Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction
Exercise A, Question 1

Question:

Prove by the method of mathematical induction, the following statement for $n \in \mathbb{Z}^+$.

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

Solution:

$$n = 1;\ \text{LHS} = \sum_{r=1}^{1} r = 1$$
$$\text{RHS} = \frac{1}{2}(1)(2) = 1$$

As LHS = RHS, the summation formula is true for $n = 1$.

Assume that the summation formula is true for $n = k$.

ie. $\sum_{r=1}^{k} r = \frac{1}{2}k(k+1)$.

With $n = k + 1$ terms the summation formula becomes:

$$\begin{aligned}\sum_{r=1}^{k+1} r &= 1 + 2 + 3 + \geq + k + (k+1) \\ &= \frac{1}{2}k(k+1) + (k+1) \\ &= \frac{1}{2}(k+1)(k+2) \\ &= \frac{1}{2}(k+1)(k+1+1)\end{aligned}$$

Therefore, summation formula is true when $n = k + 1$.

If the summation formula is true for $n = k$, then it is shown to be true for $n = k + 1$. As the result is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise A, Question 2

Question:

Prove by the method of mathematical induction, the following statement for $n \in \mathbb{Z}^+$.

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

Solution:

$$n = 1;\ \text{LHS} = \sum_{r=1}^{1} r^3 = 1$$

$$\text{RHS} = \frac{1}{4}(1)^2(2)^2 = \frac{1}{4}(4) = 1$$

As LHS = RHS, the summation formula is true for $n = 1$.

Assume that the summation formula is true for $n = k$.

ie. $\sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k+1)^2$.

With $n = k + 1$ terms the summation formula becomes:

$$\begin{aligned}\sum_{r=1}^{k+1} r^3 &= 1^3 + 2^3 + 3^3 + \geq + k^3 + (k+1)^3 \\ &= \frac{1}{4}k^2(k+1)^2 + (k+1)^3 \\ &= \frac{1}{4}(k+1)^2\left[k^2 + 4(k+1)\right] \\ &= \frac{1}{4}(k+1)^2(k^2 + 4k + 4) \\ &= \frac{1}{4}(k+1)^2(k+2)^2 \\ &= \frac{1}{4}(k+1)^2(k+1+1)^2\end{aligned}$$

Therefore, summation formula is true when $n = k + 1$.

If the summation formula is true for $n = k$, then it is shown to be true for $n = k + 1$. As the result is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise A, Question 3

Question:

Prove by the method of mathematical induction, the following statement for $n \in \mathbb{Z}^+$.

$$\sum_{r=1}^{n} r(r-1) = \frac{1}{3}n(n+1)(n-1)$$

Solution:

$$n = 1;\ \text{LHS} = \sum_{r=1}^{1} r(r-1) = 1(0) = 0$$

$$\text{RHS} = \frac{1}{3}(1)(2)(0) = 0$$

As LHS = RHS, the summation formula is true for $n = 1$.

Assume that the summation formula is true for $n = k$.

ie. $\sum_{r=1}^{k} r(r-1) = \frac{1}{3}k(k+1)(k-1)$.

With $n = k + 1$ terms the summation formula becomes:

$$\begin{aligned}\sum_{r=1}^{k+1} r(r-1) &= 1(0) + 2(1) + 3(2) + \geq + k(k-1) + (k+1)k \\ &= \frac{1}{3}k(k+1)(k-1) + (k+1)k \\ &= \frac{1}{3}k(k+1)[(k-1)+3] \\ &= \frac{1}{3}k(k+1)(k+2) \\ &= \frac{1}{3}(k+1)(k+2)k \\ &= \frac{1}{3}(k+1)(k+1+1)(k+1-1)\end{aligned}$$

Therefore, summation formula is true when $n = k + 1$.

If the summation formula is true for $n = k$, then it is shown to be true for $n = k + 1$. As the result is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise A, Question 4

Question:

Prove by the method of mathematical induction, the following statement for $n \in \mathbb{Z}^+$.

$(1 \times 6) + (2 \times 7) + (3 \times 8) + \geq + n(n+5) = \frac{1}{3}n(n+1)(n+8)$

Solution:

The identity $(1 \times 6) + (2 \times 7) + (3 \times 8) + \geq + n(n+5) = \frac{1}{3}n(n+1)(n+8)$ can be rewritten as $\sum_{r=1}^{n} r(r+5) = \frac{1}{3}n(n+1)(n+8)$.

$$n = 1;\ \text{LHS} = \sum_{r=1}^{1} r(r+5) = 1(6) = 6$$
$$\text{RHS} = \frac{1}{3}(1)(2)(9) = \frac{1}{3}(18) = 6$$

As LHS = RHS, the summation formula is true for $n = 1$.

Assume that the summation formula is true for $n = k$.

ie. $\sum_{r=1}^{k} r(r+5) = \frac{1}{3}k(k+1)(k+8)$.

With $n = k + 1$ terms the summation formula becomes:

$$\begin{aligned}
\sum_{r=1}^{k+1} r(r+5) &= 1(6) + 2(7) + 3(8) + \geq + k(k+5) + (k+1)(k+6) \\
&= \frac{1}{3}k(k+1)(k+8) + (k+1)(k+6) \\
&= \frac{1}{3}(k+1)[k(k+8) + 3(k+6)] \\
&= \frac{1}{3}(k+1)\left[k^2 + 8k + 3k + 18\right] \\
&= \frac{1}{3}(k+1)\left[k^2 + 11k + 18\right] \\
&= \frac{1}{3}(k+1)(k+9)(k+2) \\
&= \frac{1}{3}(k+1)(k+2)(k+9) \\
&= \frac{1}{3}(k+1)(k+1+1)(k+1+8)
\end{aligned}$$

Therefore, summation formula is true when $n = k + 1$.

If the summation formula is true for $n = k$, then it is shown to be true for $n = k + 1$. As the result is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise A, Question 5

Question:

Prove by the method of mathematical induction, the following statement for $n \in \mathbb{Z}^+$.

$$\sum_{r=1}^{n} r(3r-1) = n^2(n+1)$$

Solution:

$$n = 1; \text{LHS} = \sum_{r=1}^{1} r(3r-1) = 1(2) = 2$$

$$\text{RHS} = 1^2(2) = (1)(2) = 2$$

As LHS = RHS, the summation formula is true for $n = 1$.

Assume that the summation formula is true for $n = k$.

ie. $\sum_{r=1}^{k} r(3r-1) = k^2(k+1)$.

With $n = k+1$ terms the summation formula becomes:

$$\begin{aligned}\sum_{r=1}^{k+1} r(3r-1) &= 1(2) + 2(5) + 3(8) + \geq + k(3k-1) + (k+1)(3(k+1)-1) \\ &= k^2(k+1) + (k+1)(3k+3-1) \\ &= k^2(k+1) + (k+1)(3k+2) \\ &= (k+1)\left[k^2 + 3k + 2\right] \\ &= (k+1)(k+2)(k+1) \\ &= (k+1)^2(k+2) \\ &= (k+1)^2(k+1+1)\end{aligned}$$

Therefore, summation formula is true when $n = k+1$.

If the summation formula is true for $n = k$, then it is shown to be true for $n = k+1$. As the result is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise A, Question 6

Question:

Prove by the method of mathematical induction, the following statement for $n \in \mathbb{Z}^+$.

$$\sum_{r=1}^{n}(2r-1)^2 = \frac{1}{3}n(4n^2-1)$$

Solution:

$$n=1; \text{LHS} = \sum_{r=1}^{1}(2r-1)^2 = 1^2 = 1$$
$$\text{RHS} = \frac{1}{3}(1)(4-1) = \frac{1}{3}(1)(3) = 1$$

As LHS = RHS, the summation formula is true for $n = 1$.

Assume that the summation formula is true for $n = k$.

ie. $\sum_{r=1}^{k}(2r-1)^2 = \frac{1}{3}k(4k^2-1) = \frac{1}{3}k(2k+1)(2k-1)$.

With $n = k+1$ terms the summation formula becomes:

$$\begin{aligned}
\sum_{r=1}^{k+1}(2r-1)^2 &= 1^2 + 3^2 + 5^2 + \geq + (2k-1)^2 + (2(k+1)-1)^2 \\
&= \frac{1}{3}k(4k^2-1) + (2k+2-1)^2 \\
&= \frac{1}{3}k(4k^2-1) + (2k+1)^2 \\
&= \frac{1}{3}k(2k+1)(2k-1) + (2k+1)^2 \\
&= \frac{1}{3}(2k+1)[k(2k-1) + 3(2k+1)] \\
&= \frac{1}{3}(2k+1)\left[2k^2 - k + 6k + 3\right] \\
&= \frac{1}{3}(2k+1)\left[2k^2 + 5k + 3\right] \\
&= \frac{1}{3}(2k+1)(k+1)(2k+3) \\
&= \frac{1}{3}(k+1)(2k+3)(2k+1) \\
&= \frac{1}{3}(k+1)[2(k+1)+1][2(k+1)-1] \\
&= \frac{1}{3}(k+1)\left[4(k+1)^2 - 1\right]
\end{aligned}$$

Therefore, summation formula is true when $n = k+1$.

If the summation formula is true for $n = k$, then it is shown to be true for $n = k+1$. As the result is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise A, Question 7

Question:

Prove by the method of mathematical induction, the following statement for $n \in \mathbb{Z}^+$.

$$\sum_{r=1}^{n} 2^r = 2^{n+1} - 2$$

Solution:

$$n = 1;\ \text{LHS} = \sum_{r=1}^{1} 2^r = 2^1 = 2$$
$$\text{RHS} = 2^2 - 2 = 4 - 2 = 2$$

As LHS = RHS, the summation formula is true for $n = 1$.

Assume that the summation formula is true for $n = k$.

ie. $\sum_{r=1}^{k} 2^r = 2^{k+1} - 2$.

