Solutionbank FP1

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 1

Question:

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & -1 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix}, \ \mathbf{C} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$$

Determine whether or not the following products exist. Where the product exists, evaluate the product. Where the product does not exist, give a reason for this.

a AB

b BA

c BAC

d CBA.

Solution:

a **AB** does not exist.
The matrix **A** is a 2×3 matrix.
The matrix **B** is a 2×2 matrix.
The number of columns in **A**, 3, is not equal to the number of rows in **B**, 2.

An $n\times m$ matrix can be multiplied by a $m\times p$ matrix. The number of columns in the left hand matrix must equal the number of rows in the right hand matrix.

b $$\mathbf{BA}=\begin{pmatrix}2 & 0\\ 3 & -1\end{pmatrix}\begin{pmatrix}3 & 2 & 1\\ 0 & 2 & -1\end{pmatrix}$$
$$=\begin{pmatrix}2\times3+0\times0 & 2\times2+0\times2 & 2\times1+0\times(-1)\\ 3\times3+(-1)\times0 & 2\times3+(-1)\times2 & 3\times1+(-1)\times(-1)\end{pmatrix}$$
$$=\begin{pmatrix}6+0 & 4+0 & 2-0\\ 9+0 & 6-2 & 3+1\end{pmatrix}=\begin{pmatrix}6 & 4 & 2\\ 9 & 4 & 4\end{pmatrix}$$

c $$\mathbf{BAC}=(\mathbf{BA})\mathbf{C}=\begin{pmatrix}6 & 4 & 2\\ 9 & 4 & 4\end{pmatrix}\begin{pmatrix}4\\ -3\\ 1\end{pmatrix}$$
$$=\begin{pmatrix}6\times4+4\times(-3)+2\times1\\ 9\times4+4\times(-3)+4\times1\end{pmatrix}$$
$$=\begin{pmatrix}24-12+2\\ 36-12+4\end{pmatrix}=\begin{pmatrix}14\\ 28\end{pmatrix}$$

As matrix multiplication is associative, you could work out $\mathbf{B}(\mathbf{AC})$ or $(\mathbf{BA})\mathbf{C}$ - they will give the same result. It is sensible to work out $(\mathbf{BA})\mathbf{C}$ as you have already worked out **BA** in part (b).

d **CBA** does not exist.
$\mathbf{CBA}=\mathbf{C}(\mathbf{BA})$
The matrix **C** is a 3×1 matrix.
The matrix **BA** is a 2×3 matrix.
The number of columns in **C**, 1, is not equal to the number of rows in **BA**, 2.

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Review Exercise

Exercise A, Question 2

Question:

$\mathbf{M} = \begin{pmatrix} 0 & 3 \\ -1 & 2 \end{pmatrix}$, $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Find the values of the constants a and b such that $\mathbf{M}^2 + a\mathbf{M} + b\mathbf{I} = \mathbf{O}$.

Solution:

$$\mathbf{M}^2 = \begin{pmatrix} 0 & 3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 3 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0\times 0+3\times(-1) & 0\times 3+3\times 2 \\ (-1)\times 0+2\times(-1) & (-1)\times 3+2\times 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0-3 & 0+6 \\ 0-2 & -3+4 \end{pmatrix} = \begin{pmatrix} -3 & 6 \\ -2 & 1 \end{pmatrix}$$

$\mathbf{M}^2$ is quite complicated to work out and it is sensible to calculate this before working out $\mathbf{M}^2 + a\mathbf{M} + b\mathbf{I}$.

$$\mathbf{M}^2 + a\mathbf{M} + b\mathbf{I} = \mathbf{O}$$

$$\begin{pmatrix} -3 & 6 \\ -2 & 1 \end{pmatrix} + a\begin{pmatrix} 0 & 3 \\ -1 & 2 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 6 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 3a \\ -a & 2a \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -3+b & 6+3a \\ -2-a & 1+2a+b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Equating the top left elements

$-3+b=0 \Rightarrow b=3$

Equating the top right elements

$6+3a=0 \Rightarrow a=-2$

$a=-2, b=3$

There are four elements which could be equated but you only need to equate two of them to find a and b. You could use the others to check your working. For example; if $a=-2, b=3$ then $1+2a+b=1-4+3$ which does equal 0.

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Review Exercise
Exercise A, Question 3

Question:

$\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 3 & 6 \end{pmatrix}$

Show that $\mathbf{A}^2 - 10\mathbf{A} + 21\mathbf{I} = \mathbf{O}$.

Solution:

$$\mathbf{A}^2 = \begin{pmatrix} 4 & 1 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 4\times4+1\times3 & 4\times1+1\times6 \\ 3\times4+6\times3 & 3\times1+6\times6 \end{pmatrix}$$

$$= \begin{pmatrix} 16+3 & 4+6 \\ 12+18 & 3+36 \end{pmatrix} = \begin{pmatrix} 19 & 10 \\ 30 & 39 \end{pmatrix}$$

$$\mathbf{A}^2 - 10\mathbf{A} + 21\mathbf{I} = \begin{pmatrix} 19 & 10 \\ 30 & 39 \end{pmatrix} - 10\begin{pmatrix} 4 & 1 \\ 3 & 6 \end{pmatrix} + 21\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 19 & 10 \\ 30 & 39 \end{pmatrix} - \begin{pmatrix} 40 & 10 \\ 30 & 60 \end{pmatrix} + \begin{pmatrix} 21 & 0 \\ 0 & 21 \end{pmatrix}$$

$$= \begin{pmatrix} 19-40+21 & 10-10+0 \\ 30-30+0 & 39-60+21 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$= \mathbf{O}$, as required.

To show the matrix relation given in the question, you must work on the left hand side of the equation $\mathbf{A}^2 - 10\mathbf{A} + 21\mathbf{I}$ and manipulate it until you get the right hand side of the equation $\mathbf{O}$.

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Review Exercise
Exercise A, Question 4

Question:

$$\mathbf{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Find an expression for λ, in terms of a, b, c and d, so that $\mathbf{A}^2-(a+d)\mathbf{A}=\lambda\mathbf{I}$, where $\mathbf{I}$ is the 2×2 unit matrix.

Solution:

$$\mathbf{A}^2=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}$$

$$\mathbf{A}^2-(a+d)\mathbf{A}$$

$$=\begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}-\begin{pmatrix} (a+d)a & (a+d)b \\ (a+d)c & (a+d)d \end{pmatrix}$$

$$=\begin{pmatrix} a^2+bc-a^2-ad & ab+bd-ab-bd \\ ac+cd-ac-ad & bc+d^2-ad-d^2 \end{pmatrix}$$

$$=\begin{pmatrix} bc-ad & 0 \\ 0 & bc-ad \end{pmatrix}=\lambda\mathbf{I}=\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$\mathbf{I}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so

$\lambda\mathbf{I}=\lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$.

You can write down the results of simple calculations like this without showing all of the working.

Equating the top left (or bottom right elements)

$$\lambda=bc-ad$$

Note that $\lambda=-\det(\mathbf{A})$.

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Review Exercise
Exercise A, Question 5

Question:

$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ p & -1 \end{pmatrix}$, where p is a real constant. Given that $\mathbf{A}$ is singular,

a find the value of p.

Given instead that det ($\mathbf{A}$) = 4,

b find the value of p.

Using the value of p found in **b**,

c show that $\mathbf{A}^2 - \mathbf{A} = k\mathbf{I}$, stating the value of the constant k.

Solution:

a $\det(\mathbf{A}) = 2\times(-1) - 3\times p = -2 - 3p$

If $\mathbf{A}$ is singular, $\det(\mathbf{A}) = 0$.

$-2 - 3p = 0 \Rightarrow 3p = -2 \Rightarrow p = -\frac{2}{3}$

You need to memorise that, if $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det(\mathbf{A}) = ad - bc$.

b As in part (a), $\det(\mathbf{A}) = -2 - 3p$

$-2 - 3p = 4 \Rightarrow -3p = 6 \Rightarrow p = -2$

c $\mathbf{A}^2 = \begin{pmatrix} 2 & 3 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -2 & -1 \end{pmatrix}$

$= \begin{pmatrix} 4-6 & 6-3 \\ -4+2 & -6+1 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ -2 & -5 \end{pmatrix}$

$\mathbf{A}^2 - \mathbf{A} = \begin{pmatrix} -2 & 3 \\ -2 & -2 \end{pmatrix} - \begin{pmatrix} 2 & 3 \\ -2 & -1 \end{pmatrix}$

$= \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} = -4\mathbf{I}$

This is the required result with $k = -4$.

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Review Exercise
Exercise A, Question 6

Question:

$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix}$

a Find $\mathbf{A}^{-1}$.

Given that $\mathbf{A}^5 = \begin{pmatrix} 251 & -109 \\ -327 & 142 \end{pmatrix}$,

b find $\mathbf{A}^4$.

Solution:

a $\det(\mathbf{A}) = 2\times1-(-1)\times(-3)2-3=-1$

If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

You should know this formula. It is to your advantage to quote the formula in your solution. If you make a mistake, the examiner will know that you are trying to do the right thing.

$\mathbf{A}^{-1} = \frac{1}{-1}\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -3 & -2 \end{pmatrix}$

b $\mathbf{A}^4\mathbf{A} = \mathbf{A}^5$

$\mathbf{A}^4\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^5\mathbf{A}^{-1}$

It is much quicker to multiply $\mathbf{A}^5$ by $\mathbf{A}^{-1}$ than to repeatedly multiply $\mathbf{A}$ by itself. For whole numbers, the ordinary algebraic rules for indices apply to matrices and it will help you if you remember this.

$\mathbf{A}^4 = \mathbf{A}^5\mathbf{A}^{-1} = \begin{pmatrix} 251 & -109 \\ -327 & 142 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ -3 & -2 \end{pmatrix}$

$= \begin{pmatrix} -251+327 & -251+218 \\ 327-426 & 327-284 \end{pmatrix} = \begin{pmatrix} 76 & -33 \\ -99 & 43 \end{pmatrix}$

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Review Exercise
Exercise A, Question 7

Question:

A triangle T, of area 18cm^2, is transformed into a triangle T' by the matrix **A** where, $\mathbf{A} = \begin{pmatrix} k & k-1 \\ -3 & 2k \end{pmatrix}$, $k \in \mathbb{R}$.

a Find det (**A**), in terms of k.

Given that the area of T' is $198\ \text{cm}^2$,

b find the possible values of k.

Solution:

a If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det(\mathbf{A}) = ad - bc$.

$\det(\mathbf{A}) = k \times 2k - (k-1)\times(-3)$

$= 2k^2 + 3k - 3$

b The triangle has been enlarged by a factor of

$\frac{198}{11} = 11$

So $\det(\mathbf{A}) = 11$

$2k^2 + 3k - 3 = 11$

$2k^2 + 3k - 14 = (2k+7)(k-2) = 0$

$k = -\frac{7}{2}, 2$

The determinant is the area scale factor in transformations. This is equivalent to $\frac{\text{area of image}}{\text{area of object}} = \det(\mathbf{A})$. So the scale factor in part (a) must equal the determinant in part (b).

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Review Exercise
Exercise A, Question 8

Question:

A linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$ is defined by $\mathbf{p} = \mathbf{Nq}$, where $\mathbf{N}$ is a 2×2 matrix and $\mathbf{p}$, $\mathbf{q}$ are 2×1 column vectors.

Given that $\mathbf{p} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$ when $\mathbf{q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and that $\mathbf{p} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$ when $\mathbf{q} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, find $\mathbf{N}$.

Solution:

Let $\mathbf{N} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

You need to introduce some algebraic variables to help you to obtain equations. You can use any symbols you like for the elements of the matrix, except for those already used in the question.

$$\mathbf{p} = \mathbf{Nq}$$

$$\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

Equating elements

$$a = 3, c = 7$$

You use the values for a and c which you found earlier.

$$\begin{pmatrix} 6 \\ -1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 2a - 3b \\ 2c - 3d \end{pmatrix}$$

Equating the upper elements

$$2a - 3b = 6$$

$$a = 3 \Rightarrow 6 - 3b = 6 \Rightarrow b = 0$$

Equating the lower elements

$$2c - 3d = -1$$

$$c = 7 \Rightarrow 14 - 3d = -1 \Rightarrow 3d = 15 \Rightarrow d = 5$$

$$\mathbf{N} = \begin{pmatrix} 3 & 0 \\ 7 & 5 \end{pmatrix}$$

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Review Exercise
Exercise A, Question 9

Question:

$\mathbf{A} = \begin{pmatrix} 4 & -1 \\ -6 & 2 \end{pmatrix}$, $\mathbf{B}^{-1} = \begin{pmatrix} 2 & 0 \\ 3 & p \end{pmatrix}$

a Find $\mathbf{A}^{-1}$.

b Find $(\mathbf{AB})^{-1}$, in terms of p.

Given also that $\mathbf{AB} = \begin{pmatrix} -1 & 2 \\ 3 & -4 \end{pmatrix}$,

c find the value of p.

Solution:

a If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

$\det(\mathbf{A}) = 4 \times 2 - (-1) \times (-6) = 8 - 6 = 2$

$\mathbf{A}^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ 3 & 2 \end{pmatrix}$

b $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

You need to remember this property of the inverse of matrices. The order of **A** and **B** is reversed in this formula.

$= \begin{pmatrix} 2 & 0 \\ 3 & p \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} \\ 3 & 2 \end{pmatrix}$

$= \begin{pmatrix} 2 & 1 \\ 3p+3 & 2p+\frac{3}{2} \end{pmatrix}$

c $(\mathbf{AB})(\mathbf{AB})^{-1} = \mathbf{I}$

The product of any matrix and its inverse is **I**. This applies to a product matrix, **AB** in this case, as well as to a matrix such as **A**.

$\begin{pmatrix} -1 & 2 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3p+3 & 2p+\frac{3}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Equating the upper left elements

Finding all four of the elements of the product matrix of the left hand side of this equation would be lengthy. To find p, you only need one equation, so you only need to consider one element. Here the upper left hand element has been used but you could choose any of the four elements.

