = \frac{z^2 + 4}{z}$$

$$\Rightarrow w = \frac{(x + iy)^2 + 4}{x + iy}$$

$$\Rightarrow w = \frac{x^2 + 2xyi - y^2 + 4}{x + iy}$$

$$\Rightarrow w = \frac{[(x^2 - y^2 + 4) + i(2xy)]}{x + iy}$$

$$\Rightarrow w = \frac{[(x^2 - y^2 + 4) + i(2xy)]}{(x + iy)} \times \frac{(x - iy)}{(x - iy)}$$

$$\Rightarrow w = \frac{x^3 - xy^2 + 4x + 2xy^2 + i(2x^2y - x^2y + y^3 - 4y)}{x^2 + y^2}$$

$$\Rightarrow w = \left(\frac{x^3 - xy^2 + 4x}{x^2 + y^2} \right) + i \left(\frac{y^3 - x^2y - 4y}{x^2 + y^2} \right)$$

$$\Rightarrow w = \frac{x(x^2 + y^2 + 4)}{x^2 + y^2} + \frac{iy(x^2 + y^2 - 4)}{x^2 + y^2}$$

$$\text{Apply } x^2 + y^2 = 4 \Rightarrow w = \frac{x(4 + 4)}{4} + \frac{iy(4 - 4)}{4}$$

$$\Rightarrow w = \frac{8x}{4} + \frac{iy(0)}{4}$$

$$\Rightarrow w = 2x + 0i$$

$$\Rightarrow u + iv = 2x + 0i$$

$$\Rightarrow u = 2x, v = 0$$

As $|z| = 2 \Rightarrow -2 \leq x \leq 2$

So $-4 \leq 2x \leq 4$

and $-4 \leq u \leq 4$

Therefore the transformation T maps the points on a circle $|z| = 2$ in the z -plane to points in the interval $[-4, 4]$ on the real axis in the w -plane. Hence $k = 4$.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise H, Question 16

Question:

The transformation T from the z -plane, where $z = x + iy$, to the w -plane where $w = u + iv$, is given by $w = \frac{1}{z + 3}$, $z \neq -3$.

Show that the line with equation $2x - 2y + 7 = 0$ is mapped by T onto a circle C . State the centre and the exact radius of C .

Solution:

$$T: w = \frac{1}{z+3}, z \neq -3$$

Line with equation $2x - 2y + 7 = 0$ in the z -plane

$$w = \frac{1}{z+3}$$

$$\Rightarrow w(z+3) = 1$$

$$\Rightarrow wz + 3w = 1$$

$$\Rightarrow wz = 1 - 3w$$

$$\Rightarrow z = \frac{1 - 3w}{w}$$

$$\Rightarrow z = \frac{1 - 3(u + iv)}{u + iv}$$

$$\Rightarrow z = \frac{1 - 3u - 3iv}{u + iv}$$

$$\Rightarrow z = \frac{[(1 - 3u) - (3v)i]}{[(u + iv)]} \times \frac{(u - iv)}{(u - iv)}$$

$$\Rightarrow z = \frac{(1 - 3u)u - 3v^2 - iv(1 - 3u) - i(3uv)}{u^2 + v^2}$$

$$\Rightarrow z = \frac{u - 3u^2 - 3v^2}{u^2 + v^2} + \frac{i(-v + 3uv - 3uv)}{u^2 + v^2}$$

$$\Rightarrow z = \frac{u - 3u^2 - 3v^2}{u^2 + v^2} + \frac{i(-v)}{u^2 + v^2}$$

$$\text{So, } x + iy = \frac{u - 3u^2 - 3v^2}{u^2 + v^2} + \frac{i(-v)}{u^2 + v^2}$$

$$\Rightarrow x = \frac{u - 3u^2 - 3v^2}{u^2 + v^2}$$

$$\text{and } y = \frac{-v}{u^2 + v^2}$$

As $2x - 2y + 7 = 0$, then

$$2\left(\frac{u - 3u^2 - 3v^2}{u^2 + v^2}\right) - 2\left(\frac{-v}{u^2 + v^2}\right) + 7 = 0$$

$$\Rightarrow \frac{2u - 6u^2 - 6v^2}{u^2 + v^2} + \frac{2v}{u^2 + v^2} + 7 = 0 \quad (\times (u^2 + v^2))$$

$$\Rightarrow 2u - 6u^2 - 6v^2 + 2v + 7(u^2 + v^2) = 0$$

$$\Rightarrow 2u - 6u^2 - 6v^2 + 2v + 7u^2 + 7v^2 = 0$$

$$\Rightarrow u^2 + 2u + v^2 + 2v = 0$$

$$\Rightarrow (u + 1)^2 - 1 + (v + 1)^2 - 1 = 0$$

$$\Rightarrow (u + 1)^2 + (v + 1)^2 = 2$$

$$\Rightarrow (u + 1)^2 + (v + 1)^2 = (\sqrt{2})^2$$

Therefore the transformation T maps the line $2x - 2y + 7 = 0$ in the z -plane to a circle C with

© Pearson Education Ltd 2009

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise I, Question 1

Question:

Express $\frac{(\cos 3x + i \sin 3x)^2}{\cos x - i \sin x}$ in the form $\cos nx + i \sin nx$ where n is an integer to be determined.

Solution:

$$\begin{aligned} & \frac{(\cos 3x + i \sin 3x)^2}{\cos x - i \sin x} \\ &= \frac{(\cos 3x + i \sin 3x)^2}{\cos(-x) + i \sin(-x)} \\ &= \frac{\cos 6x + i \sin 6x}{\cos(-x) + i \sin(-x)} \\ &= \cos(6x - -x) + i \sin(6x - -x) \\ &= \cos 7x + i \sin 7x \end{aligned}$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise I, Question 2

Question:

Use de Moivre's theorem to evaluate

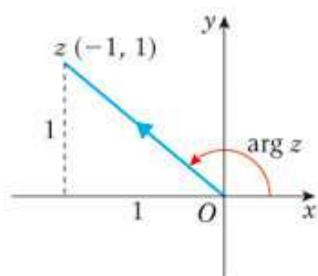
a $(-1 + i)^8$

b $\frac{1}{\left(\frac{1}{2} - \frac{1}{2}i\right)^{16}}$

Solution:

a $(-1 + i)^8$

If $z = -1 + i$, then



$$r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

$$\theta = \arg z = \pi - \tan^{-1}\left(\frac{1}{1}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$\text{So, } -1 + i = \sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$$

$$\begin{aligned} \therefore (-1 + i)^8 &= \left[\sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) \right]^8 \\ &= (\sqrt{2})^8 \left(\cos \frac{24\pi}{4} + i \sin \frac{24\pi}{4} \right) \end{aligned}$$

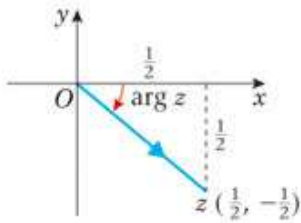
$$= 16(\cos 6\pi + i \sin 6\pi)$$

$$= 16(1 + i(0))$$

Therefore, $(-1 + i)^8 = 16$

$$\mathbf{b} \frac{1}{\left(\frac{1}{2} - \frac{1}{2}i\right)^{16}} = \left(\frac{1}{2} - \frac{1}{2}i\right)^{-16}$$

Let $z = \frac{1}{2} - \frac{1}{2}i$, then



$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$\theta = \tan^{-1}\left(\frac{-\frac{1}{2}}{\frac{1}{2}}\right) = -\frac{\pi}{4}$$

$$\text{So } \frac{1}{2} - \frac{1}{2}i = \frac{1}{\sqrt{2}} \left[\cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right) \right]$$

$$\begin{aligned} \left(\frac{1}{2} - \frac{1}{2}i\right)^{-16} &= \left[\frac{1}{\sqrt{2}} \left(\cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right) \right) \right]^{-16} \\ &= \left(2^{-\frac{1}{2}}\right)^{-16} \left(\cos\left(\frac{16\pi}{4}\right) + i \sin\left(\frac{16\pi}{4}\right) \right) \\ &= 2^8 (\cos 4\pi + i \sin 4\pi) \\ &= 256(1 + i(0)) \\ &= 256 \end{aligned}$$

$$\text{Therefore, } \frac{1}{\left(\frac{1}{2} - \frac{1}{2}i\right)^{16}} = 256$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise I, Question 3

Question:

- a** If $z = \cos \theta + i \sin \theta$, use de Moivre's theorem to show that $z^n + \frac{1}{z^n} = 2 \cos n\theta$.
- b** Express $\left(z^2 + \frac{1}{z^2}\right)^3$ in terms of $\cos 6\theta$ and $\cos 2\theta$.
- c** Hence, or otherwise, show that $\cos^3 2\theta = a \cos 6\theta + b \cos 2\theta$, where a and b are constants.
- d** Hence, or otherwise, show that $\int_0^{\frac{\pi}{6}} \cos^3 2\theta d\theta = k\sqrt{3}$, where k is a constant.