With $n = k + 1$ terms the summation formula becomes:

$$\begin{aligned}\sum_{r=1}^{k+1} 2^r &= 2^1 + 2^2 + 2^3 + \geq + 2^k + 2^{k+1}\\ &= 2^{k+1} - 2 + 2^{k+1}\\ &= 2(2^{k+1}) - 2\\ &= 2^1(2^{k+1}) - 2\\ &= 2^{1+k+1} - 2\\ &= 2^{k+1+1} - 2\end{aligned}$$

Therefore, summation formula is true when $n = k + 1$.

If the summation formula is true for $n = k$, then it is shown to be true for $n = k + 1$. As the result is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise A, Question 8

Question:

Prove by the method of mathematical induction, the following statement for $n \in \mathbb{Z}^+$.

$$\sum_{r=1}^{n} 4^{r-1} = \frac{4^n - 1}{3}$$

Solution:

$$n = 1;\ \text{LHS} = \sum_{r=1}^{1} 4^{r-1} = 4^0 = 1$$

$$\text{RHS} = \frac{4-1}{3} = \frac{3}{3} = 1$$

As LHS = RHS, the summation formula is true for $n = 1$.

Assume that the summation formula is true for $n = k$.

ie. $\sum_{r=1}^{k} 4^{r-1} = \frac{4^k - 1}{3}$.

With $n = k + 1$ terms the summation formula becomes:

$$\begin{aligned}\sum_{r=1}^{k+1} 4^{r-1} &= 4^0 + 4^1 + 4^2 + \geq + 4^{k-1} + 4^{k+1-1} \\ &= \frac{4^k - 1}{3} + 4^k \\ &= \frac{4^k - 1}{3} + \frac{3(4^k)}{3} \\ &= \frac{4^k - 1 + 3(4^k)}{3} \\ &= \frac{4(4^k) - 1}{3} \\ &= \frac{4^1(4^k) - 1}{3} \\ &= \frac{4^{k+1} - 1}{3}\end{aligned}$$

Therefore, summation formula is true when $n = k + 1$.

If the summation formula is true for $n = k$, then it is shown to be true for $n = k + 1$. As the result is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise A, Question 9

Question:

Prove by the method of mathematical induction, the following statement for $n \in \mathbb{Z}^+$.

$$\sum_{r=1}^{n} r(r!) = (n+1)! - 1$$

Solution:

$$n = 1;\ \text{LHS} = \sum_{r=1}^{1} r(r!) = 1(1!) = 1(1) = 1$$
$$\text{RHS} = 2! - 1 = 2 - 1 = 1$$

As LHS = RHS, the summation formula is true for $n = 1$.

Assume that the summation formula is true for $n = k$.

ie. $\sum_{r=1}^{k} r(r!) = (k+1)! - 1.$

With $n = k + 1$ terms the summation formula becomes:

$$\begin{aligned}\sum_{r=1}^{k+1} r(r!) &= 1(1!) + 2(2!) + 3(3!) + \geq + k(k!) + (k+1)[(k+1)!] \\ &= (k+1)! - 1 + (k+1)[(k+1)!] \\ &= (k+1)! + (k+1)[(k+1)!] - 1 \\ &= (k+1)!\,[1 + k + 1] - 1 \\ &= (k+1)!\,(k+2) - 1 \\ &= (k+2)! - 1 \\ &= (k+1+1)! - 1\end{aligned}$$

Therefore, summation formula is true when $n = k + 1$.

If the summation formula is true for $n = k$, then it is shown to be true for $n = k + 1$. As the result is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise A, Question 10

Question:

Prove by the method of mathematical induction, the following statement for $n \in \mathbb{Z}^+$.

$$\sum_{r=1}^{2n} r^2 = \frac{1}{3}n(2n+1)(4n+1)$$

Solution:

$$n = 1;\ \text{LHS} = \sum_{r=1}^{2} r^2 = 1^2 + 2^2 = 1 + 4 = 5$$

$$\text{RHS} = \frac{1}{3}(1)(3)(5) = \frac{1}{3}(15) = 5$$

As LHS = RHS, the summation formula is true for $n = 1$.

Assume that the summation formula is true for $n = k$.

ie. $\sum_{r=1}^{2k} r^2 = \frac{1}{3}k(2k+1)(4k+1)$.

With $n = k + 1$ terms the summation formula becomes:

$$\begin{aligned}\sum_{r=1}^{2(k+1)} r^2 = \sum_{r=1}^{2k+2} r^2 &= 1^2 + 2^2 + 3^2 + \geq + k^2 + (2k+1)^2 + (2k+2)^2 \\ &= \frac{1}{3}k(2k+1)(4k+1) + (2k+1)^2 + (2k+2)^2 \\ &= \frac{1}{3}k(2k+1)(4k+1) + (2k+1)^2 + 4(k+1)^2 \\ &= \frac{1}{3}(2k+1)[k(4k+1) + 3(2k+1)] + 4(k+1)^2 \\ &= \frac{1}{3}(2k+1)[4k^2 + 7k + 3] + 4(k+1)^2 \\ &= \frac{1}{3}(2k+1)(4k+3)(k+1) + 4(k+1)^2 \\ &= \frac{1}{3}(k+1)[(2k+1)(4k+3) + 12(k+1)] \\ &= \frac{1}{3}(k+1)[8k^2 + 6k + 4k + 3 + 12k + 12] \\ &= \frac{1}{3}(k+1)[8k^2 + 22k + 15] \\ &= \frac{1}{3}(k+1)(2k+3)(4k+5) \\ &= \frac{1}{3}(k+1)[2(k+1)+1][4(k+1)+1]\end{aligned}$$

Therefore, summation formula is true when $n = k + 1$.

If the summation formula is true for $n = k$, then it is shown to be true for $n = k + 1$. As the result is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction
Exercise B, Question 1

Question:

Use the method of mathematical induction to prove the following statement for $n \in \mathbb{Z}^+$.

$8^n - 1$ is divisible by 7

Solution:

Let $f(n) = 8^n - 1$, where $n \in \mathbb{Z}^+$.

$\therefore\ f(1) = 8^1 - 1 = 7$, which is divisible by 7.

$\therefore\ f(n)$ is divisible by 7 when $n = 1$.

Assume that for $n = k$,

$f(k) = 8^k - 1$ is divisible by 7 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore\ f(k+1) &= 8^{k+1} - 1\\ &= 8^k.8^1 - 1\\ &= 8(8^k) - 1\end{aligned}$$

$$\begin{aligned}\therefore\ f(k+1) - f(k) &= [8(8^k) - 1] - [8^k - 1]\\ &= 8(8^k) - 1 - 8^k + 1\\ &= 7(8^k)\end{aligned}$$

$\therefore\ f(k+1) = f(k) + 7(8^k)$

As both $f(k)$ and $7(8^k)$ are divisible by 7 then the sum of these two terms must also be divisible by 7. Therefore $f(n)$ is divisible by 7 when $n = k + 1$.

If $f(n)$ is divisible by 7 when $n = k$, then it has been shown that $f(n)$ is also divisible by 7 when $n = k + 1$. As $f(n)$ is divisible by 7 when $n = 1$, $f(n)$ is also divisible by 7 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction
Exercise B, Question 2

Question:

Use the method of mathematical induction to prove the following statement for $n \in \mathbb{Z}^+$.

$3^{2n} - 1$ is divisible by 8

Solution:

Let $f(n) = 3^{2n} - 1$, where $n \in \mathbb{Z}^+$.

$\therefore$ $f(1) = 3^{2(1)} - 1 = 9 - 1 = 8$, which is divisible by 8.

$\therefore$ $f(n)$ is divisible by 8 when $n = 1$.

Assume that for $n = k$,

$f(k) = 3^{2k} - 1$ is divisible by 8 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore f(k+1) &= 3^{2(k+1)} - 1 \\ &= 3^{2k+2} - 1 \\ &= 3^{2k}.3^2 - 1 \\ &= 9(3^{2k}) - 1\end{aligned}$$

$$\begin{aligned}\therefore f(k+1) - f(k) &= [9(3^{2k}) - 1] - [3^{2k} - 1] \\ &= 9(3^{2k}) - 1 - 3^{2k} + 1 \\ &= 8(3^{2k})\end{aligned}$$

$$\therefore f(k+1) = f(k) + 8(3^{2k})$$

As both $f(k)$ and $8(3^{2k})$ are divisible by 8 then the sum of these two terms must also be divisible by 8. Therefore $f(n)$ is divisible by 8 when $n = k + 1$.

If $f(n)$ is divisible by 8 when $n = k$, then it has been shown that $f(n)$ is also divisible by 8 when $n = k + 1$. As $f(n)$ is divisible by 8 when $n = 1$, $f(n)$ is also divisible by 8 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise B, Question 3

Question:

Use the method of mathematical induction to prove the following statement for $n \in \mathbb{Z}^+$.

$5^n + 9^n + 2$ is divisible by 4

Solution:

Let $f(n) = 5^n + 9^n + 2$, where $n \in \mathbb{Z}^+$.

$\therefore\ f(1) = 5^1 + 9^1 + 2 = 5 + 9 + 2 = 16$, which is divisible by 4.

$\therefore\ f(n)$ is divisible by 4 when $n = 1$.

Assume that for $n = k$,

$f(k) = 5^k + 9^k + 2$ is divisible by 4 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore\ f(k+1) &= 5^{k+1} + 9^{k+1} + 2\\ &= 5^k.5^1 + 9^k.9^1 + 2\\ &= 5(5^k) + 9(9^k) + 2\end{aligned}$$

$$\begin{aligned}\therefore\ f(k+1) - f(k) &= [5(5^k) + 9(9^k) + 2] - [5^k + 9^k + 2]\\ &= 5(5^k) + 9(9^k) + 2 - 5^k - 9^k - 2\\ &= 4(5^k) + 8(9^k)\\ &= 4[5^k + 2(9)^k]\end{aligned}$$

$$\therefore\ f(k+1) = f(k) + 4[5^k + 2(9)^k]$$

As both $f(k)$ and $4[5^k + 2(9)^k]$ are divisible by 4 then the sum of these two terms must also be divisible by 4. Therefore f (n) is divisible by 4 when $n = k + 1$.