$-1 \times 2 + 2(3p+3) = 1$

$-2 + 6p + 6 = 1$

$6p = -3$

$p = -\frac{1}{2}$

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Review Exercise
Exercise A, Question 10

Question:

$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 7 & -3 \end{pmatrix}$

a Show that $\mathbf{A}^3 = \mathbf{I}$.

b Deduce that $\mathbf{A}^2 = \mathbf{A}^{-1}$.

c Use matrices to solve the simultaneous equations

$$\begin{aligned} 2x - y &= 3, \\ 7x - 3y &= 2. \end{aligned}$$

Solution:

a

$$\mathbf{A}^2 = \begin{pmatrix} 2 & -1 \\ 7 & -3 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 7 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 4-7 & -2+3 \\ 14-21 & -7+9 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -7 & 2 \end{pmatrix}$$

$$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A}$$

$$= \begin{pmatrix} -3 & 1 \\ -7 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 7 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} -6+7 & 3-3 \\ -14+14 & 7-6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}\text{, as required.}$$

> $\mathbf{A}^3 = \mathbf{AAA} = \mathbf{A}^2\mathbf{A}$. Work out $\mathbf{A}^2$ first and then multiply the result by $\mathbf{A}$.

b

$\mathbf{A}^3 = \mathbf{I}$

Multiply both sides by $\mathbf{A}^{-1}$

$\mathbf{A}^3\mathbf{A}^{-1} = \mathbf{IA}^{-1}$

$\mathbf{A}^2 = \mathbf{A}^{-1}$, as required.

> It helps if you remember that, for whole numbers, the ordinary algebraic rules for indices apply to matrices. In more detail;
> $\mathbf{A}^3\mathbf{A}^{-1} = \left(\mathbf{A}^2\mathbf{A}\right)\mathbf{A}^{-1} = \mathbf{A}^2\left(\mathbf{AA}^{-1}\right) = \mathbf{A}^2\mathbf{I} = \mathbf{A}^2$

c

Writing the simultaneous equations as matrices

$$\begin{pmatrix} 2 & -1 \\ 7 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Multiply both sides of this equation on the left by $\mathbf{A}^{-1}$, which, in this case, is $\mathbf{A}^2$.

$$\mathbf{A}^2\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}^2\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\mathbf{A}^3\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{I}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -7 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -9+2 \\ -21+4 \end{pmatrix} = \begin{pmatrix} -7 \\ -17 \end{pmatrix}$$

Equating elements

$$x = -7,\ y = -17$$

> To solve simultaneous equations using matrices, you need to multiply both sides of a matrix equation by the appropriate inverse matrix. In this question, part (b) has shown that the inverse matrix is $\mathbf{A}^2$ and, as you worked this out in part (a), there is no need to work the inverse matrix out again.

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Review Exercise
Exercise A, Question 11

Question:

$\mathbf{A} = \begin{pmatrix} 5 & -2 \\ 5 & 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 & 2 \\ 5 & 1 \end{pmatrix}$

a Find $\mathbf{A}^{-1}$.

b Show that $\mathbf{A}^{-1}\mathbf{BA} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, stating the values of the constants λ_1 and λ_2.

Solution:

a $\det(\mathbf{A}) = 5 \times 5 - 5 \times (-2) = 25 + 10 = 35$

If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

$$\mathbf{A}^{-1} = \frac{1}{35}\begin{pmatrix} 5 & 2 \\ -5 & 5 \end{pmatrix}$$

This could be written as $\begin{pmatrix} \frac{1}{7} & \frac{2}{35} \\ -\frac{1}{7} & \frac{1}{7} \end{pmatrix}$.

Either form is acceptable.

b $\mathbf{A}^{-1}\mathbf{BA} = \mathbf{A}^{-1}(\mathbf{BA})$

$$= \mathbf{A}^{-1}\begin{pmatrix} 4 & 2 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ 5 & 5 \end{pmatrix}$$

$$= \mathbf{A}^{-1}\begin{pmatrix} 20+10 & -8+10 \\ 25+5 & -10+5 \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 30 & 2 \\ 30 & -5 \end{pmatrix}$$

$$= \frac{1}{35}\begin{pmatrix} 5 & 2 \\ -5 & 5 \end{pmatrix}\begin{pmatrix} 30 & 2 \\ 30 & -5 \end{pmatrix}$$

$$= \frac{1}{35}\begin{pmatrix} 150+60 & 10-10 \\ -150+150 & -10-25 \end{pmatrix}$$

$$= \frac{1}{35}\begin{pmatrix} 210 & 0 \\ 0 & -35 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & -1 \end{pmatrix}$$

This is the required form with $\lambda_1 = 6$ and $\lambda_2 = -1$.

As matrix multiplication is associative, you could work out this triple product as $(\mathbf{A}^{-1}\mathbf{B})\mathbf{A}$ but $\mathbf{A}^{-1}$ has an awkward fraction, so it is sensible to evaluate $\mathbf{BA}$ first.

If you go on to study the FP3 module, you will learn how to carry out calculations like this with larger matrices. These calculations have important applications to physics and statistics.

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Review Exercise
Exercise A, Question 12

Question:

$\mathbf{A} = \begin{pmatrix} 4p & -q \\ -3p & q \end{pmatrix}$, where p and q are non-zero constants.

a Find $\mathbf{A}^{-1}$, in terms of p and q.

Given that $\mathbf{AX} = \begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}$,

b find $\mathbf{X}$, in terms of p and q.

Solution:

a

$$\det(\mathbf{A}) = 4p \times q - (-q) \times (-3p)$$
$$= 4pq - 3pq = pq$$
$$\mathbf{A}^{-1} = \frac{1}{pq}\begin{pmatrix} q & q \\ 3p & 4p \end{pmatrix}$$

The alternative answer, multiplying the matrix by the scalar $\frac{1}{pq}$, $\begin{pmatrix} \frac{1}{p} & \frac{1}{p} \\ \frac{3}{q} & \frac{4}{q} \end{pmatrix}$ would be an equally good one.

b

$$\mathbf{AX} = \begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}$$

Multiply both sides on the left by $\mathbf{A}^{-1}$

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}$$
$$\mathbf{X} = \frac{1}{pq}\begin{pmatrix} q & q \\ 3p & 4p \end{pmatrix}\begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}$$
$$= \frac{1}{pq}\begin{pmatrix} 2pq - pq & 3q^2 + q^2 \\ 6p^2 - 4p^2 & 9pq + 4pq \end{pmatrix}$$
$$= \frac{1}{pq}\begin{pmatrix} pq & 4q^2 \\ 2p^2 & 13pq \end{pmatrix}$$

It is important to multiply by $\mathbf{A}^{-1}$ on the correct side of the expression. As shown here, multiplying on the left of $\mathbf{AX}$, you get $\mathbf{A}^{-1}\mathbf{AX} = (\mathbf{A}^{-1}\mathbf{A})\mathbf{X} = \mathbf{IX} = \mathbf{X}$, which is what you are asked to find. On the right of $\mathbf{AX}$, you would get $\mathbf{AXA}^{-1}$, which does not simplify, and no further progress can be made. Working out $\begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}\frac{1}{pq}\begin{pmatrix} q & q \\ 3p & 4p \end{pmatrix}$ instead of $\frac{1}{pq}\begin{pmatrix} q & q \\ 3p & 4p \end{pmatrix}\begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}$ is a common error.

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Review Exercise
Exercise A, Question 13

Question:

$\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & -1 \\ -4 & 5 \end{pmatrix}$

Find

a **AB**,

b **AB** – **BA**.

Given that **C** = **AB** – **BA**,

c find $\mathbf{C}^2$,

d give a geometrical interpretation of the transformation represented by $\mathbf{C}^2$.

Solution:

a $\mathbf{AB} = \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -4 & 5 \end{pmatrix}$

$= \begin{pmatrix} 12-8 & -4+10 \\ 15-12 & -5+15 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 3 & 10 \end{pmatrix}$

Matrix multiplication is not commutative and, as in this question, **AB** and **BA** can be quite different.

b $\mathbf{BA} = \begin{pmatrix} 3 & -1 \\ -4 & 5 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}$

$= \begin{pmatrix} 12-5 & 6-3 \\ -16+25 & -8+15 \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ 9 & 7 \end{pmatrix}$

$\mathbf{AB} - \mathbf{BA} = \begin{pmatrix} 4 & 6 \\ 3 & 10 \end{pmatrix} - \begin{pmatrix} 7 & 3 \\ 9 & 7 \end{pmatrix} = \begin{pmatrix} -3 & 3 \\ -6 & 3 \end{pmatrix}$

c $\mathbf{C}^2 = \begin{pmatrix} -3 & 3 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} -3 & 3 \\ -6 & 3 \end{pmatrix} = \begin{pmatrix} 9-18 & -9+9 \\ 18-18 & -18+9 \end{pmatrix}$

$= \begin{pmatrix} -9 & 0 \\ 0 & -9 \end{pmatrix}$

For all $k \neq 0$, the matrix $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ represents an enlargement, centre $(0, 0)$, scale factor k.

d Enlargement, centre $(0, 0)$, scale factor -9

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Review Exercise
Exercise A, Question 14

Question:

The matrix **A** represents reflection in the *x*-axis.

The matrix **B** represents a rotation of $135°$, in the anti-clockwise direction, about (0, 0).

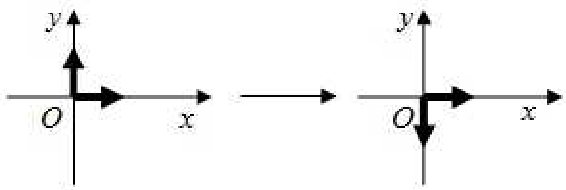
Given that **C** = **AB**,

a find the matrix **C**,

b show that $\mathbf{C}^2 = \mathbf{I}$.

Solution:

a Reflection in the x axis transforms



$$\begin{pmatrix}1\\0\end{pmatrix} \rightarrow \begin{pmatrix}1\\0\end{pmatrix}$$

$(1, 0)$ lies on the x-axis and so is not changed by reflection in the x-axis.

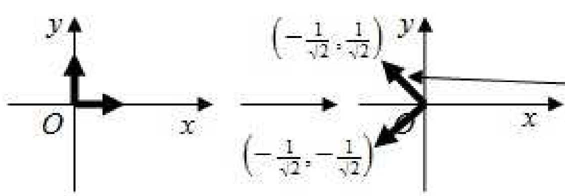
$$\begin{pmatrix}0\\1\end{pmatrix} \rightarrow \begin{pmatrix}0\\-1\end{pmatrix}$$

So

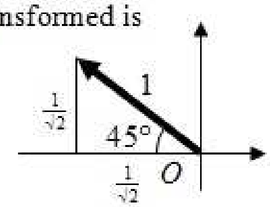
$$\mathbf{A} = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

Unless the question states otherwise, it is acceptable to write down a simple matrix like this without working.

Rotation of $+135°$ about $(0, 0)$ transforms



The geometry of the vector to which $\begin{pmatrix}1\\0\end{pmatrix}$ is transformed is



$$\begin{pmatrix}1\\0\end{pmatrix} \rightarrow \begin{pmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix} \rightarrow \begin{pmatrix}-\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}$$

Arrows have been added to this calculation so that you can see where the columns in **B** come from.

So

$$\mathbf{B} = \begin{pmatrix}-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}$$

$$\mathbf{C} = \mathbf{AB} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

b $$\mathbf{C}^2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

As an example of the calculations;
$$-\frac{1}{\sqrt{2}} \times -\frac{1}{\sqrt{2}} = +\frac{1 \times 1}{\sqrt{2} \times \sqrt{2}} = \frac{1}{2}$$

$$= \begin{pmatrix} \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) & \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) \\ \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) & \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} + \frac{1}{2} & \frac{1}{2} - \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2} & \frac{1}{2} + \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}\text{, as required.}$$

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Review Exercise
Exercise A, Question 15

Question:

The linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is represented by the matrix **M**, where $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The transformation T maps the point with coordinates (1, 0) to the point with coordinates (3, 2) and the point with coordinates (2, 1) to the point with coordinates (6, 3).

a Find the values of a, b, c and d.

b Show that $\mathbf{M}^2 = \mathbf{I}$.

The transformation T maps the point with coordinates (p, q) to the point with coordinates $(8, -3)$.

c Find the value of p and the value of q.

Solution:

a $\mathbf{M}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}$

$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}a\\c\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}$

Equating the elements

$a=3, c=2$

$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}2a+b\\2c+d\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$

Equating the upper elements

$2a+b=2$

$a=3 \Rightarrow 6+b=2 \Rightarrow b=-4$

Equating the lower elements

$2c+d=1$

$c=2 \Rightarrow 4+d=1 \Rightarrow d=-3$

$a=3, b=-4, c=2, d=-3$

In questions about transformations, you need to write the coordinates of points as column vectors. For example, the coordinate $(1, 0)$ is written as the column vector $\begin{pmatrix}1\\0\end{pmatrix}$.

b $\mathbf{M}=\begin{pmatrix}3 & -4\\2 & -3\end{pmatrix}$

$\mathbf{M}^2=\begin{pmatrix}3 & -4\\2 & -3\end{pmatrix}\begin{pmatrix}3 & -4\\2 & -3\end{pmatrix}=\begin{pmatrix}9-8 & -12+12\\6-6 & -8+9\end{pmatrix}$

$=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}=\mathbf{I}$, as required

c As $\mathbf{M}^2=\mathbf{I}$, $\mathbf{M}^{-1}\mathbf{M}^2=\mathbf{M}^{-1}\mathbf{I}$

$\mathbf{M}=\mathbf{M}^{-1}$

The matrix **M** is its own inverse. This follows from the result in part (b). In more detail; $\mathbf{M}^2=\mathbf{I}$

$\mathbf{MM}=\mathbf{I}$

$\mathbf{M}^{-1}(\mathbf{MM})=\mathbf{M}^{-1}\mathbf{I}$

$(\mathbf{M}^{-1}\mathbf{M})\mathbf{M}=\mathbf{M}^{-1}$

$\mathbf{IM}=\mathbf{M}^{-1}$

$\mathbf{M}=\mathbf{M}^{-1}$

$\mathbf{M}\begin{pmatrix}p\\q\end{pmatrix}=\begin{pmatrix}8\\-3\end{pmatrix}$

$\mathbf{M}^{-1}\mathbf{M}\begin{pmatrix}p\\q\end{pmatrix}=\mathbf{M}^{-1}\begin{pmatrix}8\\-3\end{pmatrix}$

$\mathbf{I}\begin{pmatrix}p\\q\end{pmatrix}=\mathbf{M}\begin{pmatrix}8\\-3\end{pmatrix}$

In this question, as **M** is its own inverse, you can replace $\mathbf{M}^{-1}$ by **M.**

$\begin{pmatrix}p\\q\end{pmatrix}=\begin{pmatrix}3 & -4\\2 & -3\end{pmatrix}\begin{pmatrix}8\\-3\end{pmatrix}=\begin{pmatrix}24+12\\16+9\end{pmatrix}=\begin{pmatrix}36\\25\end{pmatrix}$

Hence $p=36, q=25$

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Review Exercise
Exercise A, Question 16

Question:

16 The linear transformation T is defined by $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2y - x \\ 3y \end{pmatrix}$.