Solution:

a $z = \cos \theta + i \sin \theta$

$$\begin{aligned} z^n &= (\cos \theta + i \sin \theta)^n \\ &= \cos n\theta + i \sin n\theta \end{aligned}$$

de Moivre's Theorem.

$$\begin{aligned} \frac{1}{z^n} &= z^{-n} = (\cos \theta + i \sin \theta)^{-n} \\ &= \cos(-n\theta) + i \sin(-n\theta) \\ &= \cos n\theta - i \sin n\theta \end{aligned}$$

de Moivre's Theorem.

$$\begin{aligned} \cos(-n\theta) &= \cos n\theta \\ \sin(-n\theta) &= -\sin n\theta \end{aligned}$$

Therefore $z^n + \frac{1}{z^n} = \cos n\theta + i \sin n\theta + \cos n\theta - i \sin n\theta$

i.e. $z^n + \frac{1}{z^n} = 2 \cos n\theta$ (as required)

b $\left(z^2 + \frac{1}{z^2}\right)^3 = (z^2)^3 + {}^3C_1(z^2)^2\left(\frac{1}{z^2}\right) + {}^3C_2(z^2)\left(\frac{1}{z^2}\right)^2 + \left(\frac{1}{z^2}\right)^3$

$$\begin{aligned} &= z^6 + 3z^4\left(\frac{1}{z^2}\right) + 3z^2\left(\frac{1}{z^4}\right) + \frac{1}{z^6} \\ &= z^6 + 3z^2 + \frac{3}{z^2} + \frac{1}{z^6} \\ &= \left(z^6 + \frac{1}{z^6}\right) + 3\left(z^2 + \frac{1}{z^2}\right) \\ &= 2 \cos 6\theta + 3(2) \cos 2\theta \\ &= 2 \cos 6\theta + 6 \cos 2\theta \end{aligned}$$

Hence, $\left(z^2 + \frac{1}{z^2}\right)^3 = 2 \cos 6\theta + 6 \cos 2\theta$

$$\mathbf{c} \quad \left(z^2 + \frac{1}{z^2}\right)^3 = (2 \cos 2\theta)^3 = 8 \cos^3 2\theta = 2 \cos 6\theta + 6 \cos 2\theta$$

$$\therefore \cos^3 2\theta = \frac{2}{8} \cos 6\theta + \frac{6}{8} \cos 2\theta$$

$$\text{Hence, } \cos^3 2\theta = \frac{1}{4} \cos 6\theta + \frac{3}{4} \cos 2\theta$$

$$\mathbf{d} \quad \int_0^{\frac{\pi}{6}} \cos^3 2\theta \, d\theta = \int_0^{\frac{\pi}{6}} \frac{1}{4} \cos 6\theta + \frac{3}{4} \cos 2\theta \, d\theta$$

$$= \left[\frac{1}{24} \sin 6\theta + \frac{3}{8} \sin 2\theta \right]_0^{\frac{\pi}{6}}$$

$$= \left(\frac{1}{24} \sin \pi + \frac{3}{8} \sin \left(\frac{\pi}{3} \right) \right) - \left(\frac{1}{24} \sin 0 + \frac{3}{8} \sin 0 \right)$$

$$= \left(\frac{1}{24} (0) + \frac{3}{8} \left(\frac{\sqrt{3}}{2} \right) \right) - (0)$$

$$= \frac{3}{16} \sqrt{3}$$

$$\text{So, } \int_0^{\frac{\pi}{6}} \cos^3 2\theta \, d\theta = \frac{3}{16} \sqrt{3}$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise I, Question 4

Question:

- a** Use de Moivre's theorem to show that $\cos 5\theta = \cos \theta(16 \cos^4 \theta - 20 \cos^2 \theta + 5)$.
- b** By solving the equation $\cos 5\theta = 0$, deduce that $\cos^2\left(\frac{\pi}{10}\right) = \frac{5 + \sqrt{5}}{8}$.
- c** Hence, or otherwise, write down the exact values of $\cos^2\left(\frac{3\pi}{10}\right)$, $\cos^2\left(\frac{7\pi}{10}\right)$ and $\cos^2\left(\frac{9\pi}{10}\right)$.

Solution:

$$\mathbf{a} \quad (\cos \theta + i \sin \theta)^5 = \cos 5\theta + i \sin 5\theta$$

de Moivre's Theorem.

$$= \cos^5 \theta + {}^5C_1 \cos^4 \theta (i \sin \theta) + {}^5C_2 \cos^3 \theta (i \sin \theta)^2 \\ + {}^5C_3 \cos^2 \theta (i \sin \theta)^3 + {}^5C_4 \cos \theta (i \sin \theta)^4 + (i \sin \theta)^5$$

$$= \cos^5 \theta + 5i \cos^4 \theta \sin \theta + 10i^2 \cos^3 \theta \sin^2 \theta \\ + 10i^3 \cos^2 \theta \sin^3 \theta + 5i^4 \cos \theta \sin^4 \theta + i^5 \sin^5 \theta$$

Binomial expansion.

Hence,

$$\cos 5\theta + i \sin 5\theta = \cos^5 \theta + 5i \cos^4 \theta \sin \theta - 10 \cos^3 \theta \sin^2 \theta \\ - 10i \cos^2 \theta \sin^3 \theta + 5 \cos \theta \sin^4 \theta + i \sin^5 \theta$$

Equating the real parts gives,

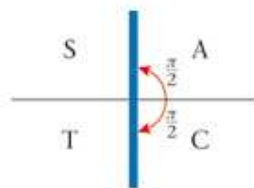
$$\cos 5\theta = \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta \\ = \cos \theta (\cos^4 \theta - 10 \cos^2 \theta \sin^2 \theta + 5 \sin^4 \theta) \\ = \cos \theta (\cos^4 \theta - 10 \cos^2 \theta (1 - \cos^2 \theta) + 5 (1 - \cos^2 \theta)^2) \\ = \cos \theta (\cos^4 \theta - 10 \cos^2 \theta + 10 \cos^4 \theta + 5 (1 - 2 \cos^2 \theta + \cos^4 \theta)) \\ = \cos \theta (\cos^4 \theta - 10 \cos^2 \theta + 10 \cos^4 \theta + 5 - 10 \cos^2 \theta + 5 \cos^4 \theta) \\ = \cos \theta (16 \cos^4 \theta - 20 \cos^2 \theta + 5)$$

Applying
 $\sin^2 \theta = 1 - \cos^2 \theta$.

Hence, $\cos 5\theta = \cos \theta (16 \cos^4 \theta - 20 \cos^2 \theta + 5)$ (as required)

$$\mathbf{b} \quad \cos 5\theta = 0$$

$$\alpha = \frac{\pi}{2}$$



$$\text{So } 5\theta = \left\{ \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2} \right\}$$

$$\theta = \left\{ \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10} \right\}$$

$$\theta = \left\{ \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10} \right\} \quad \text{for } 0 < \theta \leq \pi$$

$$\cos 5\theta = 0 \Rightarrow \cos \theta (16 \cos^4 \theta - 20 \cos^2 \theta + 5) = 0$$

$$\text{Five solutions must come from: } \cos \theta (16 \cos^4 \theta - 20 \cos^2 \theta + 5) = 0$$

$$\text{Solution ① } \cos \theta = 0$$

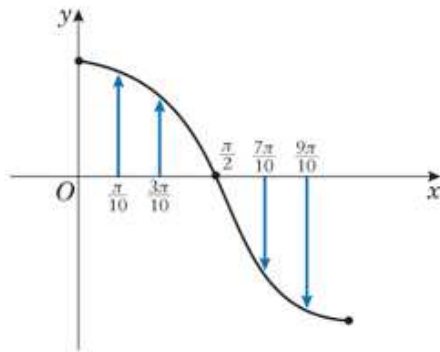
$$\alpha = \frac{\pi}{2}$$

For $0 < \theta \leq \pi$, $\theta = \frac{\pi}{2}$ (as found earlier)