If $f(n)$ is divisible by 4 when $n = k$, then it has been shown that $f(n)$ is also divisible by 4 when $n = k + 1$. As $f(n)$ is divisible by 4 when $n = 1$, $f(n)$ is also divisible by 4 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise B, Question 4

Question:

Use the method of mathematical induction to prove the following statement for $n \in \mathbb{Z}^+$.

$2^{4n} - 1$ is divisible by 15

Solution:

Let $f(n) = 2^{4n} - 1$, where $n \in \mathbb{Z}^+$.

$\therefore\ f(1) = 2^{4(1)} - 1 = 16 - 1 = 15$, which is divisible by 15.

$\therefore\ f(n)$ is divisible by 15 when $n = 1$.

Assume that for $n = k$,

$f(k) = 2^{4k} - 1$ is divisible by 15 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore\ f(k+1) &= 2^{4(k+1)} - 1 \\ &= 2^{4k+4} - 1 \\ &= 2^{4k}.2^4 - 1 \\ &= 16(2^{4k}) - 1\end{aligned}$$

$$\begin{aligned}\therefore\ f(k+1) - f(k) &= [16(2^{4k}) - 1] - [2^{4k} - 1] \\ &= 16(2^{4k}) - 1 - 2^{4k} + 1 \\ &= 15(8^k)\end{aligned}$$

$$\therefore\ f(k+1) = f(k) + 15(8^k)$$

As both $f(k)$ and $15(8^k)$ are divisible by 15 then the sum of these two terms must also be divisible by 15. Therefore f(n) is divisible by 15 when $n = k + 1$.

If f(n) is divisible by 15 when $n = k$, then it has been shown that f(n) is also divisible by 15 when $n = k + 1$. As f(n) is divisible by 15 when $n = 1$, f(n) is also divisible by 15 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise B, Question 5

Question:

Use the method of mathematical induction to prove the following statement for $n \in \mathbb{Z}^+$.

$3^{2n-1} + 1$ is divisible by 4

Solution:

Let $f(n) = 3^{2n-1} + 1$, where $n \in \mathbb{Z}^+$.

$\therefore\ f(1) = 3^{2(1)-1} + 1 = 3 + 1 = 4$, which is divisible by 4.

$\therefore\ f(n)$ is divisible by 4 when $n = 1$.

Assume that for $n = k$,

$f(k) = 3^{2k-1} + 1$ is divisible by 4 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore\ f(k+1) &= 3^{2(k+1)-1} + 1 \\ &= 3^{2k+2-1} + 1 \\ &= 3^{2k-1}.3^2 + 1 \\ &= 9\left(3^{2k-1}\right) + 1\end{aligned}$$

$$\begin{aligned}\therefore\ f(k+1) - f(k) &= \left[9(3^{2k-1}) + 1\right] - \left[3^{2k-1} + 1\right] \\ &= 9\left(3^{2k-1}\right) + 1 - 3^{2k-1} - 1 \\ &= 8\left(3^{2k-1}\right)\end{aligned}$$

$$\therefore\ f(k+1) = f(k) + 8\left(3^{2k-1}\right)$$

As both $f(k)$ and $8\left(3^{2k-1}\right)$ are divisible by 4 then the sum of these two terms must also be divisible by 4. Therefore $f(n)$ is divisible by 4 when $n = k + 1$.

If $f(n)$ is divisible by 4 when $n = k$, then it has been shown that $f(n)$ is also divisible by 4 when $n = k + 1$. As $f(n)$ is divisible by 4 when $n = 1$, $f(n)$ is also divisible by 8 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise B, Question 6

Question:

Use the method of mathematical induction to prove the following statement for $n \in \mathbb{Z}^+$.

$n^3 + 6n^2 + 8n$ is divisible by 3

Solution:

Let $f(n) = n^3 + 6n^2 + 8n$, where $n \geq 1$ and $n \in \mathbb{Z}^+$.

$\therefore$ $f(1) = 1 + 6 + 8 = 15$, which is divisible by 3.

$\therefore$ $f(n)$ is divisible by 3 when $n = 1$.

Assume that for $n = k$,

$f(k) = k^3 + 6k^2 + 8k$ is divisible by 3 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore\ f(k+1) &= (k+1)^3 + 6(k+1)^2 + 8(k+1)\\ &= k^3 + 3k^2 + 3k + 1 + 6(k^2 + 2k + 1) + 8(k+1)\\ &= k^3 + 3k^2 + 3k + 1 + 6k^2 + 12k + 6 + 8k + 8\\ &= k^3 + 9k^2 + 23k + 15\end{aligned}$$

$$\begin{aligned}\therefore\ f(k+1) - f(k) &= [k^3 + 9k^2 + 23k + 15] - [k^3 + 6k^2 + 8k]\\ &= 3k^2 + 15k + 15\\ &= 3(k^2 + 5k + 5)\end{aligned}$$

$$\therefore\ f(k+1) = f(k) + 3\left(k^2 + 5k + 5\right)$$

As both $f(k)$ and $3\left(k^2 + 5k + 5\right)$ are divisible by 3 then the sum of these two terms must also be divisible by 3.

Therefore $f(n)$ is divisible by 3 when $n = k + 1$.

If $f(n)$ is divisible by 3 when $n = k$, then it has been shown that $f(n)$ is also divisible by 3 when $n = k + 1$. As $f(n)$ is divisible by 3 when $n = 1$, $f(n)$ is also divisible by 3 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction
Exercise B, Question 7

Question:

Use the method of mathematical induction to prove the following statement for $n \in \mathbb{Z}^+$.

$n^3 + 5n$ is divisible by 6

Solution:

Let $f(n) = n^3 + 5n$, where $n \geq 1$ and $n \in \mathbb{Z}^+$.

$\therefore\ f(1) = 1 + 5 = 6$, which is divisible by 6.

$\therefore\ f(n)$ is divisible by 6 when $n = 1$.

Assume that for $n = k$,

$f(k) = k^3 + 5k$ is divisible by 6 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore\ f(k+1) &= (k+1)^3 + 5(k+1) \\ &= k^3 + 3k^2 + 3k + 1 + 5(k+1) \\ &= k^3 + 3k^2 + 3k + 1 + 5k + 5 \\ &= k^3 + 3k^2 + 8k + 6\end{aligned}$$

$$\begin{aligned}\therefore\ f(k+1) - f(k) &= [k^3 + 3k^2 + 8k + 6] - [k^3 + 5k] \\ &= 3k^2 + 3k + 6 \\ &= 3k(k+1) + 6 \\ &= 3(2m) + 6 \\ &= 6m + 6 \\ &= 6(m+1)\end{aligned}$$

Let $k(k+1) = 2m, m \in \mathbb{Z}^+$, as the product of two consecutive integers must be even.

$\therefore\ f(k+1) = f(k) + 6(m+1)$.

As both $f(k)$ and $6(m+1)$ are divisible by 6 then the sum of these two terms must also be divisible by 6. Therefore $f(n)$ is divisible by 6 when $n = k + 1$.

If $f(n)$ is divisible by 6 when $n = k$, then it has been shown that $f(n)$ is also divisible by 6 when $n = k + 1$. As $f(n)$ is divisible by 6 when $n = 1$, $f(n)$ is also divisible by 6 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise B, Question 8

Question:

Use the method of mathematical induction to prove the following statement for $n \in \mathbb{Z}^+$.

$2^n.3^{2n} - 1$ is divisible by 17

Solution:

Let $f(n) = 2^n.3^{2n} - 1$, where $n \in \mathbb{Z}^+$.

$\therefore\ f(1) = 2^1.3^{2(1)} - 1 = 2(9) - 1 = 18 - 1 = 17$, which is divisible by 17.

$\therefore\ f(n)$ is divisible by 17 when $n = 1$.

Assume that for $n = k$,

$f(k) = 2^k.3^{2k} - 1$ is divisible by 17 for $k \in \mathbb{Z}^+$.

$$\begin{aligned} \therefore\ f(k+1) &= 2^{k+1}.3^{2(k+1)} - 1 \\ &= 2^k(2)^1(3)^{2k}(3)^2 - 1 \\ &= 2^k(2)^1(3)^{2k}(9) - 1 \\ &= 18(2^k.3^{2k}) - 1 \end{aligned}$$

$$\begin{aligned} \therefore\ f(k+1) - f(k) &= \left[18(2^k.3^{2k}) - 1\right] - \left[2^k.3^{2k} - 1\right] \\ &= 18(2^k.3^{2k}) - 1 - 2^k.3^{2k} + 1 \\ &= 17(2^k.3^{2k}) \end{aligned}$$

$\therefore\ f(k+1) = f(k) + 17(2^k.3^{2k})$

As both $f(k)$ and $17(2^k.3^{2k})$ are divisible by 17 then the sum of these two terms must also be divisible by 17.

Therefore $f(n)$ is divisible by 17 when $n = k + 1$.

If $f(n)$ is divisible by 17 when $n = k$, then it has been shown that $f(n)$ is also divisible by 17 when $n = k + 1$. As $f(n)$ is divisible by 17 when $n = 1$, $f(n)$ is also divisible by 17 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise B, Question 9

Question:

$f(n) = 13^n - 6^n$, $n \in \mathbb{Z}^+$.

a Express for $k \in \mathbb{Z}^+$, $f(k+1) - 6f(k)$ in terms of k, simplifying your answer.

b Use the method of mathematical induction to prove that $f(n)$ is divisible by 7 for all $n \in \mathbb{Z}^+$.

Solution:

a

$$\begin{aligned} f(k+1) &= 13^{k+1} - 6^{k+1} \\ &= 13^k.13^1 - 6^k.6^1 \\ &= 13(13^k) - 6(6^k) \end{aligned}$$

$$\begin{aligned} \therefore\ f(k+1) - 6f(k) &= \left[13(13^k) - 6(6^k)\right] - 6\left[13^k - 6^k\right] \\ &= 13(13^k) - 6(6^k) - 6(13^k) + 6(6^k) \\ &= 7(13^k) \end{aligned}$$

b $f(n) = 13^n - 6^n$, where $n \in \mathbb{Z}^+$.