The linear transformation T is represented by the matrix **C**.

a Find **C**.

The quadrilateral $OABC$ is mapped by T to the quadrilateral $OA'B'C'$, where the coordinates of A', B' and C' are (0, 3), (10, 15) and (10, 12) respectively.

b Find the coordinates of A, B and C.

c Sketch the quadrilateral $OABC$ and verify that $OABC$ is a rectangle.

Solution:

a $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2y-x \\ 3y \end{pmatrix} = \begin{pmatrix} -1x+2y \\ 0x+3y \end{pmatrix}$

$$= \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

So $\mathbf{C} = \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}$

b $\det(\mathbf{C}) = -1\times 3 - 3\times 0 = -3$

$$\mathbf{C}^{-1} = \frac{1}{-3}\begin{pmatrix} 3 & -2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & \frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$$

You are given the results of transforming the points by T and are asked to find the original points. You are "working backwards" to the original points and you will need the inverse matrix.

Let the coordinates of A, B and C be (x_A, y_A), (x_B, y_B) and (x_C, y_C) respectively.

$$\mathbf{C}\begin{pmatrix} x_A & x_B & x_C \\ y_A & y_B & y_C \end{pmatrix} = \begin{pmatrix} 0 & 10 & 10 \\ 3 & 15 & 12 \end{pmatrix}$$

$$\mathbf{C}^{-1}\mathbf{C}\begin{pmatrix} x_A & x_B & x_C \\ y_A & y_B & y_C \end{pmatrix} = \mathbf{C}^{-1}\begin{pmatrix} 0 & 10 & 10 \\ 3 & 15 & 12 \end{pmatrix}$$

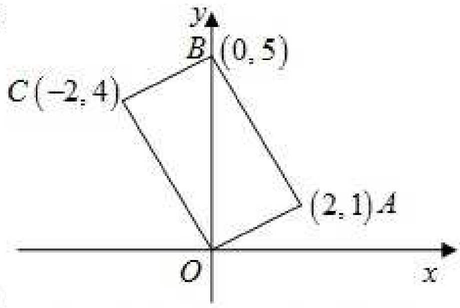
$$\begin{pmatrix} x_A & x_B & x_C \\ y_A & y_B & y_C \end{pmatrix} = \begin{pmatrix} -1 & \frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 0 & 10 & 10 \\ 3 & 15 & 12 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -10+10 & -10+8 \\ 1 & 5 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & -2 \\ 1 & 5 & 4 \end{pmatrix}$$

Hence $A: (2, 1)$, $B: (0, 5)$, $C: (-2, 4)$

c



d

Considering the gradients of the sides

$m_{OA} = \frac{1}{2}$; $m_{CB} = \frac{5-4}{0-(-2)} = \frac{1}{2}$

So OA is parallel to CB.

$m_{OC} = \frac{4-0}{-2-0} = -2$; $m_{AB} = \frac{5-1}{0-2} = \frac{4}{-2} = -2$

Using the properties of quadrilaterals you learnt for GCSE, there are many alternative ways of showing that $OABC$ is a rectangle. This is just one of many possibilities, using the result you learnt in the C1 module that the gradient of the line joining (x_1, y_1) to (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$.

So OC is parallel to AB.
The opposite sides of $OABC$ are parallel to each other and so $OABC$ is a parallelogram.

Also $m_{OA} \times m_{OC} = \frac{1}{2} \times -2 = -1$.

So OA is perpendicular to OC.
So the parallelogram $OABC$ contains a right angle and, hence, $OABC$ is a right angle.

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Review Exercise
Exercise A, Question 17

Question:

$\mathbf{A} = \begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0.8 & -0.4 \\ 0.2 & -0.6 \end{pmatrix}$ and $\mathbf{C} = \mathbf{AB}$.

a Find **C**.

b Give a geometrical interpretation of the transformation represented by **C**.

The square *OXYZ*, where the coordinates of *X* and *Y* are (0, 3) and (3, 3), is transformed into the quadrilateral $OX'Y'Z'$, by the transformation represented by **C**.

c Find the coordinates of Z'.

Solution:

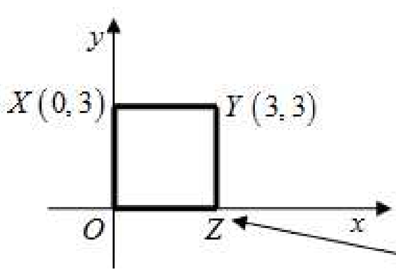
a
$$\mathbf{C} = \mathbf{BA} = \begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 0.8 & -0.4 \\ 0.2 & -0.6 \end{pmatrix}$$
$$= \begin{pmatrix} 2.4-0.4 & -1.2+1.2 \\ -0.8+0.8 & 0.4-2.4 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

b
$$\mathbf{C} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

So the transformation can be interpreted as reflection in the x-axis followed by an enlargement, centre $(0, 0)$, scale factor 2.

These transformations have the same effect with their order reversed so "enlargement, centre $(0, 0)$, scale factor 2 followed by reflection in the x-axis" is an equally good answer.

c



You have been asked to find Z' - that is the point to which Z is transformed. You have not been given the coordinates of Z. Drawing a quick sketch makes it clear that Z has coordinates $(3, 0)$.

The coordinates of Z are $(3, 0)$.

To find the coordinates of Z'

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

The coordinates of Z' are $(6, 0)$.

Alternatively you can argue, using the answer to part (b), that reflecting $(3, 0)$ in the x-axis leaves the point unchanged as it lies on the x-axis. An enlargement of scale factor 2 then leads to $(6, 0)$.

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Review Exercise
Exercise A, Question 18

Question:

Given that $\mathbf{A} = \begin{pmatrix} 5 & 3 \\ -2 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, find the matrices **C** and **D** such that

a $\mathbf{AC} = \mathbf{B}$,

b $\mathbf{DA} = \mathbf{B}$.

A linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$ is defined by the matrix **B**.

c Prove that the line with equation $y = mx$ is mapped onto another line through the origin O under this transformation.

d Find the gradient of this second line in terms of m.

Solution:

a
$$\mathbf{AC} = \mathbf{B}$$
$$\mathbf{A}^{-1}\mathbf{AC} = \mathbf{A}^{-1}\mathbf{B}$$
So $\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}$

$$\det(\mathbf{A}) = 5\times(-1) - 3\times(-2) = -5 + 6 = 1$$
$$\mathbf{A}^{-1} = \frac{1}{1}\begin{pmatrix} -1 & -3 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} -1 & -3 \\ 2 & 5 \end{pmatrix}$$
$$\mathbf{C} = \mathbf{A}^{-1}\mathbf{B} = \begin{pmatrix} -1 & -3 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -1-6 \\ 2 & 2+10 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & -7 \\ 2 & 12 \end{pmatrix}$$

$\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{D} = \mathbf{BA}^{-1}$ but, as matrix multiplication is not commutative, $\mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{BA}^{-1}$ are different. You must be careful of the order in which you multiply matrices.

b
$$\mathbf{DA} = \mathbf{B}$$
$$\mathbf{DAA}^{-1} = \mathbf{BA}^{-1}$$
So $\mathbf{D} = \mathbf{BA}^{-1}$

$$\mathbf{D} = \mathbf{BA}^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & -3 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} -1+2 & -3+5 \\ 4 & 10 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 \\ 4 & 10 \end{pmatrix}$$

c Let the general point on $y = mx$ have coordinates (t, mt)

If $x = t$, as $y = mx$, then $y = mt$. The variable t is being used as a parameter. You used parameters in Chapter 3 to solve questions involving parabolas and rectangular hyperbolas.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} t \\ mt \end{pmatrix} = \begin{pmatrix} t+mt \\ 2mt \end{pmatrix}$$

Equating the elements of the 2×1 matrices
$$x = t + mt,\ y = 2mt$$
$$y = 2mt \Rightarrow t = \frac{y}{2m}$$
Substituting into the equation for x
$$x = \frac{y}{2m} + m\times\frac{y}{2m} = \frac{y}{2m}(1+m)$$
Making y the subject of the formula
$$y = \frac{2m}{1+m}x$$

Eliminating t between these two expressions gives a linear equation relating y and x. The equation has no x^2, y^2, xy or higher powered terms. Therefore, the equation represents a straight line.

Comparing with the standard form of a line, $y = mx + c$,
$c = 0$, the line goes through the origin.
The line with equation $y = mx$ is transformed to another line passing through O.

d The gradient of this second line is $\frac{2m}{1+m}$.

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Review Exercise
Exercise A, Question 19

Question:

Referred to an origin O and coordinate axes Ox and Oy, transformations from $\mathbb{R}^2 \to \mathbb{R}^2$ are represented by the matrices **L**, **M** and **N**, where

$\mathbf{L} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\mathbf{M} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

a Explain the geometrical effect of the transformations **L** and **M**.

b Show that $\mathbf{LM} = \mathbf{N}^2$.

The transformation represented by the matrix **N** consists of a rotation of angle θ about O, followed by an enlargement, centre O, with positive scale factor k.

c Find the value of θ and the value of k.

d Find $\mathbf{N}^8$.

Solution:

a **L** represents rotation through $90°$, anti-clockwise, about the origin O.
M represents an enlargement, centre O, scale factor 2.

If you do not specify anticlockwise, positive angles are, conventionally, taken as anticlockwise and negative angles as clockwise. So, in this case, if you omitted "anti-clockwise", you would still be correct. Often $+90°$ is written to emphasize that the angle is anti-clockwise.

b $\mathbf{LM}=\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}=\begin{pmatrix}0 & -2\\2 & 0\end{pmatrix}$

$\mathbf{N}^2=\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}=\begin{pmatrix}1-1 & -1-1\\1+1 & -1+1\end{pmatrix}$

$=\begin{pmatrix}0 & -2\\2 & 0\end{pmatrix}$

So $\mathbf{LM}=\mathbf{N}^2$, they are both equal to $\begin{pmatrix}0 & -2\\2 & 0\end{pmatrix}$

c The result of part (b) can be interpreted as showing that the transformation represented by **N** applied twice is equivalent to rotation through $+90°$ about O followed by an enlargement, centre O, scale factor 2.
So the transformation represented by **N** applied once is equivalent to rotation through $+45°$ about O followed by an enlargement, centre O, scale factor $\sqrt{2}$.
$\theta=+45°$, $k=\sqrt{2}$.

Alternatively, it is possible to solve part (c) using matrices. The matrix representing a rotation of $+45°$ about O is $\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$ and the critical step is showing that
$\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\sqrt{2} & 0\\0 & \sqrt{2}\end{pmatrix}=\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}=\mathbf{N}$
.

d $\mathbf{N}^8$ represents the transformation represented by **N** applied eight times. This will rotate about the origin $8\times 45°=360°$ (which is the identity transformation), followed by an enlargement, centre O, scale factor $(\sqrt{2})^8=16$.

Hence $\mathbf{N}^8=\begin{pmatrix}16 & 0\\0 & 16\end{pmatrix}$.

Again, this can be done by matrices. You already know $\mathbf{N}^2$ from part (b) and you could then use
$\mathbf{N}^4=\mathbf{N}^2\mathbf{N}^2$
and $\mathbf{N}^8=\mathbf{N}^4\mathbf{N}^4$
to reach $\mathbf{N}^8$. Unless a question specifies a particular method, any correct alternative method can be used.

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Review Exercise
Exercise A, Question 20

Question:

A, **B** and **C** are 2×2 matrices.

a Given that **AB** = **AC**, and that **A** is not singular, prove that **B** = **C**.

b Given that **AB** = **AC**, where $\mathbf{A} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$, find a matrix **C** whose elements are all non-zero.

Solution:

a

$$\mathbf{AB} = \mathbf{AC}$$

Multiplying both sides on the left by $\mathbf{A}^{-1}$

$$\mathbf{A}^{-1}(\mathbf{AB}) = \mathbf{A}^{-1}(\mathbf{AC})$$

As matrices are associative

$$(\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = (\mathbf{A}^{-1}\mathbf{A})\mathbf{C}$$

Using $\mathbf{AA}^{-1} = \mathbf{I}$

$$\mathbf{IB} = \mathbf{IC}$$

As the identity matrix does not change another matrix

$\mathbf{B} = \mathbf{C}$, as required.

> When you are asked to prove a result, you must give each essential step in the argument. Your argument here must include multiplying by $\mathbf{A}^{-1}$ on the left of both sides of the equation and showing where you use $\mathbf{AA}^{-1} = \mathbf{I}$.

b

$$\mathbf{AB} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 21 \\ 1 & 7 \end{pmatrix}$$

Let $\mathbf{C} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\mathbf{AC} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3a+6c & 3b+6d \\ a+2c & b+2d \end{pmatrix}$$

$$\mathbf{AB} = \mathbf{AC}$$

$$\begin{pmatrix} 3 & 21 \\ 1 & 7 \end{pmatrix} = \begin{pmatrix} 3a+6c & 3b+6d \\ a+2c & b+2d \end{pmatrix}$$

> $\det(\mathbf{A}) = 6 - 6 = 0$ so $\mathbf{A}$ is singular. This question is a good example of the difficulties which can arise with non-singular matrices. There is no inverse matrix $\mathbf{A}^{-1}$ here and so you cannot use the usual rules of matrix algebra to remove $\mathbf{A}$.

Equating the upper left elements

$3 = 3a + 6c$ … … ❶

Equating the lower left elements

$1 = a + 2c$ … … ❷

❶ and ❷ are the same equation (❶ is ❷ $\times 3$)

Apart from the condition that the elements are non-zero, there is a free choice of a.

Let $a = 3$, then substituting in ❷,

$1 = 3 + 2c \Rightarrow c = -1$

> This pair of equations are satisfied by infinitely many pairs of numbers. You just have to choose any two non-zero numbers which satisfy them. For example, $a = -1, c = 1$ would do just as well.

Equating the upper right elements

$21 = 3b + 6d$ … … ❸

Equating the lower right elements

$7 = b + 2d$ … … ❹

❸ and ❹ are the same equation (❸ is ❹ $\times 3$)

Apart from the condition that the elements are non-zero, there is a free choice of b.

Let $b = 1$, then substituting in ❷,

$7 = 1 + 2d \Rightarrow d = 3$

$$\mathbf{C} = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$$

Check:

$$\mathbf{AC} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 21 \\ 1 & 7 \end{pmatrix} = \mathbf{AB}\text{, as required.}$$

> This is an unusual question and it is a good idea to check that your answer does give the correct result. You may well have a different $\mathbf{C}$ from that shown here, but you can check your answer by finding $\mathbf{AC}$. If you obtain $\begin{pmatrix} 3 & 21 \\ 1 & 7 \end{pmatrix}$, your answer is correct.