The final 4 solutions come from: $16 \cos^4 \theta - 20 \cos^2 \theta + 5 = 0$

$$\begin{aligned} \cos^2 \theta &= \frac{20 \pm \sqrt{400 - 4(16)(5)}}{32} \\ &= \frac{20 \pm \sqrt{400 - 320}}{32} \\ &= \frac{20 \pm \sqrt{80}}{32} \\ &= \frac{20 \pm \sqrt{16}\sqrt{5}}{32} \\ &= \frac{20 \pm 4\sqrt{5}}{32} \end{aligned}$$

$$\therefore \cos^2 \theta = \frac{5 \pm \sqrt{5}}{8}$$



Due to symmetry and as $\cos\left(\frac{\pi}{10}\right) > \cos\left(\frac{3\pi}{10}\right)$

$$\cos^2\left(\frac{\pi}{10}\right) = \cos^2\left(\frac{9\pi}{10}\right) > \cos^2\left(\frac{3\pi}{10}\right) = \cos^2\left(\frac{7\pi}{10}\right)$$

$$\therefore \cos^2\left(\frac{7\pi}{10}\right) = \frac{5 + \sqrt{5}}{8}$$

$$\mathbf{c} \quad \cos^2\left(\frac{3\pi}{10}\right) = \frac{5 - \sqrt{5}}{8}$$

$$\cos^2\left(\frac{7\pi}{10}\right) = \cos^2\left(\frac{3\pi}{10}\right) = \frac{5 - \sqrt{5}}{8}$$

$$\cos^2\left(\frac{9\pi}{10}\right) = \cos^2\left(\frac{\pi}{10}\right) = \frac{5 + \sqrt{5}}{8}$$

$$\text{Therefore, } \cos^2\left(\frac{3\pi}{10}\right) = \frac{5 - \sqrt{5}}{8}, \cos^2\left(\frac{7\pi}{10}\right) = \frac{5 - \sqrt{5}}{8}, \cos^2\left(\frac{9\pi}{10}\right) = \frac{5 + \sqrt{5}}{8}$$

Solutionbank FP2

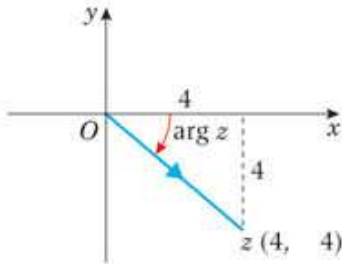
Edexcel AS and A Level Modular Mathematics

Exercise I, Question 5

Question:

- a** Express $4 - 4i$ in the form $r(\cos \theta + i \sin \theta)$, where $r > 0$, $-\pi < \theta \leq \pi$, where r and θ are exact values.
- b** Hence, or otherwise, solve the equation $z^5 = 4 - 4i$ leaving your answers in the form $z = Re^{ik\pi}$, where R is the modulus of z and k is a rational number such that $-1 \leq k \leq 1$.
- c** Show on an Argand diagram the points representing your solutions.

Solution:

a $4 - 4i$ 

$$\text{modulus } r = \sqrt{4^2 + (-4)^2} = \sqrt{16 + 16} = \sqrt{32} = \sqrt{16\sqrt{2}} = 4\sqrt{2}$$

$$\text{argument} = \theta = -\tan^{-1}\left(\frac{4}{4}\right) = -\frac{\pi}{4}$$

$$\therefore 4 - 4i = 4\sqrt{2} \left(\cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right) \right)$$

b $z^5 = 4 - 4i$

$$\text{for } 4 - 4i, r = 4\sqrt{2}, \theta = -\frac{\pi}{4}$$

$$\text{So, } z^5 = 4\sqrt{2} e^{i\left(-\frac{\pi}{4}\right)}$$

$$z^5 = 4\sqrt{2} e^{i\left(-\frac{\pi}{4} + 2k\pi\right)}, k \in \mathbb{Z}$$

$$\text{Hence, } z = \left[4\sqrt{2} e^{i\left(-\frac{\pi}{4} + 2k\pi\right)} \right]^{\frac{1}{5}}$$

$$= (4\sqrt{2})^{\frac{1}{5}} e^{i\left(\frac{-\frac{\pi}{4} + 2k\pi}{5}\right)}$$

$$= \sqrt{2} e^{i\left(-\frac{\pi}{20} + \frac{2k\pi}{5}\right)}$$

$$k = 0, z_1 = \sqrt{2} e^{i\left(-\frac{\pi}{20}\right)}$$

$$k = 1, z_2 = \sqrt{2} e^{i\left(\frac{7\pi}{20}\right)}$$

$$k = 2, z_3 = \sqrt{2} e^{i\left(\frac{3\pi}{4}\right)}$$

$$k = -1, z_4 = \sqrt{2} e^{i\left(-\frac{9\pi}{20}\right)}$$

$$k = -2, z_5 = \sqrt{2} e^{i\left(-\frac{17\pi}{20}\right)}$$

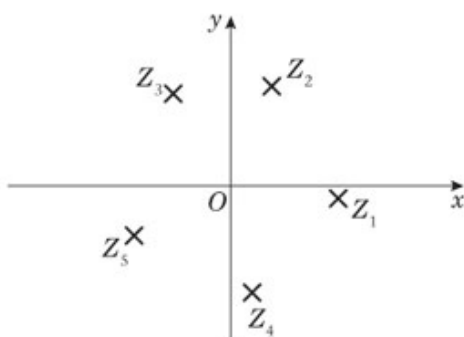
$$\text{Therefore, } z = \sqrt{2} e^{-\frac{\pi i}{20}}, \sqrt{2} e^{\frac{7\pi i}{20}}, \sqrt{2} e^{\frac{3\pi i}{4}}, \sqrt{2} e^{-\frac{9\pi i}{20}}, \sqrt{2} e^{-\frac{17\pi i}{20}}$$

de Moivre's Theorem.

$$4\sqrt{2} = 2^{\frac{5}{2}}$$

$$\text{So, } (4\sqrt{2})^{\frac{1}{5}} = (2^{\frac{5}{2}})^{\frac{1}{5}}$$

$$= 2^{\frac{1}{2}} = \sqrt{2}$$

c

© Pearson Education Ltd 2009

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise I, Question 6

Question:

a Find the Cartesian equations of

i the locus of points representing $|z - 3 + i| = |z - 1 - i|$,

ii the locus of points representing $|z - 2| = 2\sqrt{2}$.

b Find the two values of z that satisfy both $|z - 3 + i| = |z - 1 - i|$ and $|z - 2| = 2\sqrt{2}$.

c Hence on the same Argand diagram sketch:

i the locus of points representing $|z - 3 + i| = |z - 1 - i|$,

ii the locus of points representing $|z - 2| = 2\sqrt{2}$.

The region R is defined by the inequalities $|z - 3 + i| \geq |z - 1 - i|$ and $|z + 2| \leq 2\sqrt{2}$.

d On your sketch in part **c**, identify, by shading, the region R .

Solution:

a i Let $|z - 3 + i| = |z - 1 - i|$

$$\Rightarrow |x + iy - 3 + i| = |x + iy - 1 - i|$$

$$\Rightarrow |(x - 3) + i(y + 1)| = |(x - 1) + i(y - 1)|$$

$$\Rightarrow |(x - 3) + i(y + 1)|^2 = |(x - 1) + i(y - 1)|^2$$

$$\Rightarrow (x - 3)^2 + (y + 1)^2 = (x - 1)^2 + (y - 1)^2$$

$$\Rightarrow x^2 - 6x + 9 + y^2 + 2y + 1 = x^2 - 2x + 1 + y^2 - 2y + 1$$

$$\Rightarrow -6x + 2y + 10 = -2x - 2y + 2$$

$$\Rightarrow -4x + 4y + 8 = 0$$

$$\Rightarrow 4y = 4x - 8$$

$$\Rightarrow y = x - 2$$

The Cartesian equation of the locus of points representing $|z - 3 + i| = |z - 1 - i|$ is $y = x - 2$.