$\therefore\ f(1) = 13^1 - 6^1 = 7$, which is divisible by 7.

$\therefore\ f(n)$ is divisible by 7 when $n = 1$.

Assume that for $n = k$,

$f(k) = 13^k - 6^k$ is divisible by 7 for $k \in \mathbb{Z}^+$.

From (a), $f(k+1) = 6f(k) + 7(13^k)$

As both $6f(k)$ and $7(13^k)$ are divisible by 7 then the sum of these two terms must also be divisible by 7. Therefore $f(n)$ is divisible by 7 when $n = k + 1$.

If $f(n)$ is divisible by 7 when $n = k$, then it has been shown that $f(n)$ is also divisible by 7 when $n = k + 1$. As $f(n)$ is divisible by 7 when $n = 1$, $f(n)$ is also divisible by 7 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise B, Question 10

Question:

$g(n) = 5^{2n} - 6n + 8,\ n \in \mathbb{Z}^+$.

a Express for $k \in \mathbb{Z}^+$, $g(k+1) - 25g(k)$ in terms of k, simplifying your answer.

b Use the method of mathematical induction to prove that $g(n)$ is divisible by 9 for all $n \in \mathbb{Z}^+$.

Solution:

a

$$\begin{aligned} g(k+1) &= 5^{2(k+1)} - 6(k+1) + 8 \\ &= 5^{2k}.5^2 - 6k - 6 + 8 \\ &= 25(5^{2k}) - 6k + 2 \end{aligned}$$

$$\begin{aligned} \therefore\ g(k+1) - 25g(k) &= \left[25(5^{2k}) - 6k + 2\right] - 25\left[5^{2k} - 6k + 8\right] \\ &= 25(5^{2k}) - 6k + 2 - 25(5^{2k}) + 150k - 200 \\ &= 144k - 198 \end{aligned}$$

b

$g(n) = 5^{2n} - 6n + 8$, where $n \in \mathbb{Z}^+$.

$\therefore\ g(1) = 5^2 - 6(1) + 8 = 25 - 6 + 8 = 27$, which is divisible by 9.

$\therefore\ g(n)$ is divisible by 9 when $n = 1$.

Assume that for $n = k$,

$g(k) = 5^{2k} - 6k + 8$ is divisible by 9 for $k \in \mathbb{Z}^+$.

$$\begin{aligned} \text{From(a)},\ g(k+1) &= 25g(k) + 144n - 198 \\ &= 25g(k) + 18(8n - 11) \end{aligned}$$

As both $25g(k)$ and $18(8n - 11)$ are divisible by 9 then the sum of these two terms must also be divisible by 9. Therefore $g(n)$ is divisible by 9 when $n = k + 1$.

If $g(n)$ is divisible by 9 when $n = k$, then it has been shown that $g(n)$ is also divisible by 9 when $n = k + 1$. As $g(n)$ is divisible by 9 when $n = 1$, $g(n)$ is also divisible by 9 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise B, Question 11

Question:

Use the method of mathematical induction to prove that $8^n - 3^n$ is divisible by 5 for all $n \in \mathbb{Z}^+$.

Solution:

$f(n) = 8^n - 3^n$, where $n \in \mathbb{Z}^+$.

$\therefore\ f(1) = 8^1 - 3^1 = 5$, which is divisible by 5.

$\therefore\ f(n)$ is divisible by 5 when $n = 1$.

Assume that for $n = k$,

$f(k) = 8^k - 3^k$ is divisible by 5 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore\ f(k+1) &= 8^{k+1} - 3^{k+1} \\ &= 8^k.8^1 - 3^k.3^1 \\ &= 8(8^k) - 3(3^k)\end{aligned}$$

$$\begin{aligned}\therefore\ f(k+1) - 3f(k) &= \left[8(8^k) - 3(3^k)\right] - 3\left[8^k - 3^k\right] \\ &= 8(8^k) - 3(3^k) - 3(8^k) + 3(3^k) \\ &= 5(8^k)\end{aligned}$$

From (a), $f(k+1) = f(k) + 5(8^k)$

As both $f(k)$ and $5(8^k)$ are divisible by 5 then the sum of these two terms must also be divisible by 5. Therefore $f(n)$ is divisible by 5 when $n = k + 1$.

If $f(n)$ is divisible by 5 when $n = k$, then it has been shown that $f(n)$ is also divisible by 5 when $n = k + 1$. As $f(n)$ is divisible by 5 when $n = 1$, $f(n)$ is also divisible by 5 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise B, Question 12

Question:

Use the method of mathematical induction to prove that $3^{2n+2}+8n-9$ is divisible by 8 for all $n \in \mathbb{Z}^+$.

Solution:

$f(n) = 3^{2n+2}+8n-9$, where $n \in \mathbb{Z}^+$.

$\therefore\ f(1) = 3^{2(1)+2}+8(1)-9$

$= 3^4+8-9 = 81-1 = 80$, which is divisible by 8.

$\therefore$ f(n) is divisible by 8 when $n = 1$.

Assume that for $n = k$,

$f(k) = 3^{2k+2}+8k-9$ is divisible by 8 for $k \in \mathbb{Z}^+$.

$$\begin{aligned} f(k+1) &= 3^{2(k+1)+2}+8(k+1)-9 \\ &= 3^{2k+2+2}+8(k+1)-9 \\ &= 3^{2k+2}.(3^2)+8k+8-9 \\ &= 9(3^{2k+2})+8k-1 \end{aligned}$$

$$\begin{aligned} \therefore\ f(k+1)-f(k) &= \left[9(3^{2k+2})+8k-1\right]-\left[3^{2k+2}+8k-9\right] \\ &= 9(3^{2k+2})+8k-1-3^{2k+2}-8k+9 \\ &= 8(3^{2k+2})+8 \\ &= 8\left[3^{2k+2}+1\right] \end{aligned}$$

$$\therefore\ f(k+1) = f(k)+8\left[3^{2k+2}+1\right]$$

As both f(k) and $8\left[3^{2k+2}+1\right]$ are divisible by 8 then the sum of these two terms must also be divisible by 8. Therefore f(n) is divisible by 8 when $n = k+1$.

If f(n) is divisible by 8 when $n = k$, then it has been shown that f(n) is also divisible by 8 when $n = k+1$. As f(n) is divisible by 8 when $n = 1$, f(n) is also divisible by 8 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise B, Question 13

Question:

Use the method of mathematical induction to prove that $2^{6n} + 3^{2n-2}$ is divisible by 5 for all $n \in \mathbb{Z}^+$.

Solution:

$f(n) = 2^{6n} + 3^{2n-2}$, where $n \in \mathbb{Z}^+$.

$\therefore\ f(1) = 2^{6(1)} + 3^{2(1)-2} = 2^6 + 3^0 = 64 + 1 = 65$, which is divisible by 5.

$\therefore\ f(n)$ is divisible by 5 when $n = 1$.

Assume that for $n = k$,

$f(k) = 2^{6k} + 3^{2k-2}$ is divisible by 5 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore\ f(k+1) &= 2^{6(k+1)} + 3^{2(k+1)-2}\\ &= 2^{6k+6} + 3^{2k+2-2}\\ &= 2^6\left(2^{6k}\right) + 3^2\left(3^{2k-2}\right)\\ &= 64\left(2^{6k}\right) + 9\left(3^{2k-2}\right)\end{aligned}$$

$$\begin{aligned}\therefore\ f(k+1) - f(k) &= \left[64\left(2^{6k}\right) + 9\left(3^{2k-2}\right)\right] - \left[2^{6k} + 3^{2k-2}\right]\\ &= 64\left(2^{6k}\right) + 9\left(3^{2k-2}\right) - 2^{6k} - 3^{2k-2}\\ &= 63\left(2^{6k}\right) + 8\left(3^{2k-2}\right)\\ &= 63\left(2^{6k}\right) + 63\left(3^{2k-2}\right) - 55\left(3^{2k-2}\right)\\ &= 63\left[2^{6k} + 3^{2k-2}\right] - 55\left(3^{2k-2}\right)\end{aligned}$$

$$\begin{aligned}\therefore\ f(k+1) &= f(k) + 63\left[2^{6k} + 3^{2k-2}\right] - 55\left(3^{2k-2}\right)\\ &= f(k) + 63f(k) - 55\left(3^{2k-2}\right)\\ &= 64f(k) - 55\left(3^{2k-2}\right)\\ \therefore\ f(k+1) &= 64f(k) - 55\left(3^{2k-2}\right)\end{aligned}$$

As both $64f(k)$ and $-55\left(3^{2k-2}\right)$ are divisible by 5 then the sum of these two terms must also be divisible by 5. Therefore $f(n)$ is divisible by 5 when $n = k + 1$.

If $f(n)$ is divisible by 5 when $n = k$, then it has been shown that $f(n)$ is also divisible by 5 when $n = k + 1$. As $f(n)$ is divisible by 5 when $n = 1$, $f(n)$ is also divisible by 5 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction
Exercise C, Question 1

Question:

Given that $u_{n+1} = 5u_n + 4,\ u_1 = 4$, prove by induction that $u_n = 5^n - 1$.

Solution:

$n = 1$; $u_1 = 5^1 - 1 = 4$, as given.

$n = 2$; $u_2 = 5^2 - 1 = 24$, from the general statement.

and $u_2 = 5u_1 + 4 = 5(4) + 4 = 24$, from the recurrence relation.

So u_n is true when $n = 1$ and also true when $n = 2$.

Assume that for $n = k$ that, $u_k = 5^k - 1$ is true for $k \in \mathbb{Z}^+$.

$$\begin{aligned} \text{Then } u_{k+1} &= 5u_k + 4 \\ &= 5\left(5^k - 1\right) + 4 \\ &= 5^{k+1} - 5 + 4 \\ &= 5^{k+1} - 1 \end{aligned}$$

Therefore, the general statement, $u_n = 5^n - 1$ is true when $n = k + 1$.