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Review Exercise
Exercise A, Question 21

Question:

Use standard formulae to show that $\sum_{r=1}^{n} 3r(r-1) = n(n^2-1)$.

Solution:

$$\sum_{r=1}^{n} 3r(r-1) = \sum_{r=1}^{n}(3r^2 - 3r)$$

$$= 3\sum_{r=1}^{n} r^2 - 3\sum_{r=1}^{n} r$$

$$= \frac{\cancel{3}^1 n(n+1)(2n+1)}{\cancel{6}^2} - \frac{3n(n+1)}{2}$$

$$= \frac{n(n+1)(2n+1)}{2} - \frac{3n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}[(2n+1)-3]$$

$$= \frac{n(n+1)(2n-2)}{2}$$

$$= \frac{n(n+1)\cancel{2}(n-1)}{\cancel{2}}$$

$= n(n^2-1)$, as required.

Multiply out the brackets and write the expression in terms of $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r$.

You can then use the standard formulae.

After "cancelling" the fractions, look for the common factors in your expressions, here shown in **bold**;

$$\frac{\boldsymbol{n(n+1)}(2n+1)}{2} - \frac{3\boldsymbol{n(n+1)}}{2}.$$

These, with the 2, are then taken outside a bracket;

$$\frac{\boldsymbol{n(n+1)}}{2}[(2n+1)-3].$$

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Review Exercise
Exercise A, Question 22

Question:

Use standard formulae to show that $\sum_{r=1}^{n}(r^2-1)=\frac{n}{6}(2n+5)(n-1)$.

Solution:

$$\sum_{r=1}^{n}(r^2-1)=\sum_{r=1}^{n}r^2-\sum_{r=1}^{n}1$$
$$=\frac{n(n+1)(2n+1)}{6}-n$$
$$=\frac{n(n+1)(2n+1)}{6}-\frac{6n}{6}$$
$$=\frac{n}{6}[(n+1)(2n+1)-6]$$
$$=\frac{n}{6}[2n^2+3n+1-6]$$
$$=\frac{n}{6}[2n^2+3n-5]$$
$$=\frac{n}{6}(2n+5)(n-1)\text{, as required.}$$

$\sum_{r=1}^{n}1=\underbrace{1+1+1+\ldots+1}_{n\text{ times}}=n$

It is a common error to write $\sum_{r=1}^{n}1=1$.

You put both terms over a common denominator, here 6.

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 23

Question:

Use standard formulae to show that $\sum_{r=1}^{n}(2r-1)^2 = \frac{1}{3}n(4n^2-1)$.

Solution:

$$\sum_{r=1}^{n}(2r-1)^2 = \sum_{r=1}^{n}(4r^2-4r+1)$$

$$= 4\sum_{r=1}^{n}r^2 - 4\sum_{r=1}^{n}r + \sum_{r=1}^{n}1$$

$$= \frac{\cancel{4}^2 n(n+1)(2n+1)}{\cancel{6}_3} - \frac{\cancel{4}^2 n(n+1)}{\cancel{2}} + n$$

$$= \frac{2n(n+1)(2n+1)}{3} - \frac{6n(n+1)}{3} + \frac{3n}{3}$$

$$= \frac{n}{3}\left[2(n+1)(2n+1) - 6(n+1) + 3\right]$$

$$= \frac{n}{3}\left[4n^2 + 6n + 2 - 6n - 6 + 3\right]$$

$$= \frac{1}{3}n(4n^2-1) \text{, as required.}$$

$\sum_{r=1}^{n}1 = \underbrace{1+1+1+\ldots+1}_{n \text{ times}} = n$

It is a common error to write $\sum_{r=1}^{n}1 = 1$.

After "cancelling" the fractions, you should put all terms over a common denominator, here 3.

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Review Exercise
Exercise A, Question 24

Question:

Use standard formulae to show that $\sum_{r=1}^{n} r(r^2-3)=\frac{1}{4}n(n+1)(n-2)(n+3)$.

Solution:
img src=

$$\sum_{r=1}^{n} r\left(r^2-3\right)=\sum_{r=1}^{n} r^3-3\sum_{r=1}^{n} r$$

$$=\frac{n^2(n+1)^2}{4}-\frac{3n(n+1)}{2}$$

$$=\frac{n^2(n+1)^2}{4}-\frac{6n(n+1)}{4}$$

$$=\frac{n(n+1)}{4}\left[n(n+1)-6\right]$$

$$=\frac{n(n+1)}{4}\left[n^2+n-6\right]$$

$$=\frac{1}{4}n(n+1)(n-2)(n+3)\text{, as required.}$$

After putting both terms over a common denominator, look for the common factors of the terms, here shown in **bold** ;

$$\frac{\boldsymbol{n}^2(\boldsymbol{n}+\mathbf{1})^2}{4}-\frac{6\boldsymbol{n}(\boldsymbol{n}+\mathbf{1})}{4}.$$

You take these, together with the common denominator 4, outside a bracket;

$$\frac{\boldsymbol{n}(\boldsymbol{n}+\mathbf{1})}{4}\left[n(n+1)-6\right].$$

You need to be careful with the squared terms.

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Review Exercise
Exercise A, Question 25

Question:

a Use standard formulae to show that $\sum_{r=1}^{n} r(2r-1) = \frac{n(n+1)(4n-1)}{6}$.

b Hence, evaluate $\sum_{r=11}^{30} r(2r-1)$.

Solution:

a

$$\sum_{r=1}^{n} r(2r-1) = \sum_{r=1}^{n}(2r^2 - r)$$
$$= 2\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r$$
$$= \frac{2n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$$
$$= \frac{2n(n+1)(2n+1)}{6} - \frac{3n(n+1)}{6}$$
$$= \frac{n(n+1)}{6}[2(2n+1)-3]$$
$$= \frac{n(n+1)}{6}[4n+2-3]$$
$$= \frac{n(n+1)(4n-1)}{6}, \text{ as required.}$$

You put the expressions over a common denominator, here 6, and then look for the common factors of the expressions, here n and $(n+1)$.

b

$$\sum_{r=11}^{30} r(2r-1) = \sum_{r=1}^{30} r(2r-1) - \sum_{r=1}^{10} r(2r-1)$$

$\sum_{r=11}^{30} f(r) = \sum_{r=1}^{30} f(r) - \sum_{r=1}^{10} f(r)$.
You find the sum from the 11th to the 30th term by subtracting the sum from the first to the 10th term from the sum from the first to the 30th term. It is a common error to subtract one term too many, in this case the 11th term. The sum you are finding starts with the 11th term. You must not subtract it from the series – you have to leave it in the series.

Substituting $n = 30$ and $n = 10$ into the result in part (a).

$$\sum_{r=11}^{30} r(2r-1) = \frac{30\times31\times119}{6} - \frac{10\times11\times39}{6}$$
$$= 18\,445 - 715$$
$$= 17\,730$$

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Review Exercise
Exercise A, Question 26

Question:

Evaluate $\sum_{r=0}^{12}(r^2+2^r)$.

Solution:

$$\sum_{r=0}^{12}\left(r^2+2^r\right)=\sum_{r=0}^{12}r^2+\sum_{r=0}^{12}2^r$$

This question asks you to carry out two different sums. The first involves $\sum_{r=1}^{n}r^2$, which you learnt in Chapter 5 of this book. The other is a Geometric Series which you can find in Chapter 7 of Edexcel AS and A-Level Modular Mathematics, Core Mathematics 2.

$$\sum_{r=0}^{12}r^2=\sum_{r=1}^{12}r^2$$

$$\sum_{r=1}^{n}r^2=\frac{n(n+1)(2n+1)}{6}$$

Substituting $n=12$ into the standard formula.

The first term in this series, corresponding to $r=0$ is 0^2. This obviously does not add anything to the series, so you can start the summation from 1 and use the standard formula with $n=12$.

$$\sum_{r=1}^{12}r^2=\frac{12(12+1)(2\times12+1)}{6}=\frac{12\times13\times25}{6}$$
$$=650$$

$$\sum_{r=0}^{12}2^r=2^0+2^1+2^2+\ \dots+2^{12}$$

The first term in the Geometric Series is $2^0=1$ and you must include this in the sum. With this, there are 13 terms in the series.

This is a Geometric Series with $a=2^0=1, r=2$ and $n=13$.

Using the formula $S=\dfrac{a\left(r^n-1\right)}{r-1}$,

$$\sum_{r=0}^{12}2^r=\frac{1\left(2^{13}-1\right)}{2-1}=2^{13}-1=8191$$

Combining the two results

$$\sum_{r=0}^{12}\left(r^2+2^r\right)=650+8191=8841$$

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Review Exercise
Exercise A, Question 27

Question:

Evaluate $\sum_{r=1}^{50}(r+1)(r+2)$.

Solution:

$$\sum_{r=1}^{n}(r+1)(r+2)=\sum_{r=1}^{n}r^2+3\sum_{r=1}^{n}r+\sum_{r=1}^{n}2$$

You have not been asked to show that any particular formula in n is true but you have to get an expression for the summation in terms of n and then substitute $n=50$ into it.

$$=\frac{n(n+1)(2n+1)}{6}+\frac{3n(n+1)}{2}+2n$$

As you have not been asked to show that any formula is true, you need not look for any common factors in these terms. You can use the whole expression as it is.

Substituting $n=50$

$$\sum_{r=1}^{50}(r+1)(r+2)=\frac{50\times51\times101}{6}+\frac{3\times50\times51}{2}+2\times50$$

$$=42\,925+3825+100=46\,850$$

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Review Exercise
Exercise A, Question 28

Question:

Use standard formulae to show that $\sum_{r=1}^{n} r(r^2 - n) = \frac{n^2(n^2-1)}{4}$.

Solution:

$$\sum_{r=1}^{n} r(r^2 - n) = \sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} nr$$

$$= \sum_{r=1}^{n} r^3 - n\sum_{r=1}^{n} r$$

$$= \frac{n^2(n+1)^2}{4} - n \times \frac{n(n+1)}{2}$$

$$= \frac{n^2(n+1)^2}{4} - \frac{2n^2(n+1)}{4}$$

$$= \frac{n^2(n+1)}{4}[(n+1)-2]$$

$$= \frac{n^2(n+1)(n-1)}{4}$$

$$= \frac{n^2(n^2-1)}{4}, \text{ as required.}$$

In $\sum_{r=1}^{n} nr$, the r ranges from 1 to n but the n does not change; n is a constant. So

$$\sum_{r=1}^{n} nr = n \times 1 + n \times 2 + n \times 3 + \ldots + n \times n$$

$$= n \times (1 + 2 + 3 + \ldots + n) = n \times \frac{n(n+1)}{2}$$

After putting both terms over a common denominator, look for the common factors of the terms, here shown in **bold**;

$$\frac{\mathbf{n^2(n+1)}^2}{\mathbf{4}} - \frac{2\mathbf{n^2(n+1)}}{\mathbf{4}}.$$

You take these, together with the common denominator 4, outside a bracket;

$$\frac{\mathbf{n^2(n+1)}}{\mathbf{4}}[(n+1)-2].$$

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Review Exercise
Exercise A, Question 29

Question:

a Use standard formulae to show that $\sum_{r=1}^{n} r(3r+1) = n(n+1)^2$.

b Hence evaluate $\sum_{r=40}^{100} r(3r+1)$.

Solution:

a $\sum_{r=1}^{n} r(3r+1) = 3\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$

$= \frac{3n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$

$= \frac{n(n+1)}{2}[(2n+1)+1]$

$= \frac{n(n+1)(2n+2)}{2} = \frac{n(n+1)2(n+1)}{2}$

$= n(n+1)^2$, as required.

b $\sum_{r=40}^{100} r(3r+1) = \sum_{r=1}^{100} r(3r+1) - \sum_{r=1}^{39} r(3r+1)$

Substituting $n = 100$ and $n = 39$ into the result in part (a).

$\sum_{r=40}^{100} r(3r+1) = 100 \times 101^2 - 39 \times 40^2$

$= 1\,020\,100 - 62\,400$

$= 957\,700$

$\sum_{r=40}^{100} f(r) = \sum_{r=1}^{100} f(r) - \sum_{r=1}^{39} f(r)$.

You find the sum from the 40th to the 100th term by subtracting the sum from the first to the 39th term from the sum from the first to the 100th term.
It is a common error to subtract one term too many, in this case the 40th term. The sum you are finding starts with the 40th term. You must not subtract the 40th term from the series – you have to leave it in the series.

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Review Exercise
Exercise A, Question 30

Question:

a Show that $\sum_{r=1}^{n}(2r-1)(2r+3)=\frac{n}{3}(4n^2+12n-1)$.

b Hence find $\sum_{r=5}^{35}(2r-1)(2r+3)$.

Solution:

a $\sum_{r=1}^{n}(2r-1)(2r+3)=\sum_{r=1}^{n}(4r^2+4r-3)$

$=4\sum_{r=1}^{n}r^2+4\sum_{r=1}^{n}r-\sum_{r=1}^{n}3$

$=\frac{\not{4}^2 n(n+1)(2n+1)}{\not{6}^3}+\frac{\not{4}^2 n(n+1)}{\not{2}}-3n$

$=\frac{2n(n+1)(2n+1)}{3}+\frac{6n(n+1)}{3}-\frac{9n}{3}$

$=\frac{n}{3}\left[2(n+1)(2n+1)+6(n+1)-9n\right]$

$=\frac{n}{3}\left[4n^2+6n+2+6n+6-9\right]$

$=\frac{n}{3}(4n^2+12n-1)$, as required.

$\sum_{r=1}^{n}3=3+3+3+\ \dots\ +3=3n$

It is a common error to write $\sum_{r=1}^{n}3=3$.

After "cancelling" fractions, put all of the expressions over a common denominator, here 3.
You then look for any factors common to all three expressions. Here there is only one, n.

b $\sum_{r=5}^{35}(2r-1)(2r+3)$

$=\sum_{r=1}^{35}(2r-1)(2r+3)-\sum_{r=1}^{4}(2r-1)(2r+3)$

You find the sum from the 5th to the 35th term by subtracting the sum from the first to the 4th term from the sum from the first to the 35th term.

Substituting $n=35$ and $n=4$ into the result in part (a)

$\sum_{r=5}^{35}(2r-1)(2r+3)$

$=\frac{35}{3}(4\times35^2+12\times35-1)-\frac{4}{3}(4\times4^2+12\times4-1)$

$=62\,055-148=61\,907$

You use the expression you have proved in part (a) to complete the question.

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Review Exercise
Exercise A, Question 31

Question:

a Use standard formulae to show that $\sum_{r=1}^{n}(6r^2+4r-5)=n(2n^2+5n-2)$.

b Hence calculate the value of $\sum_{r=10}^{25}(6r^2+4r-5)$.