METHOD ① **i** $|z - 3 + i| = |z - 1 - i|$

As $|z - 3 + i| = |z - 1 - i|$ is a perpendicular bisector of the line joining $A(3, -1)$ to $B(1, 1)$,

$$\text{then } m_{AB} = \frac{1 - (-1)}{1 - 3} = \frac{2}{-2} = -1$$

$$\text{and perpendicular gradient} = \frac{-1}{-1} = 1$$

$$\text{mid-point of } AB \text{ is } \left(\frac{3 + 1}{2}, \frac{-1 + 1}{2} \right)$$

$$= (2, 0)$$

$$\Rightarrow y - 0 = 1(x - 2)$$

$$y = x - 2$$

The Cartesian equation of the locus of points representing $|z - 3 + i| = |z - 1 - i|$ is $y = x - 2$.

ii $|z - 2| = 2\sqrt{2}$

$$\Rightarrow \text{circle centre } (2, 0), \text{ radius } 2\sqrt{2}.$$

$$\Rightarrow \text{equation of circle is } (x - 2)^2 + y^2 = (2\sqrt{2})^2$$

$$\Rightarrow (x - 2)^2 + y^2 = 8$$

The Cartesian equation of the locus of points representing $|z - 2| = 2\sqrt{2}$ is $(x - 2)^2 + y^2 = 8$.

$$\mathbf{b} \quad |z - 3 + i| = |z - 1 + i| \Rightarrow y = x - 2 \quad \textcircled{1}$$

$$|z - 2| = 2\sqrt{2} \Rightarrow (x - 2)^2 + y^2 = 8 \quad \textcircled{2}$$

$$\textcircled{1} \wedge \textcircled{2} \Rightarrow (x - 2)^2 + (x - 2)^2 = 8$$

$$\Rightarrow 2(x - 2)^2 = 8$$

$$\Rightarrow (x - 2)^2 = 4$$

$$\Rightarrow x - 2 = \pm\sqrt{4}$$

$$\Rightarrow x - 2 = \pm 2$$

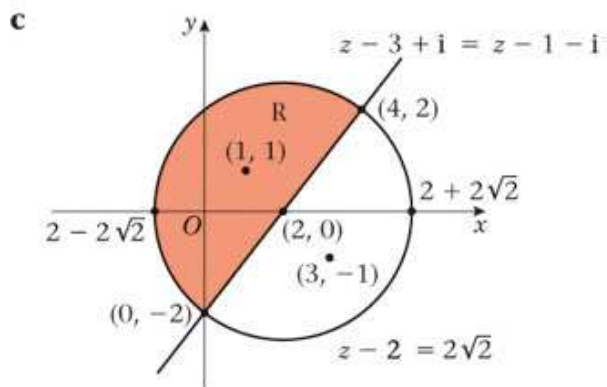
$$\Rightarrow x = 2 \pm 2$$

$$\Rightarrow x = 0, 4$$

when $x = 0$, $y = 0 - 2 = -2 \Rightarrow z = 0 - 2i$

when $x = 4$, $y = 4 - 2 = 2 \Rightarrow z = 4 + 2i$

The values of z are $-2i$ and $4 + 2i$



Note that $|z - 3 + i| = |z - 1 - i| \Rightarrow y = x - 2$ goes through the point (2, 0) and so is a diameter of $|z - 2| = 2\sqrt{2}$.

d The region R is shaded on the Argand diagram in part **i**, which satisfies

$$|z - 3 + i| \geq |z - 1 - i| \text{ and } |z - 2| \leq 2\sqrt{2}.$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise I, Question 7

Question:

- a** Find the Cartesian equation of the locus of points representing $|z + 2| = |2z - 1|$.
- b** Find the value of z which satisfies both $|z + 2| = |2z - 1|$ and $\arg z = \frac{\pi}{4}$.
- c** Hence shade in the region R on an Argand diagram which satisfies both $|z + 2| \geq |2z - 1|$ and $\frac{\pi}{4} \leq \arg z \leq \pi$.

Solution:

$$\mathbf{a} \quad |z + 2| = |2z - 1|$$

$$\Rightarrow |x + iy + 2| = |2(x + iy) - 1|$$

$$\Rightarrow |x + iy + 2| = |2x + 2iy - 1|$$

$$\Rightarrow |(x + 2) + iy| = |(2x - 1) + i(2y)|$$

$$\Rightarrow |(x + 2) + iy|^2 = |(2x - 1) + i(2y)|^2$$

$$\Rightarrow (x + 2)^2 + y^2 = (2x - 1)^2 + i(2y)^2$$

$$\Rightarrow x^2 + 4x + 4 + y^2 = 4x^2 - 4x + 1 + 4y^2$$

$$\Rightarrow 0 = 3x^2 - 8x + 3y^2 + 1 - 4$$

$$\Rightarrow 3x^2 - 8x + 3y^2 - 3 = 0$$

$$\Rightarrow x^2 - \frac{8}{3}x + y^2 - 1 = 0$$

$$\Rightarrow \left(x - \frac{4}{3}\right)^2 - \frac{16}{9} + y^2 - 1 = 0$$

$$\Rightarrow \left(x - \frac{4}{3}\right)^2 + y^2 = \frac{16}{9} + 1$$

$$\Rightarrow \left(x - \frac{4}{3}\right)^2 + y^2 = \frac{25}{9}$$

$$\Rightarrow \left(x - \frac{4}{3}\right)^2 + y^2 = \left(\frac{5}{3}\right)^2$$

This is a circle, centre $\left(\frac{4}{3}, 0\right)$, radius $\frac{5}{3}$.

The Cartesian equation of the locus of points representing $|z + 2| = |2z - 1|$ is

$$\left(x - \frac{4}{3}\right)^2 + y^2 = \frac{25}{9}.$$

$$\mathbf{b} \quad |z + 2| = |2z - 1| \Rightarrow \left(x - \frac{4}{3}\right)^2 + y^2 = \frac{25}{9} \quad \textcircled{1}$$

$$\arg z = \frac{\pi}{4} \Rightarrow \arg(x + iy) = \frac{\pi}{4}$$

$$\Rightarrow \frac{y}{x} = \tan \frac{\pi}{4}$$

$$\Rightarrow \frac{y}{x} = 1$$

$$\Rightarrow y = x \quad \text{where } x > 0, y > 0 \quad \textcircled{2}$$

$$\begin{aligned}
\textcircled{2} \wedge \textcircled{1}: \quad & \left(x - \frac{4}{3}\right)^2 + x^2 = \frac{25}{9} \\
& \Rightarrow x^2 - \frac{4}{3}x - \frac{4}{3}x + \frac{16}{9} + x^2 = \frac{25}{9} \\
& \Rightarrow 2x^2 - \frac{8}{3}x = \frac{25}{9} - \frac{16}{9} \\
& \Rightarrow 2x^2 - \frac{8}{3}x = \frac{9}{9} \\
& \Rightarrow 2x^2 - \frac{8}{3}x = 1 \quad (\times 3) \\
& \Rightarrow 6x^2 - 8x = 3 \\
& \Rightarrow 6x^2 - 8x - 3 = 0 \\
& \Rightarrow x = \frac{8 \pm \sqrt{64 - 4(6)(-3)}}{2(6)} \\
& \Rightarrow x = \frac{8 \pm \sqrt{136}}{12} \\
& \Rightarrow x = \frac{8 \pm 2\sqrt{34}}{12} \\
& \Rightarrow x = \frac{4 \pm \sqrt{34}}{6}
\end{aligned}$$

As $x > 0$ then we reject $x = \frac{4 - \sqrt{34}}{6}$

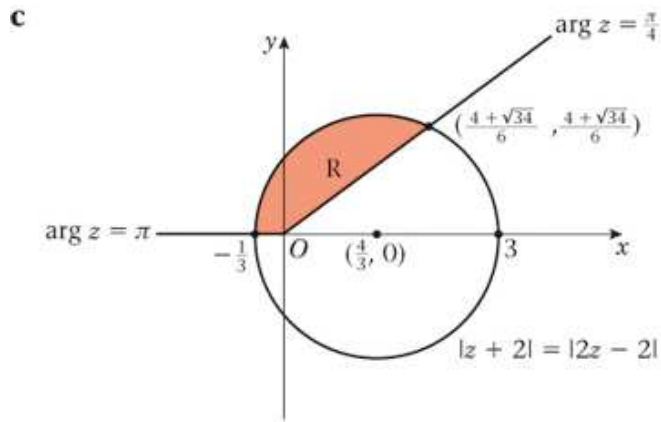
and accept $x = \frac{4 + \sqrt{34}}{6}$

as $y = x$, then $y = \frac{4 + \sqrt{34}}{6}$

So $z = \left(\frac{4 + \sqrt{34}}{6}\right) + \left(\frac{4 + \sqrt{34}}{6}\right)i$

The value of z satisfying $|z + 2| = |2z - 1|$ and $\arg z = \frac{\pi}{4}$

is $z = \left(\frac{4 + \sqrt{34}}{6}\right) + \left(\frac{4 + \sqrt{34}}{6}\right)i$ OR $z = 1.64 + 1.64i$ (2 d.p.)