If u_n is true when $n = k$, then it has been shown that $u_n = 5^n - 1$ is also true when $n = k + 1$. As u_n is true for $n = 1$ and $n = 2$, then u_n is true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

Solutionbank FP1

Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction

Exercise C, Question 2

Question:

Given that $u_{n+1} = 2u_n + 5$, $u_1 = 3$, prove by induction that $u_n = 2^{n+2} - 5$.

Solution:

$n = 1$; $u_1 = 2^{1+2} - 5 = 8 - 5 = 3$, as given.

$n = 2$; $u_2 = 2^4 - 5 = 16 - 5 = 11$, from the general statement.

and $u_2 = 2u_1 + 5 = 2(3) + 5 = 11$, from the recurrence relation.

So u_n is true when $n = 1$ and also true when $n = 2$.

Assume that for $n = k$ that, $u_k = 2^{k+2} - 5$ is true for $k \in \mathbb{Z}^+$.

$$\begin{aligned} \text{Then } u_{k+1} &= 2u_k + 5 \\ &= 2\left(2^{k+2} - 5\right) + 5 \\ &= 2^{k+3} - 10 + 5 \\ &= 2^{k+1+2} - 5 \end{aligned}$$

Therefore, the general statement, $u_n = 2^{n+2} - 5$ is true when $n = k + 1$.

If u_n is true when $n = k$, then it has been shown that $u_n = 2^{n+2} - 5$ is also true when $n = k + 1$. As u_n is true for $n = 1$ and $n = 2$, then u_n is true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise C, Question 3

Question:

Given that $u_{n+1} = 5u_n - 8,\ u_1 = 3$, prove by induction that $u_n = 5^{n-1} + 2$.

Solution:

$n = 1$; $u_1 = 5^{1-1} + 2 = 1 + 2 = 3$, as given.

$n = 2$; $u_2 = 5^{2-1} + 2 = 5 + 2 = 7$, from the general statement.

and $u_2 = 5u_1 - 8 = 5(3) - 8 = 7$, from the recurrence relation.

So u_n is true when $n = 1$ and also true when $n = 2$.

Assume that for $n = k$ that, $u_k = 5^{k-1} + 2$ is true for $k \in \mathbb{Z}^+$.

$$
\begin{aligned}
\text{Then } u_{k+1} &= 5u_k - 8 \\
&= 5\left(5^{k-1} + 2\right) - 8 \\
&= 5^{k-1+1} + 10 - 8 \\
&= 5^k + 2 \\
&= 5^{k+1-1} + 2
\end{aligned}
$$

Therefore, the general statement, $u_n = 5^{n-1} + 2$ is true when $n = k + 1$.

If u_n is true when $n = k$, then it has been shown that $u_n = 5^{n-1} + 2$ is also true when $n = k + 1$. As u_n is true for $n = 1$ and $n = 2$, then u_n is true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise C, Question 4

Question:

Given that $u_{n+1} = 3u_n + 1$, $u_1 = 1$, prove by induction that $u_n = \frac{3^n - 1}{2}$.

Solution:

$n = 1$; $u_1 = \frac{3^1 - 1}{2} = \frac{2}{2} = 1$, as given.

$n = 2$; $u_2 = \frac{3^2 - 1}{2} = \frac{8}{2} = 4$, from the general statement.

and $u_2 = 3u_1 + 1 = 3(1) + 1 = 4$, from the recurrence relation.

So u_n is true when $n = 1$ and also true when $n = 2$.

Assume that for $n = k$ that, $u_k = \frac{3^k - 1}{2}$ is true for $k \in \mathbb{Z}^+$.

$$
\begin{aligned}
\text{Then } u_{k+1} &= 3u_k + 1 \\
&= 3\left(\frac{3^k - 1}{2}\right) + 1 \\
&= \left(\frac{3(3^k) - 3}{2}\right) + \frac{2}{2} \\
&= \frac{3^{k+1} - 3 + 2}{2} \\
&= \frac{3^{k+1} - 1}{2}
\end{aligned}
$$

Therefore, the general statement, $u_n = \frac{3^n - 1}{2}$ is true when $n = k + 1$.

If u_n is true when $n = k$, then it has been shown that $u_n = \frac{3^n - 1}{2}$ is also true when $n = k + 1$. As u_n is true for $n = 1$ and $n = 2$, then u_n is true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise C, Question 5

Question:

Given that $u_{n+2} = 5u_{n+1} - 6u_n,\ u_1 = 1,\ u_2 = 5$ prove by induction that $u_n = 3^n - 2^n$.

Solution:

$n = 1;\ u_1 = 3^1 - 2^1 = 3 - 2 = 1$, as given.

$n = 2;\ u_2 = 3^2 - 2^2 = 9 - 4 = 5$, as given.

$n = 3;\ u_3 = 3^3 - 2^3 = 27 - 8 = 19$, from the general statement.

and $u_3 = 5u_2 - 6u_1 = 5(5) - 6(1)$

$= 25 - 6 = 19$, from the recurrence relation.

So u_n is true when $n = 1$, $n = 2$ and also true when $n = 3$.

Assume that for $n = k$ and $n = k + 1$,

both $u_k = 3^k - 2^k$ and $u_{k+1} = 3^{k+1} - 2^{k+1}$ are true for $k \in \mathbb{Z}^+$.

$$\begin{aligned}
\text{Then } u_{k+2} &= 5u_{k+1} - 6u_k \\
&= 5\left(3^{k+1} - 2^{k+1}\right) - 6\left(3^k - 2^k\right) \\
&= 5\left(3^{k+1}\right) - 5\left(2^{k+1}\right) - 6\left(3^k\right) + 6\left(2^k\right) \\
&= 5\left(3^{k+1}\right) - 5\left(2^{k+1}\right) - 2\left(3^1\right)\left(3^k\right) + 3\left(2^1\right)\left(2^k\right) \\
&= 5\left(3^{k+1}\right) - 5\left(2^{k+1}\right) - 2\left(3^{k+1}\right) + 3\left(2^{k+1}\right) \\
&= 3\left(3^{k+1}\right) - 2\left(2^{k+1}\right) \\
&= \left(3^1\right)\left(3^{k+1}\right) - \left(2^1\right)\left(2^{k+1}\right) \\
&= 3^{k+2} - 2^{k+2}
\end{aligned}$$

Therefore, the general statement, $u_n = 3^n - 2^n$ is true when $n = k + 2$.

If u_n is true when $n = k$ and $n = k + 1$ then it has been shown that $u_n = 3^n - 2^n$ is also true when $n = k + 2$. As u_n is true for $n = 1, n = 2$ and $n = 3$, then u_n is true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise C, Question 6

Question:

Given that $u_{n+2} = 6u_{n+1} - 9u_n$, $u_1 = -1$, $u_2 = 0$, prove by induction that $u_n = (n-2)3^{n-1}$.

Solution:

$n = 1$; $u_1 = (1-2)3^{1-1} = (-1)(1) = -1$, as given.

$n = 2$; $u_2 = (2-2)3^{2-1} = (0)(3) = 0$, as given.

$n = 3$; $u_3 = (3-2)3^{3-1} = (1)(9) = 9$, from the general statement.

and $u_3 = 6u_2 - 9u_1 = 6(0) - 9(-1)$
$= 0 - -9 = 9$, from the recurrence relation.

So u_n is true when $n = 1, n = 2$ and also true when $n = 3$.

Assume that for $n = k$ and $n = k+1$,

both $u_k = (k-2)3^{k-1}$

and $u_{k+1} = (k+1-2)3^{k+1-1} = (k-1)3^k$ are true for $k \in \mathbb{Z}^+$.

Then
$$\begin{aligned} u_{k+2} &= 6u_{k+1} - 9u_k \\ &= 6\left((k-1)3^k\right) - 9\left((k-2)3^{k-1}\right) \\ &= 6(k-1)\left(3^k\right) - 3(k-2).3\left(3^{k-1}\right) \\ &= 6(k-1)\left(3^k\right) - 3(k-2)\left(3^{k-1+1}\right) \\ &= 6(k-1)\left(3^k\right) - 3(k-2)\left(3^k\right) \\ &= \left(3^k\right)[6(k-1) - 3(k-2)] \\ &= \left(3^k\right)[6k - 6 - 3k + 6] \\ &= 3k\left(3^k\right) \\ &= k\left(3^{k+1}\right) \\ &= (k+2-2)\left(3^{k+2-1}\right) \end{aligned}$$

Therefore, the general statement, $u_n = (n-2)3^{n-1}$ is true when $n = k+2$.

If u_n is true when $n = k$ and $n = k+1$ then it has been shown that $u_n = (n-2)3^{n-1}$ is also true when $n = k+2$. As u_n is true for $n = 1, n = 2$ and $n = 3$, then u_n is true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction
Exercise C, Question 7

Question:

Given that $u_{n+2} = 7u_{n+1} - 10u_n$, $u_1 = 1$, $u_2 = 8$, prove by induction that $u_n = 2(5^{n-1}) - 2^{n-1}$.

Solution:

$n = 1$; $u_1 = 2(5^0) - (2^0) = 2 - 1 = 1$, as given.

$n = 2$; $u_2 = 2(5^1) - (2^1) = 10 - 2 = 8$, as given.

$n = 3$; $u_3 = 2(5^2) - (2^2) = 50 - 4 = 46$, from the general statement.

and $u_3 = 7u_2 - 10u_1 = 7(8) - 10(1)$
$= 56 - 10 = 46$, from the recurrence relation.

So u_n is true when $n = 1$, $n = 2$ and also true when $n = 3$.