Solution:

a
$$\sum_{r=1}^{n}(6r^2+4r-5)=6\sum_{r=1}^{n}r^2+4\sum_{r=1}^{n}r-\sum_{r=1}^{n}5$$
$$=\frac{6n(n+1)(2n+1)}{6}+\frac{4^2n(n+1)}{2}-5n$$
$$=n(n+1)(2n+1)+2n(n+1)-5n$$
$$=n[(n+1)(2n+1)+2(n+1)-5]$$
$$=n[2n^2+3n+1+2n+2-5]$$
$$=n(2n^2+5n-2)\text{, as required.}$$

A common error with the last term is to write $-\sum_{r=1}^{n}5=-5$. Correctly:
$$-\sum_{r=1}^{n}5=-(5+5+5+\ldots+5)$$
$$=-5(\underbrace{1+1+1+\ldots+1}_{n\text{ times}})$$
$$=-5n$$

b
$$\sum_{r=10}^{25}(6r^2+4r-5)=\sum_{r=1}^{25}(6r^2+4r-5)-\sum_{r=1}^{9}(6r^2+4r-5)$$
Substituting $n=25$ and $n=9$ into the result in part (a)
$$\sum_{r=10}^{25}(6r^2+4r-5)$$
$$=25(2\times25^2+5\times25-2)-9(2\times9^2+5\times9-2)$$
$$=34\,325-1845=32\,480$$

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Review Exercise
Exercise A, Question 32

Question:

a Use standard formulae to show that $\sum_{r=1}^{n}(r+1)(r+5)=\frac{1}{6}n(n+7)(2n+7)$.

b Hence calculate the value of $\sum_{r=10}^{40}(r+1)(r+5)$.

Solution:

a $\sum_{r=1}^{n}(r+1)(r+5)=\sum_{r=1}^{n}(r^2+6r+5)$

$$=\sum_{r=1}^{n}r^2+6\sum_{r=1}^{n}r+\sum_{r=1}^{n}5$$

$$=\frac{n(n+1)(2n+1)}{6}+\frac{6n(n+1)}{2}+5n$$

$$=\frac{n(n+1)(2n+1)}{6}+\frac{18n(n+1)}{6}+\frac{30n}{6}$$

$$=\frac{n}{6}\left[(n+1)(2n+1)+18(n+1)+30\right]$$

$$=\frac{n}{6}\left[2n^2+3n+1+18n+18+30\right]$$

$$=\frac{n}{6}\left(n^2+21n+49\right)$$

$$=\frac{1}{5}n(n+7)(2n+7)\text{, as required.}$$

As the question prints the answer, factorising the quadratic expression gives no difficulty, but you should check your solution by multiplying out the brackets in the answer. This helps you to correct any errors that you may have made in your working. In this case, the check is

$$(n+7)(2n+7)=2n^2+7n+14n+49$$
$$=2n^2+21n+49.$$

This checks and you can be confident the working is correct.

b $\sum_{r=10}^{40}(r+1)(r+5)=\sum_{r=1}^{40}(r+1)(r+5)-\sum_{r=1}^{9}(r+1)(r+5)$

Substituting $n=40$ and $n=9$ into the result in part (a)

$$\sum_{r=10}^{40}(r+1)(r+5)=\frac{1}{6}\times40\times47\times87-\frac{1}{6}\times9\times16\times25$$

$$=27\,260-600=26\,660$$

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Review Exercise
Exercise A, Question 33

Question:

a Use standard formulae to show that $\sum_{r=1}^{n} r^2(r+1) = \frac{n(n+1)(3n^2+7n+2)}{12}$.

b Find $\sum_{r=4}^{30} (2r)^2(2r+2)$.

Solution:

a $\sum_{r=1}^{n} r^2(r+1) = \sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} r^2$

$$= \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{3n^2(n+1)^2}{12} + \frac{2n(n+1)(2n+1)}{12}$$

$$= \frac{n(n+1)}{12}\left[3n(n+1) + 2(2n+1)\right]$$

After putting both terms over the common denominator 12, find the common factors of the terms, here shown in **bold**;

$$\frac{3\boldsymbol{n}^2(\boldsymbol{n}+\mathbf{1})^2}{12} + \frac{2\boldsymbol{n}(\boldsymbol{n}+\mathbf{1})(2n+1)}{12}$$

$$= \frac{\boldsymbol{n}(\boldsymbol{n}+\mathbf{1})}{12}\left[3n(n+1) + 2(2n+1)\right].$$

$$= \frac{n(n+1)}{12}\left[3n^2 + 3n + 4n + 2\right]$$

$$= \frac{n(n+1)(3n^2+7n+2)}{12}, \text{ as required.}$$

b $(2r)^2(2r+2) = 4r^2 \times 2(r+1) = 8r^2(r+1)$

$$\sum_{r=4}^{30}(2r)^2(2r+2) = 8\sum_{r=4}^{30} r^2(r+1)$$

Each term, $(2r)^2(2r+2)$, in the summation in part (b) is eight times the corresponding term, $r^2(r+1)$, in part (a). The key idea is then to find $\sum_{r=4}^{30} r^2(r+1)$ and multiply this by 8.

$$= 8\sum_{r=4}^{30} r^2(r+1) = 8\left(\sum_{r=1}^{30} r^2(r+1) - \sum_{r=1}^{3} r^2(r+1)\right)$$

Substituting $n = 30$ and $n = 3$ into the result in part (a)

$$\sum_{r=4}^{30}(2r)^2(2r+2)$$

$$= 8\left(\frac{30\times31\times(3\times30^2+7\times30+2)}{12} - \frac{3\times4\times(3\times3^2+7\times3+2)}{12}\right)$$

$$= 8(225\,680 - 50) = 8\times225\,630 = 1\,805\,040$$

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Review Exercise
Exercise A, Question 34

Question:

Using the formula $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$,

a show that $\sum_{r=1}^{n}(4r^2 - 1) = \frac{n}{3}(4n^2 + 6n - 1)$.

Given that $\sum_{r=1}^{12}(4r^2 + kr - 1) = 2120$, where k is a constant,

b find the value of k.

Solution:

a $\sum_{r=1}^{n}(4r^2-1) = 4\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} 1$

$= \frac{\not{4}^2 n(n+1)(2n+1)}{\not{6}^3} - n$

$= \frac{2n(n+1)(2n+1)}{3} - \frac{3n}{3}$

$= \frac{n}{3}\left[2(n+1)(2n+1) - 3\right]$

$= \frac{n}{3}\left[4n^2 + 6n + 2 - 3\right]$

$= \frac{n}{3}(4n^2 + 6n - 1)$, as required.

> $\sum_{r=1}^{n} 1 = 1 + 1 + 1 + \ldots + 1 = n$
>
> It is a common error to write $\sum_{r=1}^{n} 1 = 1$.

> After "cancelling" fractions, put all of the expressions over a common denominator, here 3.
> You then look for any factors common to all three expressions. Here there is only one, n.

b $\sum_{r=1}^{12}(4r^2 + kr - 1) = 2120$

$\sum_{r=1}^{12}(4r^2 - 1) + k\sum_{r=1}^{12} r = 2120 \quad \ldots \ast$

Using the result in part (a) with $n = 12$

$\sum_{r=1}^{12}(4r^2 - 1) = \frac{12}{3}(4 \times 12^2 + 6 \times 12 - 1)$

$= 2588$

Using the standard result $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

with $n = 12$

$\sum_{r=1}^{12} r = \frac{12(12+1)}{2} = 78$

Equation $\ast$ now becomes

$2588 + 78k = 2120$

$k = \frac{2120 - 2588}{78} = -6$

> $(4r^2 + kr - 1)$ can be written as $(4r^2 - 1) + kr$.
>
> You can then evaluate $\sum_{r=1}^{12}(4r^2 - 1)$, using part (a),
>
> and $\sum_{r=1}^{12} r$, using a formula you learnt in Chapter 5.
>
> The relation in the question then becomes an equation in k which you can solve.

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Review Exercise
Exercise A, Question 35

Question:

a Use standard formulae to show that $\sum_{r=1}^{n} r(3r-5) = n(n+1)(n-2)$.

b Hence show that $\sum_{r=n}^{2n} r(3r-5) = 7n(n^2-1)$.

Solution:

a $\sum_{r=1}^{n} r(3r-5) = 3\sum_{r=1}^{n} r^2 - 5\sum_{r=1}^{n} r$

$= \frac{3n(n+1)(2n+1)}{6} - \frac{5n(n+1)}{2}$

$= \frac{n(n+1)}{2}[2n+1-5]$

$= \frac{n(n+1)2(n-2)}{2}$

$= n(n+1)(n-2)$, as required.

Look for the common factors of the terms, here shown in **bold**;

$\frac{\mathbf{n(n+1)}(2n+1)}{2} - \frac{5\mathbf{n(n+1)}}{2}$.

Take the common factors, together with the common denominator 2, outside a bracket;

$\frac{\mathbf{n(n+1)}}{2}[(2n+1)-5]$.

b $\sum_{r=n}^{2n} r(3r-5) = \sum_{r=1}^{2n} r(3r-5) - \sum_{r=1}^{n-1} r(3r-5)$

Using the result in part (a), replacing n by $2n$ and $n-1$.

$\sum_{r=n}^{2n} r(3r-5) = 2n(2n+1)(2n-2) - (n-1)n(n-3)$

$= 4n(2n+1)(n-1) - (n-1)n(n-3)$

$= n(n-1)[4(2n+1)-(n-3)]$

$= n(n-1)[8n+4-n+3]$

$= n(n-1)(7n+7)$

$= 7n(n-1)(n+1)$

$= 7n(n^2-1)$, as required.

$\sum_{r=n}^{2n} r(3r-5) = \sum_{r=1}^{2n} f(r) - \sum_{r=1}^{n-1} f(r)$

To find an expression for $\sum_{r=1}^{2n} f(r)$, you replace the n in the result in part (a) by $2n$;

$n(n+1)(n-2)$

becomes $2n(2n+1)(2n-2)$.

To find an expression for $\sum_{r=1}^{n-1} f(r)$, you replace the n in the result in part (a) by $n-1$;

$n(n+1)(n-2)$

becomes $(n-1)((n-1)+1)((n-1)-2)$

$= (n-1)n(n-3)$.

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Review Exercise
Exercise A, Question 36

Question:

a Use standard formulae to show that $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$.

b Hence, or otherwise, show that $\sum_{r=n}^{3n} r(r+1) = \frac{1}{3}n(2n+1)(pn+q)$, stating the values of the integers p and q.

Solution:

a $\sum_{r=1}^{n} r(r+1) = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$

$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$

$= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{6}$

After putting the expressions over a common denominator 6, you look for any factors common to both expressions. Here there are two, n and $(n+1)$.

$= \frac{n(n+1)}{6}[2n+1+3]$

$= \frac{n(n+1)2^1(n+2)}{6^3}$

$= \frac{1}{3}n(n+1)(n+2)$, as required.

b $\sum_{r=n}^{3n} r(r+1) = \sum_{r=1}^{3n} r(r+1) - \sum_{r=1}^{n-1} r(r+1)$

To find an expression for $\sum_{r=1}^{n-1} r(r+1)$, you replace the n in the result in part (a) by $n-1$; $\frac{1}{3}n(n+1)(n+2)$ becomes $\frac{1}{3}(n-1)((n-1)+1)((n-1)+2)$ $= \frac{1}{3}(n-1)n(n+1)$.

$= \frac{1}{3}3n(3n+1)(3n+2) - \frac{1}{3}(n-1)n(n+1)$

$= \frac{1}{3}n[3(3n+1)(3n+2)-(n-1)(n+1)]$

$= \frac{1}{3}n[27n^2+27n+6-(n^2-1)]$

$= \frac{1}{3}n(26n^2+27n+7)$

$= \frac{1}{3}n(2n+1)(13n+7)$

$p = 13, q = 7$

As you are given that $(2n+1)$ is one factor of $26n^2+27n+7$, the other can just be written down. $(2n+1)(pn+q) = 26n^2+27n+7$, only if $2p = 26$ and $1q = 7$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 37

Question:

Given that $\sum_{r=1}^{n} r^2(r-1) = \frac{1}{12}n(n+1)(pn^2+qn+r)$,

a find the values of p, q and r.

b Hence evaluate $\sum_{r=50}^{100} r^2(r-1)$.

Solution:

a $\sum_{r=1}^{n} r^2(r-1) = \sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} r^2$

$$= \frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{3n^2(n+1)^2}{12} - \frac{2n(n+1)(2n+1)}{12}$$

$$= \frac{n(n+1)}{12}\left[3n(n+1) - 2(2n+1)\right]$$

$$= \frac{n(n+1)}{12}\left[3n^2 + 3n - 4n - 2\right]$$

$$= \frac{1}{12}n(n+1)(3n^2 - n - 2)$$

$p = 3,\ q = -1,\ r = -2$

After putting the expressions over a common denominator 12, you look for any factors common to both expressions. Here there are two, n and $(n+1)$.

b $\sum_{r=50}^{100} r^2(r-1) = \sum_{r=1}^{100} r^2(r-1) - \sum_{r=1}^{49} r^2(r-1)$

$$= \frac{1}{12} \times 100 \times 101 \times (3 \times 100^2 - 100 - 2)$$

$$- \frac{1}{12} \times 49 \times 50 \times (3 \times 49^2 - 49 - 2)$$

$$= 25\,164\,150 - 1\,460\,200$$

$$= 23\,703\,950$$

$\sum_{r=50}^{100} f(r) = \sum_{r=1}^{100} f(r) - \sum_{r=1}^{49} f(r)$.

You find the sum from the 50th to the 100th term by subtracting the sum from the first to the 49th term from the sum from the first to the 100th term.
It is a common error to subtract one term too many, in this case the 50th term. The sum you are finding starts with the 50th term. You must not remove it from the series.

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 38

Question:

a Use standard formula to show that $\sum_{r=1}^{n} r(r+2) = \frac{1}{6}n(n+1)(2n+7)$.

b Hence, or otherwise, find the value of $\sum_{r=1}^{10} (r+2)\log_4 2^r$.