The region R (shaded) satisfies both $|z + 2| \geq |2z - 1|$ and $\frac{\pi}{4} \leq \arg z \leq \pi$.

Note that $|z + 2| \geq |2z - 1|$

$$\Rightarrow (x + 2)^2 + y^2 \geq (2x - 1)^2 + (2y)^2$$

$$\Rightarrow 0 \geq 3x^2 - 8x + 3y^2 - 3$$

$$\Rightarrow 0 \geq \left(x - \frac{4}{3}\right)^2 - \frac{16}{9} + y^2 - 1$$

$$\Rightarrow \frac{25}{9} \geq \left(x - \frac{4}{3}\right)^2 + y^2$$

$$\Rightarrow \left(x - \frac{4}{3}\right)^2 + y^2 \leq \frac{25}{9}$$

represents region inside and bounded by the circle, centre $(\frac{4}{3}, 0)$, radius $\frac{5}{3}$.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise I, Question 8

Question:

The point P represents a complex number z in an Argand diagram.
Given that $|z + 1 - i| = 1$

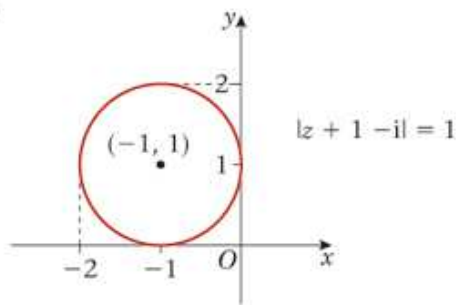
- a** find a Cartesian equation for the locus of P ,
- b** sketch the locus of P on an Argand diagram,
- c** find the greatest and least values of $|z|$,
- d** find the greatest and least values of $|z - 1|$.

Solution:

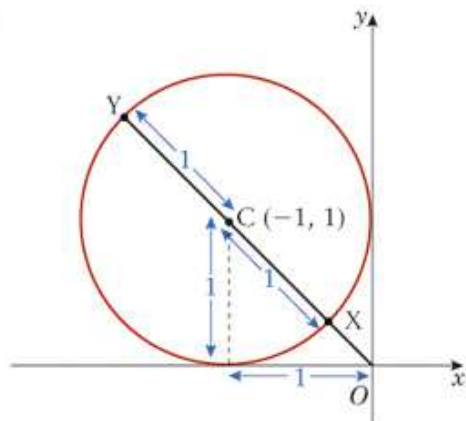
a $|z + 1 - i| = 1$ is a circle, centre $(-1, 1)$, radius 1.

The Cartesian equation for the locus of P is $(x + 1)^2 + (y - 1)^2 = 1$.

b



c



$|z|$ is the distance from $(0, 0)$ to the locus of points.

From the Argand diagram,

$|z|_{\max}$ is the distance OY

$|z|_{\min}$ is the distance OX

Note that radius = $CX = CY = 1$

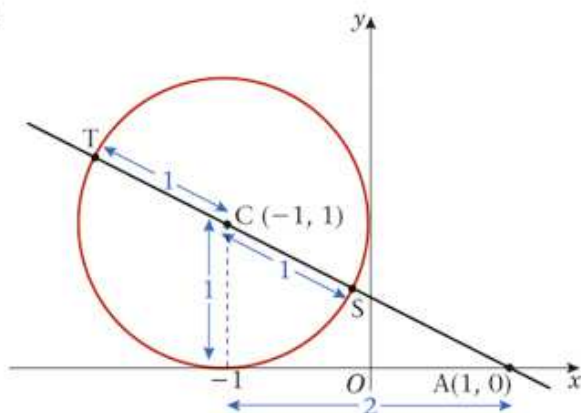
and $OC = \sqrt{1^2 + 1^2} = \sqrt{2}$

$|z|_{\max} = OC + CY = \sqrt{2} + 1$

$|z|_{\min} = OC - CX = \sqrt{2} - 1$

The greatest value of $|z|$ is $\sqrt{2} + 1$ and the least value of $|z|$ is $\sqrt{2} - 1$.

d



$|z - 1|$ is the distance from $A(1, 0)$ to the locus of points.

From the Argand diagram,

$|z - 1|_{\max}$ is the distance AS

© Pearson Education Ltd 2009

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise I, Question 9

Question:

Given that $\arg\left(\frac{z - 4 - 2i}{z - 6i}\right) = \frac{\pi}{2}$,

a sketch the locus of $P(x, y)$ which represents z on an Argand diagram,

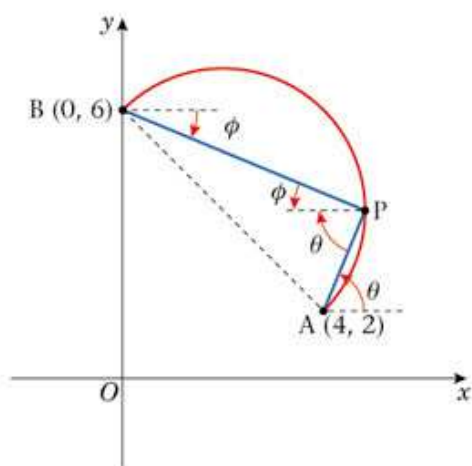
b deduce the exact value of $|z - 2 - 4i|$.

Solution:

a $\arg\left(\frac{z - 4 - 2i}{z - 6i}\right) = \frac{\pi}{2}$

$$\Rightarrow \arg(z - 4 - 2i) - \arg(z - 6i) = \frac{\pi}{2}$$

$$\Rightarrow \theta - \phi = \frac{\pi}{2}, \text{ where } \arg(z - 4 - 2i) = \theta \text{ and } \arg(z - 6i) = \phi.$$



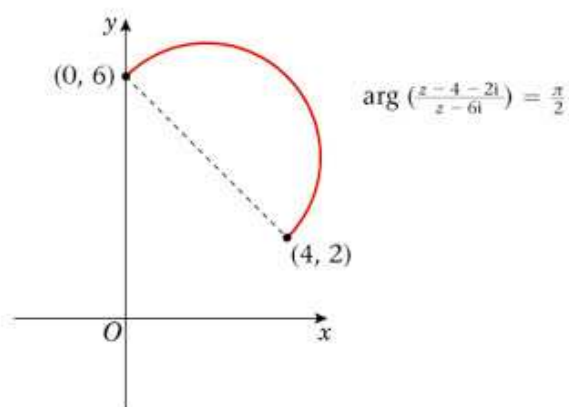
Using geometry,

$$\Rightarrow \hat{APB} = -\phi + \theta$$

$$\Rightarrow \hat{APB} = \theta - \phi$$

$$\Rightarrow \hat{APB} = \frac{\pi}{2}$$

The locus of z is the arc of a circle (in this case, a semi-circle) cut off at $(4, 2)$ and $(0, 6)$ as shown below.



b $|z - 2 - 4i|$ is the distance from the point $(2, 4)$ to the locus of points P .

Note, as the locus is a semi-circle, its centre is $\left(\frac{4+0}{2}, \frac{2+6}{2}\right) = (2, 4)$.

Therefore $|z - 2 - 4i|$ is the distance from the centre of the semi-circle to points on the locus of points P .

Hence $|z - 2 - 4i| = \text{radius of semi-circle}$

$$= \sqrt{(0 - 2)^2 + (6 - 4)^2}$$

$$= \sqrt{4 + 4}$$

$$= \sqrt{8}$$

$$= 2\sqrt{2}$$

The exact value of $|z - 2 - 4i|$ is $2\sqrt{2}$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise I, Question 10

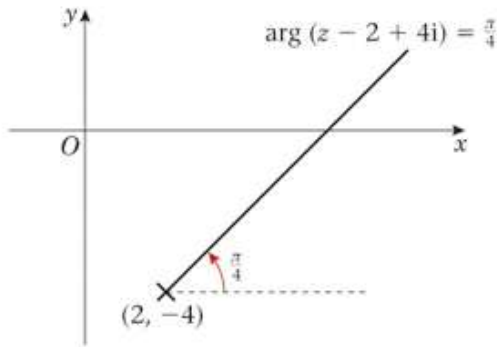
Question:

Given that $\arg(z - 2 + 4i) = \frac{\pi}{4}$,

- a** sketch the locus of $P(x, y)$ which represents z on an Argand diagram,
- b** find the minimum value of $|z|$ for points on this locus.