Assume that for $n = k$ and $n = k + 1$,

both $u_k = 2(5^{k-1}) - 2^{k-1}$

and $u_{k+1} = 2(5^{k+1-1}) - 2^{k+1-1} = 2(5^k) - 2^k$ are true for $k \in \mathbb{Z}^+$.

$$\begin{aligned}
\text{Then } u_{k+2} &= 7u_{k+1} - 10u_k \\
&= 7\left(2(5^k) - 2^k\right) - 10\left(2(5^{k-1}) - 2^{k-1}\right) \\
&= 14\left(5^k\right) - 7\left(2^k\right) - 20\left(5^{k-1}\right) + 10\left(2^{k-1}\right) \\
&= 14\left(5^k\right) - 7\left(2^k\right) - 4\left(5^1\right)\left(5^{k-1}\right) + 5\left(2^1\right)\left(2^{k-1}\right) \\
&= 14\left(5^k\right) - 7\left(2^k\right) - 4\left(5^{k-1+1}\right) + 5\left(2^{k-1+1}\right) \\
&= 14\left(5^k\right) - 7\left(2^k\right) - 4\left(5^k\right) + 5\left(2^k\right) \\
&= 2\left(5^1\right)\left(5^k\right) - \left(2^1\right)\left(2^k\right) \\
&= 2\left(5^{k+1}\right) - \left(2^{k+1}\right) \\
&= 2\left(5^{k+2-1}\right) - \left(2^{k+2-1}\right)
\end{aligned}$$

Therefore, the general statement, $u_n = 2(5^{n-1}) - 2^{n-1}$ is true when $n = k + 2$.

If u_n is true when $n = k$ and $n = k + 1$ then it has been shown that $u_n = 2(5^{n-1}) - 2^{n-1}$ is also true when $n = k + 2$. As u_n is true for $n = 1$, $n = 2$ and $n = 3$, then u_n is true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction
Exercise C, Question 8

Question:

Given that $u_{n+2} = 6u_{n+1} - 9u_n$, $u_1 = 3$, $u_2 = 36$, prove by induction that $u_n = (3n-2)3^n$.

Solution:

$n = 1$; $u_1 = (3(1)-2)(3^1) = (1)(3) = 3$, as given.

$n = 2$; $u_2 = (3(2)-2)(3^2) = (4)(9) = 36$, as given.

$n = 3$; $u_3 = (3(3)-2)(3^3) = (7)(27) = 189$, from the general statement.

and $u_3 = 6u_2 - 9u_1 = 6(36) - 9(3)$
$= 216 - 27 = 189$, from the recurrence relation.

So u_n is true when $n = 1, n = 2$ and also true when $n = 3$.

Assume that for $n = k$ and $n = k+1$,

both $u_k = (3k-2)(3^k)$

and $u_{k+1} = (3(k+1)-2)(3^{k+1}) = (3k+1)(3^{k+1})$ are true for $k \in \mathbb{Z}^+$.

$$\begin{aligned}
\text{Then } u_{k+2} &= 6u_{k+1} - 9u_k \\
&= 6\left((3k+1)(3^{k+1})\right) - 9\left((3k-2)(3^k)\right) \\
&= 6(3k+1)3^1\left(3^k\right) - 9(3k-2)\left(3^k\right) \\
&= 18(3k+1)\left(3^k\right) - 9(3k-2)(3^k) \\
&= 9\left(3^k\right)[2(3k+1) - (3k-2)] \\
&= 9\left(3^k\right)[6k+2-3k+2] \\
&= 9\left(3^k\right)[3k+4] \\
&= 3^2\left(3^k\right)[3k+4] \\
&= (3k+4)\left(3^{k+2}\right) \\
&= (3(k+2)-2)\left(3^{k+2}\right)
\end{aligned}$$

Therefore, the general statement, $u_n = (3n-2)3^n$ is true when $n = k+2$.

If u_n is true when $n = k$ and $n = k+1$ then it has been shown that $u_n = (3n-2)3^n$ is also true when $n = k+2$. As u_n is true for $n = 1, n = 2$ and $n = 3$, then u_n is true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction
Exercise D, Question 1

Question:

Prove by the method of mathematical induction the following statement for $n \in \mathbb{Z}^+$.

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$$

Solution:

$$n = 1;\ \text{LHS} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\text{RHS} = \begin{pmatrix} 1 & 2(1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

As LHS = RHS, the matrix equation is true for $n = 1$.

Assume that the matrix equation is true for $n = k$.

ie. $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix}$

With $n = k + 1$ the matrix equation becomes

$$\begin{aligned} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{k+1} &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^k \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1+0 & 2+2k \\ 0+0 & 0+1 \end{pmatrix}. \\ &= \begin{pmatrix} 1 & 2(k+1) \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Therefore the matrix equation is true when $n = k + 1$.

If the matrix equation is true for $n = k$, then it is shown to be true for $n = k + 1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction
Exercise D, Question 2

Question:

Prove by the method of mathematical induction the following statement for $n \in \mathbb{Z}^+$.

$$\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^n = \begin{pmatrix} 2n+1 & -4n \\ n & -2n+1 \end{pmatrix}$$

Solution:

$$n = 1;\ \text{LHS} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^1 = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$$
$$\text{RHS} = \begin{pmatrix} 2(1)+1 & -4(1) \\ 1 & -2(1)+1 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$$

As LHS = RHS, the matrix equation is true for $n = 1$.

Assume that the matrix equation is true for $n = k$.

ie. $\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^k = \begin{pmatrix} 2k+1 & -4k \\ k & -2k+1 \end{pmatrix}$.

With $n = k+1$ the matrix equation becomes

$$\begin{aligned}
\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^{k+1} &= \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^k \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 2k+1 & -4k \\ k & -2k+1 \end{pmatrix} \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 6k+3-4k & -8k-4+4k \\ 3k-2k+1 & -4k+2k-1 \end{pmatrix} \\
&= \begin{pmatrix} 2k+3 & -4k-4 \\ k+1 & -2k-1 \end{pmatrix} \\
&= \begin{pmatrix} 2(k+1)+1 & -4(k+1) \\ (k+1) & -2(k+1)+1 \end{pmatrix}
\end{aligned}$$

Therefore the matrix equation is true when $n = k+1$.

If the matrix equation is true for $n = k$, then it is shown to be true for $n = k+1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise D, Question 3

Question:

Prove by the method of mathematical induction the following statement for $n \in \mathbb{Z}^+$.

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}$$

Solution:

$$n = 1; \text{LHS } = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}^1 = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\text{RHS } = \begin{pmatrix} 2^1 & 0 \\ 2^1 - 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

As LHS = RHS, the matrix equation is true for $n = 1$.

Assume that the matrix equation is true for $n = k$.

ie. $\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}^k = \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix}$

With $n = k + 1$ the matrix equation becomes

$$\begin{aligned}
\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}^{k+1} &= \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}^k \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2(2^k) + 0 & 0 + 0 \\ 2(2^k) - 2 + 1 & 0 + 1 \end{pmatrix} \\
&= \begin{pmatrix} 2^1(2^k) & 0 \\ 2^1(2^k) - 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2^{k+1} & 0 \\ 2^{k+1} - 1 & 1 \end{pmatrix}
\end{aligned}$$

Therefore the matrix equation is true when $n = k + 1$.

If the matrix equation is true for $n = k$, then it is shown to be true for $n = k + 1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise D, Question 4

Question:

Prove by the method of mathematical induction the following statement for $n \in \mathbb{Z}^+$.

$$\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^n = \begin{pmatrix} 4n+1 & -8n \\ 2n & 1-4n \end{pmatrix}$$

Solution:

$$n = 1; \text{LHS} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^1 = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$$

$$\text{RHS} = \begin{pmatrix} 4(1)+1 & -8(1) \\ 2(1) & 1-4(1) \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$$

As LHS = RHS, the matrix equation is true for $n = 1$.

Assume that the matrix equation is true for $n = k$.

ie. $\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^k = \begin{pmatrix} 4k+1 & -8k \\ 2k & 1-4k \end{pmatrix}$.

With $n = k+1$ the matrix equation becomes

$$\begin{aligned}
\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^{k+1} &= \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^k \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix} \\
&= \begin{pmatrix} 4k+1 & -8k \\ 2k & 1-4k \end{pmatrix} \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix} \\
&= \begin{pmatrix} 20k+5-16k & -32k-8+24k \\ 10k+2-8k & -16k-3+12k \end{pmatrix} \\
&= \begin{pmatrix} 4k+5 & -8k-8 \\ 2k+2 & -4k-3 \end{pmatrix} \\
&= \begin{pmatrix} 4(k+1)+1 & -8(k+1) \\ 2(k+1) & 1-4(k+1) \end{pmatrix}
\end{aligned}$$

Therefore the matrix equation is true when $n = k+1$.

If the matrix equation is true for $n = k$, then it is shown to be true for $n = k+1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise D, Question 5

Question:

Prove by the method of mathematical induction the following statement for $n \in \mathbb{Z}^+$.

$$\begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 2^n & 5(2^n - 1) \\ 0 & 1 \end{pmatrix}$$

Solution:

$$n = 1;\ \text{LHS} = \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}^1 = \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}$$

$$\text{RHS} = \begin{pmatrix} 2^1 & 5(2^1 - 1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}$$

As LHS = RHS, the matrix equation is true for $n = 1$.

Assume that the matrix equation is true for $n = k$.

ie. $\begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 2^k & 5(2^k - 1) \\ 0 & 1 \end{pmatrix}$

With $n = k + 1$ the matrix equation becomes

$$\begin{aligned}
\begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}^{k+1} &= \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}^k \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2^k & 5(2^k - 1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2(2^k) + 0 & 5(2^k) + 5(2^k - 1) \\ 0 + 0 & 0 + 1 \end{pmatrix} \\
&= \begin{pmatrix} 2^1(2^k) & 5(2^k) + 5(2^k) - 5 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2^{k+1} & 5(2^1)(2^k) - 5 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2^{k+1} & 5(2^{k+1}) - 5 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2^{k+1} & 5(2^{k+1} - 1) \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

Therefore the matrix equation is true when $n = k + 1$.

If the matrix equation is true for $n = k$, then it is shown to be true for $n = k + 1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction
Exercise E, Question 1

Question:

Prove by induction that $9^n - 1$ is divisible by 8 for $n \in \mathbb{Z}^+$.