Solution:

a $\sum_{r=1}^{n} r(r+2) = \sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r$

$= \frac{n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2}$

$= \frac{n(n+1)(2n+1)}{6} + \frac{6n(n+1)}{6}$

$= \frac{n(n+1)}{6}[2n+1+6]$

$= \frac{1}{6}n(n+1)(2n+7)$, as required.

b $\sum_{r=1}^{10} (r+2)\log_4 2^r = \sum_{r=1}^{10} (r+2)r\log_4 2$

$= \log_4 2 \sum_{r=1}^{10} r(r+2)$

$= \log_4 4^{\frac{1}{2}} \sum_{r=1}^{10} r(r+2)$

$= \frac{1}{2}\log_4 4 \sum_{r=1}^{10} r(r+2)$

$= \frac{1}{2}\sum_{r=1}^{10} r(r+2)$

$= \frac{1}{2} \times \frac{1}{6} \times 10 \times 11 \times 27$

$= 247\frac{1}{2}$

In part (b), you need to use the properties of logarithms you learnt in the C2 course. You can find this material in Chapter 3 of Edexcel AS and A-Level Modular Mathematics, Core Mathematics 2.

The power law of logarithms, $\log_a x^k = k\log_a x$, is used twice.

$\log_a a = 1$, for any positive a.

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Review Exercise
Exercise A, Question 39

Question:

Use the method of mathematical induction to prove that, for all positive integers n, $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$.

Solution:

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$$

All inductions need to be shown to be true for a small number, usually 1.

Let $n = 1$.
The left-hand side becomes

$$\sum_{r=1}^{1} \frac{1}{r(r+1)} = \frac{1}{1\times 2} = \frac{1}{2}$$

$\sum_{r=1}^{1} \frac{1}{r(r+1)}$ consists of just one term. That is $\frac{1}{r(r+1)}$ with 1 substituted for r.

The right-hand side becomes

$$\frac{1}{1+1} = \frac{1}{2}$$

The left-hand side and the right-hand side are equal and so the summation is true for $n = 1$.

Assume the summation is true for $n = k$.

That is $\sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}$ ... ... $*$

$$\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \sum_{r=1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+2)}$$

The sum from 1 to $k+1$ is the sum from 1 to k plus one extra term.
In this case, the extra term is found by replacing each r in $\frac{1}{r(r+1)}$ by $k+1$.

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}, \text{ using } *$$

$$= \frac{k(k+1)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2} = \frac{k+1}{(k+1)+1}$$

Keep in mind what you are aiming for as you work out the algebra. You are looking to prove that the summation is true for $n = k+1$, so you are trying to reach $\frac{n}{n+1}$ with the n replaced by $k+1$.

This is the result obtained by substituting $n = k+1$ into the right-hand side of the summation and so the summation is true for $n = k+1$.

The summation is true for $n = 1$, and, if it is true for $n = k$, then it is true for $n = k+1$.

By mathematical induction the summation is true for all positive integers n.

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Review Exercise
Exercise A, Question 40

Question:

Use the method of mathematical induction to prove that $\sum_{r=1}^{n} r(r+3) = \frac{1}{3}n(n+1)(n+5)$.

Solution:

$$\sum_{r=1}^{n} r(r+3) = \frac{1}{3}n(n+1)(n+5)$$

Let $n = 1$.
The left-hand side becomes

$$\sum_{r=1}^{1} r(r+3) = 1(1+3) = 4$$

$\sum_{r=1}^{1} r(r+3)$ consists of just one term. That is $r(r+3)$ with 1 substituted for r.

The right-hand side becomes

$$\frac{1}{3}\times 1(1+1)(1+5) = \frac{1}{3}\times 2\times 6 = 4$$

The left-hand side and the right-hand side are equal and so the summation is true for $n = 1$.

Assume the summation is true for $n = k$.

That is $\sum_{r=1}^{k} r(r+3) = \frac{1}{3}k(k+1)(k+5)$ ... ... ✱

This is often called the **induction hypothesis**.

$$\sum_{r=1}^{k+1} r(r+3) = \sum_{r=1}^{k} r(r+3) + (k+1)(k+4)$$

The sum from 1 to $k+1$ is the sum from 1 to k plus one extra term. In this case, the extra term is found by replacing each r in $r(r+3)$ by $k+1$.

$$= \frac{1}{3}k(k+1)(k+5) + (k+1)(k+4), \text{ using ✱}$$

$$= \frac{1}{3}k(k+1)(k+5) + \frac{3}{3}(k+1)(k+4)$$

$$= \frac{1}{3}(k+1)\left[k(k+5) + 3(k+4)\right]$$

$$= \frac{1}{3}(k+1)\left[k^2 + 5k + 3k + 12\right]$$

$$= \frac{1}{3}(k+1)\left[k^2 + 8k + 12\right]$$

$$= \frac{1}{3}(k+1)(k+2)(k+6)$$

$$= \frac{1}{3}(k+1)((k+1)+1)((k+1)+5)$$

Multiplying out the brackets would give you an awkward cubic expression which would be difficult to factorise. You should try to simplify the working by looking for any common factors and taking them outside a bracket. Here $(k+1)$ is a common factor.

This expression is $\frac{1}{3}n(n+1)(n+5)$ with each n replaced by $k+1$.

This is the result obtained by substituting $n = k+1$ into the right-hand side of the summation and so the summation is true for $n = k+1$.

The summation is true for $n = 1$, and, if it is true for $n = k$, then it is true for $n = k+1$.

By mathematical induction the summation is true for all positive integers n.

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Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 41

Question:

Prove by induction that, for $n \, \varepsilon \, \mathbb{Z}^+$, $\sum_{r=1}^{n} (2r-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$.

Solution:

$$\sum_{r=1}^{n}(2r-1)^2=\frac{1}{3}n(2n-1)(2n+1)$$

Let $n=1$.

The left-hand side becomes

$$\sum_{r=1}^{1}(2r-1)^2=(2-1)^2=1^2=1$$

$\sum_{r=1}^{1}(2r-1)^2$ consists of just one term. That is $(2r-1)^2$ with 1 substituted for r.

The right-hand side becomes

$$\frac{1}{3}\times1(2-1)(2+1)=\frac{1}{3}\times1\times1\times3=1$$

The left-hand side and the right-hand side are equal and so the summation is true for $n=1$.

Assume the summation is true for $n=k$.

That is $\sum_{r=1}^{k}(2r-1)^2=\frac{1}{3}k(2k-1)(2k+1)$ ... ... *

$$\sum_{r=1}^{k+1}(2r-1)^2=\sum_{r=1}^{k}(2r-1)^2+(2k+1)^2$$

The sum from 1 to $k+1$ is the sum from 1 to k plus one extra term. In this case, the extra term is found by replacing the r in $(2r-1)^2$ by $k+1$. Giving $(2(k+1)-1)^2=(2k+2-1)^2=(2k+1)^2$

$$=\frac{1}{3}k(2k-1)(2k+1)+\frac{3}{3}(2k+1)^2\text{, using *}$$

$$=\frac{1}{3}(2k+1)\left[k(2k-1)+3(2k+1)\right]$$

Multiplying out the brackets would give you an awkward cubic expression which would be difficult to factorise. Look for any common factors and take them outside a bracket. Here $(2k+1)$ is a common factor.

$$=\frac{1}{3}(2k+1)\left[2k^2+5k+3\right]$$

$$=\frac{1}{3}(2k+1)(k+1)(2k+3)$$

$$=\frac{1}{3}(k+1)(2(k+1)-1)(2(k+1)+1)$$

This expression is $\frac{1}{3}n(2n-1)(2n+1)$ with each n replaced by $k+1$.

This is the result obtained by substituting $n=k+1$ into the right-hand side of the summation and so the summation is true for $n=k+1$.

The summation is true for $n=1$, and, if it is true for $n=k$, then it is true for $n=k+1$.

By mathematical induction the summation is true for all positive integers n.

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Review Exercise
Exercise A, Question 42

Question:

The rth term a_r in a series is given by $a_r = r(r+1)(2r+1)$.

Prove, by mathematial induction, that the sum of the first n terms of the series is $\frac{1}{2}n(n+1)^2(n+2)$.

Solution:

$$\sum_{r=1}^{n} a_r = \sum_{r=1}^{n} r(r+1)(2r+1) = \frac{1}{2}n(n+1)^2(n+2)$$

Let $n=1$.

The left-hand side becomes

$$\sum_{r=1}^{1} r(r+1)(2r+1) = 1\times2\times3 = 6$$

The right-hand side becomes

$$\frac{1}{2}\times1\times2^2\times3 = 6$$

The left-hand side and the right-hand side are equal and so the summation is true for $n=1$.

All inductions need to be shown to be true for a small number, usually 1.

Assume the summation is true for $n=k$.

That is $\sum_{r=1}^{k} r(r+1)(2r+1) = \frac{1}{2}k(k+1)^2(k+2)$ ∗

$$\sum_{r=1}^{k+1} r(r+1)(2r+1) = \sum_{r=1}^{k} r(r+1)(2r+1) + (k+1)(k+2)(2k+3)$$

$$= \frac{1}{2}k(k+1)^2(k+2) + \frac{2}{2}(k+1)(k+2)(2k+3) \text{ , using } *$$

$$= \frac{1}{2}(k+1)(k+2)\left[k(k+1)+2(2k+3)\right]$$

$$= \frac{1}{2}(k+1)(k+2)\left[k^2+5k+6\right]$$

$$= \frac{1}{2}(k+1)(k+2)(k+2)(k+3)$$

$$= \frac{1}{2}(k+1)(k+2)^2(k+3)$$

$$= \frac{1}{2}(k+1)((k+1)+1)^2((k+1)+2)$$

Fractions need to be expressed to the same denominator before factorising. The form of the answer shows that you need to have $\frac{1}{2}$ as a common factor and it helps you to write $\frac{2}{2}$ before the second term on the right-hand side of the summation.

This expression is $\frac{1}{2}n(n+1)^2(2n+1)$ with each n replaced by $k+1$.

This is the result obtained by substituting $n=k+1$ into the right-hand side of the summation and so the summation is true for $n=k+1$.

The summation is true for $n=1$, and, if it is true for $n=k$, then it is true for $n=k+1$.

By mathematical induction the summation is true for all positive integers n.

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Review Exercise
Exercise A, Question 43

Question:

Prove, by induction, that $\sum_{r=1}^{n} r^2(r-1) = \frac{1}{12}n(n-1)(n+1)(3n+2)$.

Solution:

$$\sum_{r=1}^{n} r^2(r-1) = \frac{1}{12}n(n-1)(n+1)(3n+2)$$

Let $n = 1$.

The left-hand side becomes

$$\sum_{r=1}^{1} r^2(r-1) = 1^2 \times (1-1) = 0$$

$\sum_{r=1}^{1} r^2(r-1)$ consists of just one term. That is $r^2(r-1)$ with 1 substituted for r. In this case, because of the bracket, this clearly gives 0.

The right-hand side becomes

$$\frac{1}{12} \times 1 \times (1-1) \times (1+1) \times (3+2)$$

$$= \frac{1}{12} \times 1 \times 0 \times 2 \times 5 = 0$$

The left-hand side and the right-hand side are equal and so the summation is true for $n = 1$.

Assume the summation is true for $n = k$.

That is $\sum_{r=1}^{k} r^2(r-1) = \frac{1}{12}k(k-1)(k+1)(3k+2)$ … … $\ast$

$$\sum_{r=1}^{k+1} r^2(r-1) = \sum_{r=1}^{k} r^2(r-1) + (k+1)^2(k+1-1)$$

$$= \frac{1}{12}k(k-1)(k+1)(3k+2) + \frac{12}{12}k(k+1)^2, \text{ using } \ast$$

The common factors in these two terms are $\frac{1}{12}$, k and $(k+1)$.

$$= \frac{1}{12}k(k+1)\left[(k-1)(3k+2) + 12(k+1)\right]$$

$$= \frac{1}{12}k(k+1)\left[3k^2 - k - 2 + 12k + 12\right]$$

$$= \frac{1}{12}k(k+1)\left[3k^2 + 11k + 10\right]$$

$$= \frac{1}{12}k(k+1)(k+2)(3k+5)$$

$$= \frac{1}{12}(k+1)((k+1)-1)((k+1)+1)(3(k+1)+2)$$

Rearrange this expression so that it is the right-hand side of the summation with n replaced by $k + 1$.

This is the result obtained by substituting $n = k+1$ into the right-hand side of the summation and so the summation is true for $n = k+1$.

The summation is true for $n = 1$, and, if it is true for $n = k$, then it is true for $n = k+1$.

By mathematical induction the summation is true for all positive integers n.

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Review Exercise
Exercise A, Question 44

Question:

Given that $u_1 = 8$ and $u_{n+1} = 4u_n - 9n$, use mathematical induction to prove that $u_n = 4^n + 3n + 1, n \varepsilon \mathbb{Z}^+$.

Solution:

$u_n = 4^n + 3n + 1$
Let $n = 1$
$u_1 = 4^1 + 3 \times 1 + 1 = 4 + 3 + 1 = 8$
As the question gives $u_1 = 8$, the formula is true for $n = 1$.

All inductions need to be shown to be true for a small number, usually 1. In this question $u_1 = 8$ is part of the data of the question and you have to start by showing that $u_n = 4^n + 3n + 1$ satisfies $u_1 = 8$.

Assume the formula is true for $n = k$.
That is $u_k = 4^k + 3k + 1$ ... ... ∗

The **induction hypothesis** is just the formula you are asked to prove with the ns replaced by ks.

$u_{k+1} = 4u_k - 9k$
$= 4(4^k + 3k + 1) - 9k$, using ∗
$= 4^{k+1} + 12k + 4 - 9k$
$= 4^{k+1} + 3k + 4$
$= 4^{k+1} + 3(k+1) + 1$

The induction hypothesis allows you to substitute $4^k + 3k + 1$ for u_k.

This is the result obtained by substituting $n = k+1$ into the formula $u_n = 4^n + 3n + 1$ and so the formula is true for $n = k+1$.

The formula is true for $n = 1$, and, if it is true for $n = k$, then it is true for $n = k+1$.

By mathematical induction the formula is true for all positive integers n.

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Review Exercise
Exercise A, Question 45

Question:

Given that $u_1 = 0$ and $u_{r+1} = 2r - u_r$, use mathematical induction to prove that $2u_n = 2n - 1 + (-1)^n, n \varepsilon \mathbb{Z}^+$.

Solution:

$2u_n = 2n - 1 + (-1)^n$

Let $n = 1$

$2u_1 = 2 - 1 + (-1)^1 = 2 - 1 - 1 = 0 \Rightarrow u_1 = 0$

As the question gives $u_1 = 0$, the formula is true for $n = 1$.

Assume the formula is true for $n = k$.

That is $2u_k = 2k - 1 + (-1)^k$ … … ∗

$u_{k+1} = 2k - u_k$

Replacing the r by a k in $u_{r+1} = 2r - u_r$. This question has used r in the data in the question where n has been used in the previous questions in this exercise. The letters used are symbols and which particular letter is used is makes no difference to the question or the way you solve it .