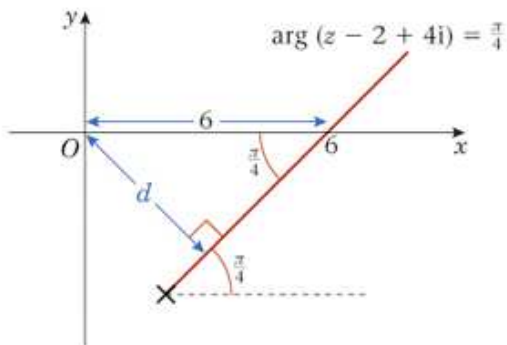
Solution:

a $\arg(z - 2 + 4i) = \frac{\pi}{4}$ is a half-line from $(2, -4)$ as shown



$$\begin{aligned} \mathbf{b} \quad \arg(z - 2 + 4i) = \frac{\pi}{4} &\Rightarrow \arg(x + iy - 2 + 4i) = \frac{\pi}{4} \\ &\Rightarrow \arg((x - 2) + i(y + 4)) = \frac{\pi}{4} \\ &\Rightarrow \frac{y + 4}{x - 2} = \tan \frac{\pi}{4} = 1 \\ &\Rightarrow y + 4 = x - 2 \\ &\Rightarrow y = x - 6, x > 0, y > 0 \end{aligned}$$

Half-line cuts x -axis at $0 = x - 6 \Rightarrow x = 6$.



$|z|$ is the distance from $(0, 0)$ to the locus of points.

$$|z|_{\min} = d \Rightarrow \frac{d}{6} = \sin\left(\frac{\pi}{4}\right) \Rightarrow d = 6 \sin\left(\frac{\pi}{4}\right) = 6\left(\frac{1}{\sqrt{2}}\right) = \frac{6\sqrt{2}}{2} = 3\sqrt{2}.$$

Therefore the minimum value of $|z|$ is $3\sqrt{2}$.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise I, Question 11

Question:

The transformation T from the z -plane, where $z = x + iy$, to the w -plane where $w = u + iv$, is given by $w = \frac{1}{z}$, $z \neq 0$.

- a** Show that the image, under T , of the line with equation $x = \frac{1}{2}$ in the z -plane is a circle C in the w -plane. Find the equation of C .
- b** Hence, or otherwise, shade and label on an Argand diagram the region R of the w -plane which is the image of $x \geq \frac{1}{2}$ under T .

Solution:

$$T: w = \frac{1}{z}$$

- a** line $x = \frac{1}{2}$ in the z -plane

$$w = \frac{1}{z}$$

$$\Rightarrow wz = 1$$

$$\Rightarrow z = \frac{1}{w}$$

$$\Rightarrow z = \frac{1}{u + iv}$$

$$\Rightarrow z = \frac{1}{(u + iv)} \times \frac{(u - iv)}{(u - iv)}$$

$$\Rightarrow z = \frac{u - iv}{u^2 + v^2}$$

$$\Rightarrow z = \frac{u}{u^2 + v^2} + i\left(\frac{-v}{u^2 + v^2}\right)$$

$$\text{So, } x + iy = \frac{u}{u^2 + v^2} + i\left(\frac{-v}{u^2 + v^2}\right)$$

$$\Rightarrow x = \frac{u}{u^2 + v^2} \text{ and } y = \frac{-v}{u^2 + v^2}$$

$$\text{As } x = \frac{1}{2}, \text{ then } \frac{1}{2} = \frac{u}{u^2 + v^2}$$

$$\Rightarrow u^2 + v^2 = 2u$$

$$\Rightarrow u^2 - 2u + v^2 = 0$$

$$\Rightarrow (u - 1)^2 - 1 + v^2 = 0$$

$$\Rightarrow (u - 1)^2 + v^2 = 1$$

Therefore the transformation T maps the line $x = \frac{1}{2}$ in the z -plane to a circle C , with centre $(1, 0)$, radius 1. The equation of C is $(u - 1)^2 + v^2 = 1$.

$$\mathbf{b} \quad x \geq \frac{1}{2} \frac{u}{u^2 + v^2} \geq \frac{1}{2}$$

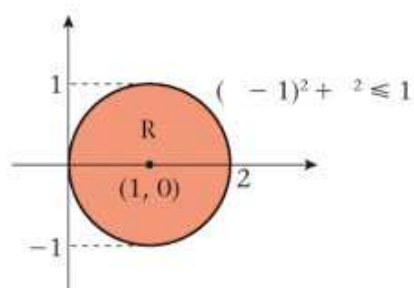
$$\Rightarrow 2u \geq u^2 + v^2$$

$$\Rightarrow 0 \geq u^2 + v^2 - 2u$$

$$\Rightarrow 0 \geq (u - 1)^2 + v^2 - 1$$

$$\Rightarrow 1 \geq (u - 1)^2 + v^2$$

$$\Rightarrow (u - 1)^2 + v^2 \leq 1$$



© Pearson Education Ltd 2009

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise I, Question 12

Question:

The point P represents the complex number z on an Argand diagram.

Given that $|z + 4i| = 2$,

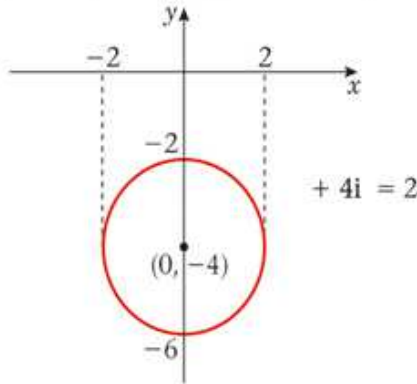
- a** sketch the locus of P on an Argand diagram.
- b** Hence find the maximum value of $|z|$.

T_1 , T_2 , T_3 and T_4 represent transformations from the z -plane to the w -plane. Describe the locus of the image of P under the transformations

- c** T_1 : $w = 2z$,
- d** T_2 : $w = iz$,
- e** T_3 : $w = -iz$,
- f** T_4 : $w = z^*$

Solution:

- a** $|z + 4i| = 2$ is represented by a circle centre $(0, -4)$, radius 2.



- b** $|z|$ represents the distance from $(0, 0)$ to points on the locus of P . Hence $|z|_{\max}$ is the distance OY .
 $|z|_{\max} = OY = 6$.
- c** $T_1: w = 2z$

METHOD ① z lies on circle with equation $|z + 4i| = 2$

$$\Rightarrow w = 2z$$

$$\Rightarrow \frac{w}{2} = z$$

$$\Rightarrow \frac{w}{2} + 4i = z + 4i$$

$$\Rightarrow \frac{w + 8i}{2} = z + 4i$$

$$\Rightarrow \left| \frac{w + 8i}{2} \right| = |z + 4i|$$

$$\Rightarrow \frac{|w + 8i|}{|2|} = |z + 4i|$$

$$\Rightarrow \frac{|w + 8i|}{2} = 2$$

$$\Rightarrow |w + 8i| = 4$$

So the locus of the image of P under T_1 is a circle centre $(0, -8)$, radius 4, with equation $u^2 + (v + 8)^2 = 16$.

METHOD ② z lies on circle centre $(0, -4)$, radius 2



enlargement scale factor 2, centre 0.

$w = 2z$ lies on a circle centre $(0, -8)$, radius 4.