Solution:

Let $\text{f}(n) = 9^n - 1$, where $n \in \mathbb{Z}^+$.

$\therefore\ \text{f}(1) = 9^1 - 1 = 8$, which is divisible by 8.

$\therefore\ \text{f}(n)$ is divisible by 8 when $n = 1$.

Assume that for $n = k$,

$\text{f}(k) = 9^k - 1$ is divisible by 8 for $k \in \mathbb{Z}^+$.

$$\begin{aligned} \therefore\ \text{f}(k+1) &= 9^{k+1} - 1 \\ &= 9^k.9^1 - 1 \\ &= 9(9^k) - 1 \end{aligned}$$

$$\begin{aligned} \therefore\ \text{f}(k+1) - \text{f}(k) &= [9(9^k) - 1] - [9^k - 1] \\ &= 9(9^k) - 1 - 9^k + 1 \\ &= 8(9^k) \end{aligned}$$

$$\therefore\ \text{f}(k+1) = \text{f}(k) + 8(9^k)$$

As both $\text{f}(k)$ and $8(9^k)$ are divisible by 8 then the sum of these two terms must also be divisible by 8. Therefore $\text{f}(n)$ is divisible by 8 when $n = k + 1$.

If $\text{f}(n)$ is divisible by 8 when $n = k$, then it has been shown that $\text{f}(n)$ is also divisible by 8 when $n = k + 1$. As $\text{f}(n)$ is divisible by 8 when $n = 1$, $\text{f}(n)$ is also divisible by 8 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Proof by mathematical induction
Exercise E, Question 2

Question:

The matrix **B** is given by $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$.

a Find $\mathbf{B}^2$ and $\mathbf{B}^3$.

b Hence write down a general statement for $\mathbf{B}^n$, for $n \in \mathbb{Z}^+$.

c Prove, by induction that your answer to part **b** is correct.

Solution:

a

$$\mathbf{B}^2 = \mathbf{BB} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1+0 & 0+0 \\ 0+0 & 0+9 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$$
$$\mathbf{B}^3 = \mathbf{B}^2\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1+0 & 0+0 \\ 0+0 & 0+27 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 27 \end{pmatrix}$$

b As $\mathbf{B}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 3^2 \end{pmatrix}$ and $\mathbf{B}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 3^3 \end{pmatrix}$, we suggest that $\mathbf{B}^n = \begin{pmatrix} 1 & 0 \\ 0 & 3^n \end{pmatrix}$.

c

$$n = 1;\ \text{LHS} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^1 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$
$$\text{RHS} = \begin{pmatrix} 1 & 0 \\ 0 & 3^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

As LHS = RHS, the matrix equation is true for $n = 1$.

Assume that the matrix equation is true for $n = k$.

ie. $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^k = \begin{pmatrix} 1 & 0 \\ 0 & 3^k \end{pmatrix}$

With $n = k + 1$ the matrix equation becomes

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^k \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 3^k \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 1+0 & 0+0 \\ 0+0 & 0+3(3^k) \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 3^{k+1} \end{pmatrix}$$

Therefore the matrix equation is true when $n = k + 1$.

If the matrix equation is true for $n = k$, then it is shown to be true for $n = k + 1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise E, Question 3

Question:

Prove by induction that for $n \in \mathbb{Z}^+$, that $\sum_{r=1}^{n} (3r+4) = \frac{1}{2}n(3n+11)$.

Solution:

$$n = 1;\ \text{LHS} = \sum_{r=1}^{1} (3r+4) = 7$$

$$\text{RHS} = \frac{1}{2}(1)(14) = \frac{1}{2}(14) = 7$$

As LHS = RHS, the summation formula is true for $n = 1$.

Assume that the summation formula is true for $n = k$.

ie. $\sum_{r=1}^{k} (3r+4) = \frac{1}{2}k(3k+11)$.

With $n = k+1$ terms the summation formula becomes:

$$\begin{aligned}\sum_{r=1}^{k+1} (3r+4) &= 7 + 10 + 13 + \geq + (3k+4) + (3(k+1)+4) \\ &= \frac{1}{2}k(3k+11) + (3(k+1)+4) \\ &= \frac{1}{2}k(3k+11) + (3k+7) \\ &= \frac{1}{2}[k(3k+11) + 2(3k+7)] \\ &= \frac{1}{2}[3k^2 + 11k + 6k + 14] \\ &= \frac{1}{2}[3k^2 + 17k + 14] \\ &= \frac{1}{2}(k+1)(3k+14) \\ &= \frac{1}{2}(k+1)[3(k+1)+11]\end{aligned}$$

Therefore, summation formula is true when $n = k+1$.

If the summation formula is true for $n = k$, then it is shown to be true for $n = k+1$. As the result is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise E, Question 4

Question:

A sequence $u_1,\ u_2,\ u_3,\ u_4\ ,\ \geq$ is defined by $u_{n+1} = 5u_n - 3(2^n),\ \ u_1 = 7$.

a Find the first four terms of the sequence.

b Prove, by induction for $n \in \mathbb{Z}^+$, that $u_n = 5^n + 2^n$.

Solution:

a $u_{n+1} = 5u_n - 3(2^n)$

Given, $u_1 = 7$.

$u_2 = 5u_1 - 3(2^1) = 5(7) - 6 = 35 - 6 = 29$

$u_3 = 5u_2 - 3(2^2) = 5(29) - 3(4) = 145 - 12 = 133$

$u_4 = 5u_3 - 3(2^3) = 5(133) - 3(8) = 665 - 24 = 641$

The first four terms of the sequence are 7, 29, 133, 641.

b

$n = 1$; $u_1 = 5^1 + 2^1 = 5 + 2 = 7$, as given.

$n = 2$; $u_2 = 5^2 + 2^2 = 25 + 4 = 29$, from the general statement.

From the recurrence relation in part (a), $u_2 = 29$.

So u_n is true when $n = 1$ and also true when $n = 2$.

Assume that for $n = k$, $u_k = 5^k + 2^k$ is true for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\text{Then }\ u_{k+1} &= 5u_k - 3(2^k)\\ &= 5(5^k + 2^k) - 3(2^k)\\ &= 5(5^k) + 5(2^k) - 3(2^k)\\ &= 5^1(5^k) + 2^1(2^k)\\ &= 5^{k+1} + 2^{k+1}\end{aligned}$$

Therefore, the general statement, $u_n = 5^n + 2^n$ is true when $n = k + 1$.

If u_n is true when $n = k$, then it has been shown that $u_n = 5^n + 2^n$ is also true when $n = k + 1$. As u_n is true for $n = 1$ and $n = 2$, then u_n is true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise E, Question 5

Question:

The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}$.

a Prove by induction that $\mathbf{A}^n = \begin{pmatrix} 8n+1 & 16n \\ -4n & 1-8n \end{pmatrix}$ for $n \in \mathbb{Z}^+$.

The matrix **B** is given by $\mathbf{B} = (\mathbf{A}^n)^{-1}$

b Hence find **B** in terms of n.

Solution:

a

$$n = 1;\ \text{LHS} = \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}^1 = \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}$$
$$\text{RHS} = \begin{pmatrix} 8(1)+1 & 16(1) \\ -4(1) & 1-8(1) \end{pmatrix} = \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}$$

As LHS = RHS, the matrix equation is true for $n = 1$.

Assume that the matrix equation is true for $n = k$.

ie. $\begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}^k = \begin{pmatrix} 8k+1 & 16k \\ -4k & 1-8k \end{pmatrix}$.

With $n = k+1$ the matrix equation becomes

$$\begin{aligned}
\begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}^{k+1} &= \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}^k \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix} \\
&= \begin{pmatrix} 8k+1 & 16k \\ -4k & 1-8k \end{pmatrix} \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix} \\
&= \begin{pmatrix} 72k+9-64k & 128k+16-112k \\ -36k-4+32k & -64k-7+56k \end{pmatrix} \\
&= \begin{pmatrix} 8k+9 & 16k+16 \\ -4k-4 & -8k-7 \end{pmatrix} \\
&= \begin{pmatrix} 8(k+1)+1 & 16(k+1) \\ -4(k+1) & 1-8(k+1) \end{pmatrix}
\end{aligned}$$

Therefore the matrix equation is true when $n = k+1$.

If the matrix equation is true for $n = k$, then it is shown to be true for $n = k+1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

b

$$\begin{aligned}
\det(\mathbf{A}^n) &= (8n+1)(1-8n) - -64n^2 \\
&= 8n - 64n^2 + 1 - 8n + 64n^2 \\
&= 1
\end{aligned}$$

$\mathbf{B} = (\mathbf{A}^n)^{-1} = \frac{1}{1}\begin{pmatrix} 1-8n & -16n \\ 4n & 8n+1 \end{pmatrix}$

So $\mathbf{B} = \begin{pmatrix} 1-8n & -16n \\ 4n & 8n+1 \end{pmatrix}$

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Proof by mathematical induction
Exercise E, Question 6

Question:

The function f is defined by $f(n) = 5^{2n-1} + 1$, where $n \in \mathbb{Z}^+$.

a Show that $f(n+1) - f(n) = \mu(5^{2n-1})$, where μ is an integer to be determined.

b Hence prove by induction that f(*n*) is divisible by 6.

Solution:

a

$$\begin{aligned} f(n+1) &= 5^{2(n+1)-1} + 1 \\ &= 5^{2n+2-1} + 1 \\ &= 5^{2n-1}.5^2 + 1 \\ &= 25(5^{2n-1}) + 1 \end{aligned}$$

$$\begin{aligned} \therefore\ f(n+1) - f(n) &= \left[25(5^{2n-1}) + 1\right] - [5^{2n-1} + 1] \\ &= 25(5^{2n-1}) + 1 - (5^{2n-1}) - 1 \\ &= 24(5^{2n-1}) \end{aligned}$$

Therefore, $\mu = 24$.

b $f(n) = 5^{2n-1} + 1$, where $n \in \mathbb{Z}^+$.