$2u_{k+1} = 4k - 2u_k = 4k - \left(2k - 1 + (-1)^k\right)$, using ∗

$= 4k - 2k + 1 - (-1)^k$

$= 2k + 1 + (-1)^{k+1}$

$-(-1)^k = (-1)(-1)^k = (-1)^1(-1)^k = (-1)^{k+1}$

$= 2(k+1) - 1 + (-1)^{k+1}$

This is the result obtained by substituting $n = k + 1$ into the formula $2u_n = 2n - 1 + (-1)^n$ and so the formula is true for $n = k + 1$.

The formula is true for $n = 1$, and, if it is true for $n = k$, then it is true for $n = k + 1$.

By mathematical induction the formula is true for all positive integers n.

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Review Exercise
Exercise A, Question 46

Question:

$f(n) = (2n + 1)7^n - 1$.

Prove by induction that, for all positive integers n, $f(n)$ is divisible by 4.

Solution:

$f(n) = (2n + 1)7^n - 1$

Let $n = 1$

$f(1) = (2 \times 1 + 1)7^1 - 1 = 3 \times 7 - 1 = 20$

20 is divisible by 4, so "f(n) is divisible by 4" is true for $n = 1$.

All inductions need to be shown to be true for a small number, usually 1. In this question, you need to check that when you substitute $n = 1$, the number you obtain is divisible exactly by 4.

Consider $f(k+1) - f(k)$

$f(k+1) - f(k) = (2(k+1)+1)7^{k+1} - 1 - ((2k+1)7^k - 1)$

$= (2k+3)7^{k+1} - (2k+1)7^k$

$= (2k+3)7 \times 7^k - (2k+1)7^k$

$= (14k+21)7^k - (2k+1)7^k$

$= (14k+21-2k-1)7^k$

$= (12k+20)7^k = 4(3k+5)7^k$ ... ✱

So 4 is a factor of $f(k+1) - f(k)$.

When you are trying to prove, by induction, that an expression $f(n)$ is divisible by a number, it is often a good start to try and show that $f(k+1) - f(k)$ is divisible by the same number. This does not always work but it is worth trying!

Showing that "$f(k+1) - f(k)$ has a factor of 4" is exactly the same thing as showing that "$f(k+1) - f(k)$ is divisible by 4".

Assume that $f(k)$ is divisible by 4.

It would follow that $f(k) = 4m$, where m is an integer.

From ✱

$f(k+1) = f(k) + 4(3k+5)7^k$

$= 4m + 4(3k+5)7^k$

$= 4(m + (3k+5)7^k)$

So $f(k+1)$ is divisible by 4.

The essential point here is that if both $f(k)$ and $4(3k+5)7^k$ are divisible by 4, then their sum, $f(k+1)$ is divisible by 4.

$f(n)$ is divisible by 4 for $n = 1$, and, if it is divisible by 4 for $n = k$, then it divisible by 4 for $n = k+1$.

By mathematical induction, $f(n)$ is divisible by 4 for all positive integers n.

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Review Exercise
Exercise A, Question 47

Question:

$\mathbf{A} = \begin{pmatrix} 1 & c \\ 0 & 2 \end{pmatrix}$, where c is a constant.

Prove by induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 1 & (2^n - 1)c \\ 0 & 2^n \end{pmatrix}$$

Solution:

$$\mathbf{A}^n = \begin{pmatrix} 1 & (2^n - 1)c \\ 0 & 2^n \end{pmatrix}$$

Let $n = 1$

$$\mathbf{A}^1 = \begin{pmatrix} 1 & (2^1 - 1)c \\ 0 & 2^1 \end{pmatrix} = \begin{pmatrix} 1 & c \\ 0 & 2 \end{pmatrix}$$

This is $\mathbf{A}$, as defined in the question, so the result is true for $n = 1$.

You need to begin by showing the result is true for $n = 1$. You substitute $n = 1$ into the printed expression for $\mathbf{A}^n$ and check that you get the matrix $\mathbf{A}$ as given in the question.

Assume the result is true for $n = k$.

That is $\mathbf{A}^k = \begin{pmatrix} 1 & (2^k - 1)c \\ 0 & 2^k \end{pmatrix}$ … … $\ast$

$$\mathbf{A}^{k+1} = \mathbf{A}^k . \mathbf{A}$$
$$= \begin{pmatrix} 1 & (2^k - 1)c \\ 0 & 2^k \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & c + 2(2^k - 1)c \\ 0 & 2.2^k \end{pmatrix}$$
$$= \begin{pmatrix} 1 & c + 2^{k+1}c - 2c \\ 0 & 2^{k+1} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2^{k+1}c - c \\ 0 & 2^{k+1} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & (2^{k+1} - 1)c \\ 0 & 2^{k+1} \end{pmatrix}$$

Keep in mind as you multiply out the matrices that you are aiming at the expression $\mathbf{A}^n = \begin{pmatrix} 1 & (2^n - 1)c \\ 0 & 2^n \end{pmatrix}$ with each n replaced by $k+1$.

$2.2^k = 2^1 2^k = 2^{k+1}$ by one of the laws of indices. You use this twice.

This is the result obtained by substituting $n = k+1$ into the result $\mathbf{A}^n = \begin{pmatrix} 1 & (2^n - 1)c \\ 0 & 2^n \end{pmatrix}$ and so the result is true for $n = k+1$.

The result is true for $n = 1$, and, if it is true for $n = k$, then it is true for $n = k+1$.

By mathematical induction the result is true for all positive integers n.

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Review Exercise
Exercise A, Question 48

Question:

Given that $u_1 = 4$ and that $2u_{r+1} + u_r = 6$, use mathematical induction to prove that $u_n = 2 - \left(-\frac{1}{2}\right)^{n-2}$, for $n \varepsilon \mathbb{Z}^+$.

Solution:

$u_n = 2 - \left(-\frac{1}{2}\right)^{n-2}$

Let $n = 1$

$u_1 = 2 - \left(-\frac{1}{2}\right)^{1-2} = 2 - \left(-\frac{1}{2}\right)^{-1}$

$= 2 - (-2) = 4$

As the question gives $u_1 = 4$, the formula is true for $n = 1$.

All inductions need to be shown to be true for a small number, usually 1. In this question $u_1 = 4$ is part of the data of the question and you have to start by showing that $u_n = 4^n + 3n + 1$ satisfies $u_1 = 4$.

Using $a^{-1} = \frac{1}{a}$, $\left(-\frac{1}{2}\right)^{-1} = \frac{1}{-\frac{1}{2}} = 1 \times -\frac{2}{1} = -2$

Assume the formula is true for $n = k$.

That is $u_k = 2 - \left(-\frac{1}{2}\right)^{k-2}$ ... ... ∗

The **induction hypothesis** is just the formula you are asked to prove, with the ns replaced by ks.

$2u_{k+1} = 6 - u_k$

$2u_{k+1} = 6 - \left[2 - \left(-\frac{1}{2}\right)^{k-2}\right] = 4 + \left(-\frac{1}{2}\right)^{k-2}$, using ∗

The **induction hypothesis** allows you to substitute $2 - \left(-\frac{1}{2}\right)^{k-2}$ for u_k.

Hence, dividing both sides of the equation by 2

$u_{k+1} = 2 + \frac{1}{2}\left(-\frac{1}{2}\right)^{k-2}$

$= 2 - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)^{k-2} = 2 - \left(-\frac{1}{2}\right)^{k-1}$

$\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)^{k-2} = \left(-\frac{1}{2}\right)^{1}\left(-\frac{1}{2}\right)^{k-2}$
$= \left(-\frac{1}{2}\right)^{1+k-2} = \left(-\frac{1}{2}\right)^{k-1}$

This is the result obtained by substituting $n = k+1$ into the formula $u_n = 2 - \left(-\frac{1}{2}\right)^{n-2}$ and so the formula is true for $n = k+1$.

The formula is true for $n = 1$, and, if it is true for $n = k$, then it is true for $n = k+1$.

By mathematical induction the formula is true for all positive integers n, that is $n \in \mathbb{Z}^+$.

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 49

Question:

Prove by induction that, for all $n \varepsilon \mathbb{Z}^+$, $\sum_{r=1}^{n} r\left(\frac{1}{2}\right)^r = 2 - \left(\frac{1}{2}\right)^n (n+2)$.

Solution:

$$\sum_{r=1}^{n} r\left(\tfrac{1}{2}\right)^r = 2-\left(\tfrac{1}{2}\right)^n(n+2)$$

Let $n=1$.
The left-hand side becomes

$$\sum_{r=1}^{1} r\left(\tfrac{1}{2}\right)^r = 1\times\tfrac{1}{2}=\tfrac{1}{2}$$

$\sum_{r=1}^{1} r\left(\tfrac{1}{2}\right)^r$ consists of just one term. That is $r\left(\tfrac{1}{2}\right)^r$ with 1 substituted for r, which gives $\tfrac{1}{2}$.

The right-hand side becomes

$$2-\left(\tfrac{1}{2}\right)^1(1+2)=2-\tfrac{1}{2}\times 3=\tfrac{1}{2}$$

The left-hand side and the right-hand side are equal and so the summation is true for $n=1$.

Assume the summation is true for $n=k$.

That is $\sum_{r=1}^{k} r\left(\tfrac{1}{2}\right)^r = 2-\left(\tfrac{1}{2}\right)^k(k+2)$ ... ... ✱

$$\sum_{r=1}^{k+1} r\left(\tfrac{1}{2}\right)^r = \sum_{r=1}^{k} r\left(\tfrac{1}{2}\right)^r 2+(k+1)\left(\tfrac{1}{2}\right)^{k+1}$$

$$= 2-\left(\tfrac{1}{2}\right)^k(k+2)+(k+1)\left(\tfrac{1}{2}\right)^{k+1}\text{, using ✱}$$

$$= 2-\left(\tfrac{1}{2}\right)^{k+1}2(k+2)+(k+1)\left(\tfrac{1}{2}\right)^{k+1}$$

$$= 2-\left(\tfrac{1}{2}\right)^{k+1}\left[2(k+2)-(k+1)\right]$$

$$= 2-\left(\tfrac{1}{2}\right)^{k+1}\left[k+3\right]$$

$$= 2-\left(\tfrac{1}{2}\right)^{k+1}((k+1)+2)$$

You are aiming at an expression where the n in $\left(\tfrac{1}{2}\right)^n$, on the right-hand side of the summation in the question, has been replaced by $k+1$. Replacing $\left(\tfrac{1}{2}\right)^k$ by the equal $\left(\tfrac{1}{2}\right)^{k+1}\times 2$ will give you $\left(\tfrac{1}{2}\right)^{k+1}$ as a common factor of the second and third terms.

This is the result obtained by substituting $n=k+1$ into the right-hand side of the summation and so the summation is true for $n=k+1$.

The summation is true for $n=1$, and, if it is true for $n=k$, then it is true for $n=k+1$.

By mathematical induction the summation is true for all positive integers n, that is $n\in\mathbb{Z}^+$.

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Review Exercise
Exercise A, Question 50

Question:

$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$

Prove by induction that, for all positive integers n,

$\mathbf{A}^n = \begin{pmatrix} 2n+1 & n \\ -4n & -2n+1 \end{pmatrix}$

Solution:

$\mathbf{A}^n = \begin{pmatrix} 2n+1 & n \\ -4n & -2n+1 \end{pmatrix}$

Let $n=1$

$\mathbf{A}^1 = \begin{pmatrix} 2+1 & 1 \\ -4 & -2+1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$

This is $\mathbf{A}$, as defined in the question, so the result is true for $n=1$.

You need to begin by showing the result is true for $n=1$. You substitute $n=1$ into the printed expression for $\mathbf{A}^n$ and check that you get the matrix $\mathbf{A}$, as given in the question.

Assume the result is true for $n=k$.

That is $\mathbf{A}^k = \begin{pmatrix} 2k+1 & k \\ -4k & -2k+1 \end{pmatrix}$

The **induction hypothesis** is the result you are asked to prove with all the *n*s replaced by *k*s.

$\mathbf{A}^{k+1} = \mathbf{A}^k.\mathbf{A}$

$= \begin{pmatrix} 2k+1 & k \\ -4k & -2k+1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$

$= \begin{pmatrix} 3(2k+1)-4k & 2k+1-k \\ -12k-4(-2k+1) & -4k-(-2k+1) \end{pmatrix}$

$= \begin{pmatrix} 2k+3 & k+1 \\ -4k-4 & -2k+1 \end{pmatrix}$

$= \begin{pmatrix} 2(k+1)+1 & k+1 \\ -4(k+1) & -2(k+1)+1 \end{pmatrix}$

$\mathbf{A}^{k+1}$ is the matrix $\mathbf{A}$, multiplied by itself k times, multiplied by $\mathbf{A}$ one more time. $\mathbf{A}^{k+1} = \mathbf{A}^k\mathbf{A}^1 = \mathbf{A}^k.\mathbf{A}$. This is one of the index laws applied to matrices.

This is the result obtained by substituting $n=k+1$ into the result $\mathbf{A}^n = \begin{pmatrix} 2n+1 & n \\ -4n & -2n+1 \end{pmatrix}$ and so the result is true for $n=k+1$.

The result is true for $n=1$, and, if it is true for $n=k$, then it is true for $n=k+1$.

By mathematical induction the result is true for all positive integers n.

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Review Exercise
Exercise A, Question 51

Question:

Given that $f(n) = 3^{4n} + 2^{4n+2}$,

a show that, for $k \in \mathbb{Z}^+$, $f(k+1) - f(k)$ is divisible by 15,

b prove that, for $n \in \mathbb{Z}^+$, $f(n)$ is divisible by 5.

Solution:

a $f(n) = 3^{4n} + 2^{4n+2}$

$$f(k+1) - f(k) = 3^{4(k+1)} + 2^{4(k+1)+2} - \left(3^{4k} + 2^{4k+2}\right)$$
$$= 3^{4k+4} - 3^{4k} + 2^{4k+6} - 2^{4k+2}$$
$$= 3^{4k}\left(3^4 - 1\right) + 2^{4k}\left(2^6 - 2^2\right)$$
$$= 3^{4k} \times 80 + 2^{4k} \times 60$$
$$= 3^{4k-1} \times 3 \times 80 + 2^{4k} \times 60$$
$$= 240 \times 3^{4k-1} + 60 \times 2^{4k}$$
$$= 15\left(16 \times 3^{4k-1} + 4 \times 2^{4k}\right) \quad *$$

For all $k \in \mathbb{Z}^+$, $\left(16 \times 3^{4k-1} + 4 \times 2^{4k}\right)$ is an integer, and, hence, $f(k+1) - f(k)$ is divisible by 15.