So the locus of the image of P under T_1 is a circle centre $(0, -8)$, radius 4, with equation $u^2 + (v + 8)^2 = 16$.

d $T_2: w = iz$ z lies on a circle with equation $|z + 4i| = 2$

$$w = iz$$

$$\Rightarrow \frac{w}{i} = z$$

$$\Rightarrow \frac{w}{i} \left(\frac{i}{i} \right) = z$$

$$\Rightarrow \frac{wi}{(-1)} = z$$

$$\Rightarrow -wi = z$$

$$\Rightarrow z = -wi$$

$$\text{Hence } |z + 4i| = 2 \Rightarrow |-wi + 4i| = 2$$

$$\Rightarrow |(-i)(w - 4)| = 2$$

$$\Rightarrow |(-i)| |w - 4| = 2$$

$$\Rightarrow |w - 4| = 2$$

So the locus of the image of P under T_2 is a circle centre $(4, 0)$, radius 2, with equation $(u - 4)^2 + v^2 = 4$.

e $T_3: w = -iz$ z lies on a circle with equation $|z + 4i| = 2$

$$w = -iz$$

$$\Rightarrow iw = i(-iz)$$

$$\Rightarrow iw = z$$

$$\Rightarrow z = iw$$

$$\text{Hence } |z + 4i| = 2 \Rightarrow |iw + 4i| = 2$$

$$\Rightarrow |i(w + 4)| = 2$$

$$\Rightarrow |i| |w + 4| = 2$$

$$\Rightarrow |w + 4| = 2 \quad \leftarrow \boxed{|i| = 1}$$

So the locus of the image of P under T_3 is a circle centre $(-4, 0)$, radius 2, with equation $(u + 4)^2 + v^2 = 4$.

f $T_4: w = z^*$ z lies on a circle with equation $|z + 4i| = 2$

$$w = z^* \Rightarrow u + iv = x - iy$$

So $u = x$, $v = -y$ and $x = u$ and $y = -v$

$$|z + 4i| = 2 \Rightarrow |x + iy + 4i| = 2$$

$$\Rightarrow |x + i(y + 4)| = 2$$

$$\Rightarrow |u + i(-v + 4)| = 2$$

$$\Rightarrow |u + i(4 - v)| = 2$$

$$\Rightarrow |u + i(4 - v)|^2 = 2^2$$

$$\Rightarrow u^2 + (4 - v)^2 = 4$$

$$\Rightarrow u^2 + (v - 4)^2 = 4$$

So the locus of the image of P under T_4 is a circle centre $(0, 4)$, radius 2, with equation $u^2 + (v - 4)^2 = 4$.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise I, Question 13

Question:

The transformation T from the z -plane, where $z = x + iy$, to the w -plane where $w = u + iv$, is given by $w = \frac{z+2}{z+i}$, $z \neq -i$.

- a** Show that the image, under T , of the imaginary axis in the z -plane is a line l in the w -plane. Find the equation of l .
- b** Show that the image, under T , of the line $y = x$ in the z -plane is a circle C in the w -plane. Find the centre of C and show that the radius of C is $\frac{1}{2}\sqrt{10}$.

Solution:

$$T: w = \frac{z+2}{z+i}, z \neq -i$$

a the imaginary axis in z -plane $\Rightarrow x = 0$

$$w = \frac{z+2}{z+i}$$

$$\Rightarrow w(z+i) = z+2$$

$$\Rightarrow wz + iw = z + 2$$

$$\Rightarrow wz - z = 2 - iw$$

$$\Rightarrow z(w-1) = 2 - iw$$

$$\Rightarrow z = \frac{2 - iw}{w-1}$$

$$\Rightarrow z = \frac{2 - i(u+iv)}{u+iv-1}$$

$$\Rightarrow z = \frac{2 - iu + v}{(u-1) + iv}$$

$$\Rightarrow z = \left[\frac{(2+v) - iu}{(u-1) + iv} \right] \times \left[\frac{(u-1) - iv}{(u-1) - iv} \right]$$

$$\Rightarrow z = \frac{(2+v)(u-1) - uv - iv(2+v) - iu(u-1)}{(u-1)^2 + v^2}$$

$$\Rightarrow z = \frac{(2+v)(u-1) - uv}{(u-1)^2 + v^2} - i \left(\frac{v(2+v) + u(u-1)}{(u-1)^2 + v^2} \right)$$

$$\text{So } x + iy = \frac{(2+v)(u-1) - uv}{(u-1)^2 + v^2} - i \left(\frac{v(2+v) + u(u-1)}{(u-1)^2 + v^2} \right)$$

$$\Rightarrow x = \frac{(2+v)(u-1) - uv}{(u-1)^2 + v^2} \text{ and } y = \frac{-v(2+v) - u(u-1)}{(u-1)^2 + v^2}$$

As $x = 0$, then

$$\frac{(2+v)(u-1) - uv}{(u-1)^2 + v^2} = 0$$

$$\Rightarrow (2+v)(u-1) - uv = 0$$

$$\Rightarrow 2u - 2 + vu - v - uv = 0$$

$$\Rightarrow 2u - 2 - v = 0$$

$$\Rightarrow v = 2u - 2$$

The transformation T maps the imaginary axis in the z -plane to the line l with equation $v = 2u - 2$ in the w -plane.

b As $y = x$, then

$$\frac{-v(2+v) - u(u-1)}{(u-1)^2 + v^2} = \frac{(2+v)(u-1) - uv}{(u-1)^2 + v^2}$$

$$\Rightarrow -v(2+v) - u(u-1) = (2+v)(u-1) - uv$$

$$\Rightarrow -2v - v^2 - u^2 + u = 2u - 2 + vu - v - uv$$

$$\Rightarrow -2v - v^2 - u^2 + u = 2u - 2 - v$$

$$\Rightarrow 0 = u^2 + v^2 + u + v - 2$$

$$\Rightarrow (u + \frac{1}{2})^2 - \frac{1}{4} + (v + \frac{1}{2})^2 - \frac{1}{4} - 2 = 0$$

$$\frac{\sqrt{5}}{\sqrt{2}} = \frac{\sqrt{5}}{\sqrt{2}}$$

$$\sqrt{5} \sqrt{2}$$

© Pearson Education Ltd 2009

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise I, Question 14

Question:

The transformation T from the z -plane, where $z = x + iy$ to the w -plane where $w = u + iv$, is given by $w = \frac{4-z}{z+i}$, $z \neq -i$.

The circle $|z| = 1$ is mapped by T onto a line l . Show that l can be written in the form $au + bv + c = 0$, where a , b and c are integers to be determined.

Solution:

$$T: w = \frac{4-z}{z+i} \quad z \neq -i$$

circle with equation $|z| = 1$ in the z -plane.

$$w = \frac{4-z}{z+i}$$

$$\Rightarrow w(z+i) = 4-z$$

$$\Rightarrow wz + iw = 4-z$$

$$\Rightarrow wz + z = 4 - iw$$

$$\Rightarrow z(w+1) = 4 - iw$$

$$\Rightarrow z = \frac{4-iw}{w+1}$$

$$\Rightarrow |z| = \left| \frac{4-iw}{w+1} \right|$$

$$\Rightarrow |z| = \frac{|4-iw|}{|w+1|}$$

$$\text{Applying } |z| = 1 \text{ gives } 1 = \frac{|4-iw|}{|w+1|}$$

$$\Rightarrow |w+1| = |4-iw|$$

$$\Rightarrow |w+1| = |-i(w+4i)|$$

$$\Rightarrow |w+1| = |-i| |w+4i|$$

$$\Rightarrow |w+1| = |w+4i|$$

$$\Rightarrow |u+iv+1| = |u+iv+4i|$$

$$\Rightarrow |(u+1)+iv| = |u+i(v+4)|$$

$$\Rightarrow |(u+1)+iv|^2 = |u+i(v+4)|^2$$

$$\Rightarrow (u+1)^2 + v^2 = u^2 + (v+4)^2$$

$$\Rightarrow u^2 + 2u + 1 + v^2 = u^2 + v^2 + 8v + 16$$

$$\Rightarrow 2u + 1 = 8v + 16$$

$$\Rightarrow 2u - 8v - 15 = 0$$

The circle $|z| = 1$ is mapped by T onto the line $l: 2u - 8v - 15 = 0$ (i.e. $a = 2$, $b = -8$, $c = -15$).

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise I, Question 15

Question:

The transformation T from the z -plane, where $z = x + iy$, to the w -plane where $w = u + iv$, is given by $w = \frac{3iz + 6}{1 - z}$, $z \neq 1$.

Show that the circle $|z| = 2$ is mapped by T onto a circle C . State the centre of C and show that the radius of C can be expressed in the form $k\sqrt{5}$ where k is an integer to be determined.