$\therefore\ f(1) = 5^{2(1)-1} + 1 = 5^1 + 1 = 6$, which is divisible by 6.

$\therefore$ f(*n*) is divisible by 6 when $n = 1$.

Assume that for $n = k$,

$f(k) = 5^{2k-1} + 1$ is divisible by 6 for $k \in \mathbb{Z}^+$.

Using (a), $f(k+1) - f(k) = 24(5^{2k-1})$

$\therefore\ f(k+1) = f(k) + 24(5^{2k-1})$

As both f(*k*) and $24(5^{2k-1})$ are divisible by 6 then the sum of these two terms must also be divisible by 6. Therefore f(*n*) is divisible by 6 when $n = k + 1$.

If f(*n*) is divisible by 6 when $n = k$, then it has been shown that f(*n*) is also divisible by 6 when $n = k + 1$. As f(*n*) is divisible by 6 when $n = 1$, f(*n*) is also divisible by 6 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise E, Question 7

Question:

Use the method of mathematical induction to prove that $7^n + 4^n + 1$ is divisible by 6 for all $n \in \mathbb{Z}^+$.

Solution:

Let $f(n) = 7^n + 4^n + 1$, where $n \in \mathbb{Z}^+$.

$\therefore$ $f(1) = 7^1 + 4^1 + 1 = 7 + 4 + 1 = 12$, which is divisible by 6.

$\therefore$ $f(n)$ is divisible by 6 when $n = 1$.

Assume that for $n = k$,

$f(k) = 7^k + 4^k + 1$ is divisible by 6 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore\ f(k+1) &= 7^{k+1} + 4^{k+1} + 1\\ &= 7^k.7^1 + 4^k.4^1 + 1\\ &= 7(7^k) + 4(4^k) + 1\end{aligned}$$

$$\begin{aligned}\therefore\ f(k+1) - f(k) &= [7(7^k) + 4(4^k) + 1] - [7^k + 4^k + 1]\\ &= 7(7^k) + 4(4^k) + 1 - 7^k - 4^k - 1\\ &= 6(7^k) + 3(4^k)\\ &= 6(7^k) + 3(4^{k-1}).4^1\\ &= 6(7^k) + 12(4^{k-1})\\ &= 6[7^k + 2(4)^{k-1}]\end{aligned}$$

$\therefore$ $f(k+1) = f(k) + 6[7^k + 2(4)^{k-1}]$

As both $f(k)$ and $6[7^k + 2(4)^{k-1}]$ are divisible by 6 then the sum of these two terms must also be divisible by 6.

Therefore $f(n)$ is divisible by 6 when $n = k + 1$.

If $f(n)$ is divisible by 6 when $n = k$, then it has been shown that $f(n)$ is also divisible by 6 when $n = k + 1$. As $f(n)$ is divisible by 6 when $n = 1$, $f(n)$ is also divisible by 6 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise E, Question 8

Question:

A sequence $u_1,\ u_2,\ u_3,\ u_4\ ,\ \geq$ is defined by $u_{n+1} = \frac{3u_n - 1}{4},\ u_1 = 2.$

a Find the first five terms of the sequence.

b Prove, by induction for $n \in \mathbb{Z}^+$, that $u_n = 4\left(\frac{3}{4}\right)^n - 1.$

Solution:

a $u_{n+1} = \frac{3u_n - 1}{4}.$

Given, $u_1 = 2$

$$u_2 = \frac{3u_1 - 1}{4} = \frac{3(2) - 1}{4} = \frac{5}{4}$$

$$u_3 = \frac{3u_2 - 1}{4} = \frac{3\left(\frac{5}{4}\right) - 1}{4} = \frac{\frac{11}{4}}{4} = \frac{11}{16}$$

$$u_4 = \frac{3u_3 - 1}{4} = \frac{3\left(\frac{11}{16}\right) - 1}{4} = \frac{\frac{17}{16}}{4} = \frac{17}{64}$$

$$u_5 = \frac{3u_4 - 1}{4} = \frac{3\left(\frac{17}{64}\right) - 1}{4} = \frac{-\frac{13}{64}}{4} = -\frac{13}{256}$$

The first five terms of the sequence are $2, \frac{5}{4}, \frac{11}{16}, \frac{17}{64}, -\frac{13}{256}$.

b

$n = 1; u_1 = 4\left(\frac{3}{4}\right)^1 - 1 = 3 - 1 = 2$, as given.

$n = 2; u_2 = 4\left(\frac{3}{4}\right)^2 - 1 = \frac{9}{4} - 1 = \frac{5}{4}$, from the general statement.

From the recurrence relation in part (a), $u_2 = \frac{5}{4}$.

So u_n is true when $n = 1$ and also true when $n = 2$.

Assume that for $n = k$, $u_k = 4\left(\frac{3}{4}\right)^k - 1$ is true for $k \in \mathbb{Z}^+$.

$$\begin{aligned}
\text{Then } u_{k+1} &= \frac{3u_k - 1}{4} \\
&= \frac{3\left[4\left(\frac{3}{4}\right)^k - 1\right] - 1}{4} \\
&= \frac{3}{4}\left[4\left(\frac{3}{4}\right)^k - 1\right] - \frac{1}{4} \\
&= 4\left(\frac{3}{4}\right)^1\left(\frac{3}{4}\right)^k - \frac{3}{4} - \frac{1}{4} \\
&= 4\left(\frac{3}{4}\right)^{k+1} - 1
\end{aligned}$$

Therefore, the general statement, $u_n = 4\left(\frac{3}{4}\right)^n - 1$ is true when $n = k + 1$.

If u_n is true when $n = k$, then it has been shown that $u_n = 4\left(\frac{3}{4}\right)^n - 1$ is also true when $n = k + 1$. As u_n is true for $n = 1$ and $n = 2$, then u_n is true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise E, Question 9

Question:

A sequence $u_1,\ u_2,\ u_3,\ u_4\ ,\ \geq$ is defined by $u_n = 3^{2n} + 7^{2n-1}$.

a Show that $u_{n+1} - 9u_n = \lambda(7^{2k-1})$, where λ is an integer to be determined.

b Hence prove by induction that u_n is divisible by 8 for all positive integers n.

Solution:

a

$$\begin{aligned} u_{n+1} &= 3^{2(n+1)} + 7^{2(n+1)-1} \\ &= 3^{2n}(3^2) + 7^{2n+2-1} \\ &= 3^{2n}(3^2) + 7^{2n-1}(7^2) \\ &= 9(3^{2n}) + 49(7^{2n-1}) \end{aligned}$$

$$\begin{aligned} \therefore\ u_{n+1} - 9u_n &= [9(3^{2n}) + 49(7^{2n-1})] - 9[3^{2n} + 7^{2n-1}] \\ &= 9(3^{2n}) + 49(7^{2n-1}) - 9(3^{2n}) - 9(7^{2n-1}) \\ &= 40(7^{2n-1}) \end{aligned}$$

Therefore, $\lambda = 40$.

b $u_n = 3^{2n} + 7^{2n-1}$, where $n \in \mathbb{Z}^+$.

$\therefore\ u_1 = 3^{2(1)} - 7^{2(1)-1} = 3^2 + 7^1 = 16$, which is divisible by 8.

$\therefore\ u_n$ is divisible by 8 when $n = 1$.

Assume that for $n = k$,

$u_k = 3^{2k} + 7^{2k-1}$ is divisible by 8 for $k \in \mathbb{Z}^+$.

Using (a), $u_{k+1} - 9u_k = 40(7^{2k-1})$

$\therefore\ u_{k+1} = 9u_k + 40(7^{2k-1})$

As both $9u_k$ and $40(7^{2k-1})$ are divisible by 8 then the sum of these two terms must also be divisible by 8. Therefore u_n is divisible by 8 when $n = k + 1$.

If u_n is divisible by 8 when $n = k$, then it has been shown that u_n is also divisible by 8 when $n = k + 1$. As u_n is divisible by 8 when $n = 1$, u_n is also divisible by 8 for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

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Proof by mathematical induction
Exercise E, Question 10

Question:

Prove by induction, for all positive integers n, that

$(1\times 5)+(2\times 6)+(3\times 7)+\geq +n(n+4)=\frac{1}{6}n(n+1)(2n+13).$

Solution:

The identity $(1\times 5)+(2\times 6)+(3\times 7)+\geq +n(n+4)=\frac{1}{6}n(n+1)(2n+13).$

can be rewritten as $\sum_{r=1}^{n} r(r+4)=\frac{1}{6}n(n+1)(2n+13).$

$n=1$; LHS $=\sum_{r=1}^{1} r(r+4)=1(5)=5$

RHS $=\frac{1}{6}(1)(2)(15)=\frac{1}{6}(30)=5$

As LHS = RHS, the summation formula is true for $n=1$.

Assume that the summation formula is true for $n=k$.

ie. $\sum_{r=1}^{k} r(r+4)=\frac{1}{6}k(k+1)(2k+13).$

With $n=k+1$ terms the summation formula becomes:

$$\begin{aligned}\sum_{r=1}^{k+1} r(r+4) &= 1(5)+2(6)+3(7)+\geq +k(k+4)+(k+1)(k+5)\\ &=\frac{1}{6}k(k+1)(2k+13)+(k+1)(k+5)\\ &=\frac{1}{6}(k+1)[k(2k+13)+6(k+5)]\\ &=\frac{1}{6}(k+1)[2k^2+13k+6k+30]\\ &=\frac{1}{6}(k+1)[2k^2+19k+30]\\ &=\frac{1}{6}(k+1)(k+2)(2k+15)\\ &=\frac{1}{6}(k+1)(k+1+1)[2(k+1)+13]\end{aligned}$$

Therefore, summation formula is true when $n=k+1$.

If the summation formula is true for $n=k$, then it is shown to be true for $n=k+1$. As the result is true for $n=1$, it is now also true for all $n\geq 1$ and $n\in \mathbb{Z}^+$ by mathematical induction.