> At this stage $f(k+1) - f(k)$ is clearly divisible by 10 (and 20) but to obtain that the expression is divisible by 15, you have to obtain a 3, to go with the 80, by writing 3^{4k} as $3^{4k-1} \times 3^1$.

> This shows that 15 is a factor of $f(k+1) - f(k)$ and this is the equivalent to showing that $f(k+1) - f(k)$ is exactly divisible by 15. Note that the result would not be true for negative integers as, for example, 4×2^{4k} would be a fraction less than one.

b Let $n = 1$

$f(1) = 3^4 + 2^6 = 81 + 64 = 145 = 5 \times 29$

So $f(n)$ is divisible by 5 for $n = 1$.

Assume that $f(k)$ is divisible by 5.

It would follow that $f(k) = 5m$, where m is an integer.

From $*$

$$f(k+1) = f(k) + 15\left(16 \times 3^{4k-1} + 4 \times 2^{4k}\right)$$
$$= 5m + 15\left(16 \times 3^{4k-1} + 4 \times 2^{4k}\right)$$
$$= 5\left(m + 3\left(16 \times 3^{4k-1} + 4 \times 2^{4k}\right)\right)$$

So $f(k+1)$ is divisible by 5.

> An expression which is divisible by 15 is certainly divisible by 5, which is all that is required for part (b).

> If both $f(k)$ and $15\left(16 \times 3^{4k-1} + 4 \times 2^{4k}\right)$ are divisible by 5, then their sum, $f(k+1)$ is divisible by 5.

$f(n)$ is divisible by 5 for $n = 1$, and, if it is divisible by 5 for $n = k$, then it divisible by 5 for $n = k+1$.

By mathematical induction, $f(n)$ is divisible by 5 for all $n \in \mathbb{Z}^+$.

> Although $f(k+1) - f(k)$ is divisible by 15, $f(n)$ is never divisible by 15 for any n. After reading part (a), you might misread the question and attempt to prove that 15 was a factor of $f(n)$. It is always necessary to read questions carefully.

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Review Exercise
Exercise A, Question 52

Question:

$f(n) = 24 \times 2^{4n} + 3^{4n}$, where n is a non-negative integer.

a Write down $f(n+1) - f(n)$.

b Prove, by induction, that $f(n)$ is divisible by 5.

Solution:

a $f(n) = 24 \times 2^{4n} + 3^{4n}$

$f(n+1) - f(n) = 24 \times 2^{4(n+1)} + 3^{4(n+1)} - 24 \times 2^{4n} - 3^{4n}$

This is an acceptable answer for part (a). However, reading ahead, the question concerns divisibility by 5. So it is sensible to further work on this expression and show that it is divisible by 5.

b $f(n+1) - f(n)$

$= 24 \times 2^{4n+4} - 24 \times 2^{4n} + 3^{4n+4} - 3^{4n}$

$= 24 \times 2^{4n}(2^4 - 1) + 3^{4n}(3^4 - 1)$

$= 24 \times 2^{4n} \times 15 + 3^{4n} \times 80$

$= 5(72 \times 2^{4n} + 16 \times 3^{4n})$ … ∗

In the middle of a question it is easy to forget that, in all inductions, you need to show that the result is true for a small number. This is usually 1 but this question asks you to show a result is true for all non-negative integers and 0 is a non-negative integer, so you should begin with 0.

Let $n = 0$

$f(0) = 24 \times 2^0 + 3^0 = 24 + 1 = 25$

So $f(n)$ is divisible by 5 for $n = 0$.

Assume that $f(k)$ is divisible by 5.

It would follow that $f(k) = 5m$, where m is an integer.

From ∗, substituting $n = k$ and rearranging,

$f(k+1) = f(k) + 5(72 \times 2^{4n} + 16 \times 3^{4n})$

$= 5m + 5(72 \times 2^{4n} + 16 \times 3^{4n})$

$= 5(m + 72 \times 2^{4n} + 16 \times 3^{4n})$

So $f(k+1)$ is divisible by 5.

$f(n)$ is divisible by 5 for $n = 0$, and, if it is divisible by 5 for $n = k$, then it divisible by 5 for $n = k+1$.

By mathematical induction, $f(n)$ is divisible by 5 for all non-negative integers n.

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Review Exercise
Exercise A, Question 53

Question:

Prove that the expression $7^n + 4^n + 1$ is divisible by 6 for all positive integers n.

Solution:

Let $f(n) = 7^n + 4^n + 1$

> If the question gives no label to the function, here $7^n + 4^n + 1$, it helps if you call it $f(n)$. You are going to have to refer to this function a number of times in your solution.

Let $n = 1$

$f(1) = 7^1 + 4^1 + 1 = 12$

12 is divisible by 6, so f(n) is divisible by 6 for $n = 1$.

Consider $f(k+1) - f(k)$

> This question gives you no hint to help you. With divisibility questions, it often helps to consider $f(k+1) - f(k)$ and try and show that this divides by the appropriate number, here 6. It does not always work and there are other methods which often work just as well or better. You should compare this question with questions **54** and **57** in this Review Exercise.

$$
\begin{aligned}
f(k+1) - f(k) &= 7^{k+1} + 4^{k+1} + 1 - \left(7^k + 4^k + 1\right) \\
&= 7^{k+1} - 7^k + 4^{k+1} - 4^k \\
&= 7^k(7-1) + 4^k(4-1) \\
&= 6 \times 7^k + 3 \times 4^k \\
&= 6 \times 7^k + 3 \times 4 \times 4^{k-1} \\
&= 6\left(7^k + 2 \times 4^{k-1}\right) \ \ldots \ *
\end{aligned}
$$

So 6 is a factor of $f(k+1) - f(k)$.

Assume that $f(k)$ is divisible by 6.

It would follow that $f(k) = 6m$, where m is an integer.

From $*$

$$
\begin{aligned}
f(k+1) &= f(k) + 6\left(7^k + 2 \times 4^{k-1}\right) \\
&= 6m + 6\left(7^k + 2 \times 4^{k-1}\right) \\
&= 6\left(m + 7^k + 2 \times 4^{k-1}\right)
\end{aligned}
$$

> If both $f(k)$ and $6\left(7^k + 2 \times 4^{k-1}\right)$ are divisible by 6, then their sum, $f(k+1)$ is divisible by 6. You could write this down instead of the working shown here.

So $f(k+1)$ is divisible by 6.

$f(n)$ is divisible by 6 for $n = 1$, and, if it is divisible by 6 for $n = k$, then it divisible by 6 for $n = k+1$.

By mathematical induction, $f(n)$ is divisible by 6 for all positive integers n.

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 54

Question:

Prove by induction that $4^n + 6n - 1$ is divisible by 9 for $n \varepsilon \mathbb{Z}^+$.

Solution:

If the question gives no label to the function, here $4^n + 6n - 1$, it helps if you call it $f(n)$. You are going to have to refer to this function a number of times in your solution.

Let $f(n) = 4^n + 6n - 1$

Let $n = 1$

$f(1) = 4^1 + 6 - 1 = 9$

So $f(n)$ is divisible by 9 for $n = 1$.

With divisibility questions, it often helps to consider $f(k+1) - f(k)$ and try and show that this divides by the appropriate number, here 9. This will work in this question. However the method shown here is, for this question, a neat one and you need to be aware of various alternative methods. No particular method works every time.

Assume that $f(k)$ is divisible by 9.

Then, for some integer m,

$f(k) = 4^k + 6k - 1 = 9m$

Rearranging

$4^k = 9m - 6k + 1 \quad \ldots \ast$

Here you substitute the expression for 4^k in $\ast$ for the 4^k in your expression for $f(k+1)$.

$$\begin{aligned} f(k+1) &= 4^{k+1} + 6(k+1) - 1 \\ &= 4 \times 4^k + 6k + 5 \\ &= 4 \times (9m - 6k + 1) + 6k + 5 \\ &= 36m - 24k + 4 + 6k + 5 = 36m - 18k + 9 \\ &= 9(4m - 2k + 1) \end{aligned}$$

This is divisible by 9.

$f(n)$ is divisible by 9 for $n = 1$, and, if it is divisible by 9 for $n = k$, then it divisible by 9 for $n = k + 1$.

By mathematical induction, $f(n)$ is divisible by 9 for all $n \in \mathbb{Z}^+$.

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 55

Question:

Prove that the expression $3^{4n-1}+2^{4n-1}+5$ is divisible by 10 for all positive integers *n*.

Solution:

Let $\text{f}(n)=3^{4n-1}+2^{4n-1}+5$

Let $n=1$

$\text{f}(1)=3^3+2^3+5=27+8+5=40=10\times 4$

So $\text{f}(n)$ is divisible by 10 for $n=1$.

Consider $\text{f}(k+1)-\text{f}(k)$

$$\begin{aligned}\text{f}(k+1)-\text{f}(k) &= 3^{4k+3}+2^{4k+3}-5-\left(3^{4k-1}+2^{4k-1}-5\right)\\ &= 3^{4k+3}-3^{4k-1}+2^{4k+3}-2^{4k-1}\\ &= 3^{4k-1}\left(3^4-1\right)+2^{4k-3}\left(2^6-2^2\right)\\ &= 3^{4k-1}\times 80+2^{4k-3}\times 30\\ &= 10\left(8\times 3^{4k-1}+3\times 2^{4k-3}\right)\ \dots\ *\end{aligned}$$

When you replace n by $k+1$ in, for example, 3^{4n-1} you get $3^{4(k+1)-1}=3^{4k+4-1}=3^{4k+3}$.

The index manipulation is quite complicated here. For example, $2^{4k-3}\times 2^6=2^{4k-3+6}=2^{4k+3}$.

Assume that $\text{f}(k)$ is divisible by 10.

It would follow that $\text{f}(k)=10m$, where m is an integer.

From *

$$\begin{aligned}\text{f}(k+1) &= \text{f}(k)+10\left(8\times 3^{4k-1}+3\times 2^{4k-3}\right)\\ &= 10m+10\left(8\times 3^{4k-1}+3\times 2^{4k-3}\right)\\ &= 10\left(m+\left(8\times 3^{4k-1}+3\times 2^{4k-3}\right)\right)\end{aligned}$$

If both $\text{f}(k)$ and $10\left(8\times 3^{4k-1}+3\times 2^{4k-3}\right)$ are divisible by 10, then their sum, $\text{f}(k+1)$ is divisible by 10. If you preferred, you could write this down instead of the working shown here.

So $\text{f}(k+1)$ is divisible by 10.

$\text{f}(n)$ is divisible by 10 for $n=1$, and, if it is divisible by 10 for $n=k$, then it divisible by 10 for $n=k+1$.

By mathematical induction, $\text{f}(n)$ is divisible by 10 for all positive integers n.

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Review Exercise
Exercise A, Question 56

Question:

a Express $\frac{6x+10}{x+3}$ in the form $p+\frac{q}{x+3}$, where p and q are integers to be found.

The sequence of real numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 = 5.2$ and $u_{n+1} = \frac{6u_n+10}{u_n+3}$.

b Prove by induction that $u_n > 5$, for $n \in \mathbb{Z}^+$.

Solution:

a $\frac{6x+10}{x+3} = \frac{6x+18-8}{x+3} = \frac{6(x+3)-8}{x+3}$

$= \frac{6(x+3)}{x+3} - \frac{8}{x+3} = 6 - \frac{8}{x+3}$

$p = 6$, $q = -8$

You may use any correct method to carry out the division in part (a). Methods can be found in Chapter 1 of Edexcel AS and A-Level Modular Mathematics, Core Mathematics 2.

b $u_1 = 5.2 > 5$

So $u_n > 5$ for $n = 1$.

It is obvious that $5.2 > 5$ but all inductions need to be shown to be true for a small number, usually 1, and you must remember to write down that $5.2 > 5$ shows that the result is true for $n = 1$.

Assume that $u_k > 5$

If $u_k > 5$, there exists a positive number ε such that $u_k = 5 + \varepsilon$.

$u_{k+1} = \frac{6u_k+10}{u_k+3} = 6 - \frac{8}{u_k+3}$, using the result in part (a)

$= 6 - \frac{8}{5+\varepsilon+3} = 6 - \frac{8}{8+\varepsilon}$

$> 6 - 1 = 5$

If $\varepsilon > 0$ then $\frac{8}{8+\varepsilon}$ is less than one – the numerator is smaller than the denominator. It follows that $6 - \frac{8}{8+\varepsilon}$ will be bigger than 5.

So $u_{k+1} > 5$

$u_1 > 5$ and, if $u_k > 5$, then $u_{k+1} > 5$.

By mathematical induction, $u_n > 5$ for all $n \in \mathbb{Z}^+$.

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Review Exercise
Exercise A, Question 57

Question:

Given that $n \varepsilon \mathbb{Z}^+$, prove, by mathematical induction, that $2(4^{2n+1}) + 3^{3n+1}$ is divisible by 11.

Solution:

Let $f(n) = 2(4^{2n+1}) + 3^{3n+1}$

Let $n = 1$

$f(1) = 2(4^{2+1}) + 3^{3+1} = 2 \times 4^3 + 3^4$

$= 2 \times 64 + 81 = 209 = 11 \times 19$

So $f(n)$ is divisible by 11 for $n = 1$.

Assume that $f(k)$ is divisible by 11.

Then, for some integer m,

$f(k) = 2(4^{2k+1}) + 3^{3k+1} = 11m$

> The method of considering $f(k+1) - f(k)$ is very difficult to make work in this question and this alternative method is easier here.

Rearranging

$2(4^{2k+1}) = 11m - 3^{3k+1}$ … ✱

$f(k+1) = 2(4^{2k+3}) + 3^{3k+4}$

$= 2(4^{2k+1} \times 4^2) + 3^{3k+4}$

$= 16 \times 2(4^{2k+1}) + 3^{3k+4}$

> Here you substitute the expression for $2(4^{2k+1})$ in ✱ for the $2(4^{2k+1})$ in your expression for $f(k+1)$.

$= 16 \times (11m - 3^{3k+1}) + 3^{3k+4}$

> $3^{3k+4} = 3^{3k+1+3} = 3^{3k+1} \times 3^3 = 27 \times 3^{3k+1}$
> This is needed to get a common factor of 3^{3k+1} in the second and third terms of the expression.

$= 176m - 16 \times 3^{3k+1} + 27 \times 3^{3k+1}$

$= 176m + 11 \times 3^{3k+1}$

$= 11(16m + 3^{3k+1})$

This is divisible by 11.

$f(n)$ is divisible by 11 for $n = 1$, and, if it is divisible by 11 for $n = k$, then it is divisible by 11 for $n = k + 1$.

By mathematical induction, $f(n)$ is divisible by 11 for all $n \in \mathbb{Z}^+$.