Solution:

$$T: w = \frac{3iz + 6}{1 - z}; \quad z \neq 1$$

circle with equation $|z| = 2$

$$w = \frac{3iz + 6}{1 - z}$$

$$\Rightarrow w(1 - z) = 3iz + 6$$

$$\Rightarrow w - wz = 3iz + 6$$

$$\Rightarrow w - 6 = 3iz + wz$$

$$\Rightarrow w - 6 = z(3i + w)$$

$$\Rightarrow \frac{w - 6}{w + 3i} = z$$

$$\Rightarrow \left| \frac{w - 6}{w + 3i} \right| = |z|$$

$$\Rightarrow \frac{|w - 6|}{|w + 3i|} = |z|$$

Applying $|z| = 2 \Rightarrow \frac{|w - 6|}{|w + 3i|} = 2$

$$\Rightarrow |w - 6| = 2|w + 3i|$$

$$\Rightarrow |u + iv - 6| = 2|u + iv + 3i|$$

$$\Rightarrow |(u - 6) + iv| = 2|u + i(v + 3)|$$

$$\Rightarrow |(u - 6) + iv|^2 = 2^2|u + i(v + 3)|^2$$

$$\Rightarrow (u - 6)^2 + v^2 = 4[u^2 + (v + 3)^2]$$

$$\Rightarrow u^2 - 12u + 36 + v^2 = 4[u^2 + v^2 + 6v + 9]$$

$$\Rightarrow u^2 - 12u + 36 + v^2 = 4u^2 + 4v^2 + 24v + 36$$

$$\Rightarrow 0 = 3u^2 + 12u + 3v^2 + 24v$$

$$\Rightarrow 0 = u^2 + 4u + v^2 + 8v$$

$$\Rightarrow 0 = (u + 2)^2 - 4 + (v + 4)^2 - 16$$

$$\Rightarrow 20 = (u + 2)^2 + (v + 4)^2$$

$$\Rightarrow (u + 2)^2 + (v + 4)^2 = (2\sqrt{5})^2$$

$$\sqrt{20} = \sqrt{4}\sqrt{5} = 2\sqrt{5}$$

Therefore the circle with equation $|z| = 2$ is mapped onto a circle C , centre $(-2, -4)$, radius $2\sqrt{5}$. So $k = 2$.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise I, Question 16

Question:

A transformation from the z -plane to the w -plane is defined by $w = \frac{az + b}{z + c}$,
where $a, b, c \in \mathbb{R}$.

Given that $w = 1$ when $z = 0$ and that $w = 3 - 2i$ when $z = 2 + 3i$,

- a** find the values of a, b and c ,
- b** find the exact values of the two points in the complex plane which remain invariant under the transformation.

Solution:

$$\mathbf{a} \quad w = \frac{az + b}{z + c} \quad a, b, c \in \mathbb{R}.$$

$$w = 1 \text{ when } z = 0 \quad \textcircled{1}$$

$$w = 3 - 2i \text{ when } z = 2 + 3i \quad \textcircled{2}$$

$$\textcircled{1} \Rightarrow 1 = \frac{a(0) + b}{0 + c} \Rightarrow 1 = \frac{b}{c} \Rightarrow c = b \quad \textcircled{3}$$

$$\textcircled{3} \Rightarrow w = \frac{az + b}{z + b}$$

$$\textcircled{2} \Rightarrow 3 - 2i = \frac{a(2 + 3i) + b}{2 + 3i + b}$$

$$3 - 2i = \frac{(2a + b) + 3ai}{(2 + b) + 3i}$$

$$(3 - 2i)[(2 + b) + 3i] = 2a + b + 3ai$$

$$6 + 3b + 9i - 4i - 2bi + 6 = 2a + b + 3ai$$

$$(12 + 3b) + (5 - 2b)i = (2a + b) + 3ai$$

$$\text{Equate real parts: } 12 + 3b = 2a + b$$

$$\Rightarrow 12 = 2a - 2b \quad \textcircled{4}$$

$$\text{Equate imaginary parts: } 5 - 2b = 3a$$

$$\Rightarrow 5 = 3a + 2b \quad \textcircled{5}$$

$$\textcircled{4} + \textcircled{5}: \quad 17 = 5a$$

$$\Rightarrow \frac{17}{5} = a$$

$$\textcircled{5} \Rightarrow 5 = \frac{51}{5} + 2b$$

$$\Rightarrow \frac{-26}{5} = 2b$$

$$\Rightarrow \frac{-13}{5} = b$$

$$\text{As } b = c \text{ then } c = \frac{-13}{5}$$

$$\text{The values are } a = \frac{17}{5}, b = \frac{-13}{5}, c = \frac{-13}{5}$$

$$\mathbf{b} \quad w = \frac{\frac{17}{5}z - \frac{13}{5}}{z - \frac{13}{5}} \quad (\times 5)$$

$$w = \frac{17z - 13}{5z - 13}$$

$$\text{invariant points } \Rightarrow z = \frac{17z - 13}{5z - 13}$$

$$z(5z - 13) = 17z - 13$$

$$5z^2 - 13z = 17z - 13$$

$$5z^2 - 30z + 13 = 0$$

$$z = \frac{30 \pm \sqrt{900 - 4(5)(13)}}{10}$$

$$z = \frac{30 \pm \sqrt{900 - 260}}{10}$$

© Pearson Education Ltd 2009

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise I, Question 17

Question:

The transformation T from the z -plane, where $z = x + iy$, to the w -plane where $w = u + iv$, is given by

$$w = \frac{z+i}{z}, z \neq 0.$$

- a** The transformation T maps the points on the line with equation $y = x$ in the z -plane other than $(0, 0)$, to points on the l in the w -plane. Find an equation of l .
- b** Show that the image, under T , of the line with equation $x + y + 1 = 0$ in the z -plane is a circle C in the w -plane, where C has equation $u^2 + v^2 - u + v = 0$.
- c** On the same Argand diagram, sketch l and C .

Solution:

$$T: w = \frac{z+i}{z}, \quad z \neq 0.$$

a the line $y = x$ in the z -plane other than $(0, 0)$

$$w = \frac{z+i}{z}$$

$$\Rightarrow wz = z + i$$

$$\Rightarrow wz - z = i$$

$$\Rightarrow z(w - 1) = i$$

$$\Rightarrow z = \frac{i}{w-1}$$

$$\Rightarrow z = \frac{i}{(u+iv)-1} = \frac{i}{(u-1)+iv}$$

$$\Rightarrow z = \left[\frac{i}{(u-1)+iv} \right] \left[\frac{(u-1)-iv}{(u-1)-iv} \right]$$

$$\Rightarrow z = \frac{i(u-1)+v}{(u-1)^2+v^2}$$

$$\Rightarrow z = \frac{v}{(u-1)^2+v^2} + i \frac{(u-1)}{(u-1)^2+v^2}$$

$$\text{So } x + iy = \frac{v}{(u-1)^2+v^2} + i \frac{(u-1)}{(u-1)^2+v^2}$$

$$\Rightarrow x = \frac{v}{(u-1)^2+v^2} \text{ and } y = \frac{u-1}{(u-1)^2+v^2}$$

$$\text{Applying } y = x, \text{ gives } \frac{u-1}{(u-1)^2+v^2} = \frac{v}{(u-1)^2+v^2}$$

$$\Rightarrow u-1 = v$$

$$\Rightarrow v = u-1$$

Therefore the line l has equation $v = u - 1$.

b the line with equation $x + y + 1 = 0$ in the z -plane

$$x + y + 1 = 0 \Rightarrow \frac{v}{(u-1)^2+v^2} + \frac{u-1}{(u-1)^2+v^2} + 1 = 0 \quad [\times(u-1)^2+v^2]$$

$$\Rightarrow v + (u-1) + (u-1)^2 + v^2 = 0$$

$$\Rightarrow v + u - 1 + u^2 - 2u + 1 + v^2 = 0$$

$$\Rightarrow u^2 + v^2 - u + v = 0$$

$$\Rightarrow \left(u - \frac{1}{2}\right)^2 - \frac{1}{4} + \left(v + \frac{1}{2}\right)^2 - \frac{1}{4} = 0$$

$$\Rightarrow \left(u - \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{1}{2}$$

$$\Rightarrow \left(u - \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \left(\frac{\sqrt{2}}{2}\right)^2$$

The image of $x + y + 1 = 0$ under T is a circle C , centre $\left(\frac{1}{2}, -\frac{1}{2}\right)$, radius $\frac{\sqrt{2}}{2}$ with equation $u^2 + v^2 - u + v = 0$, as required.

c



© Pearson Education Ltd 2009