Solutionbank C4
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 1

Question:

Express $\frac{2x-1}{(x-1)(2x-3)}$ in partial fractions. ***E***

Solution:

$$\frac{2x-1}{(x-1)(2x-3)} \equiv \frac{A}{x-1}+\frac{B}{2x-3}$$

$$\equiv \frac{A(2x-3)+B(x-1)}{(x-1)(2x-3)}$$

Set $\frac{2x-1}{(x-1)(2x-3)}$ identical to $\frac{A}{x-1}+\frac{B}{2x-3}$.

Use a common denominator and add the two fractions.

Compare numerators of fractions

$2x-1 \equiv A(2x-3)+B(x-1)$ *

Because the fractions are equivalent, the numerators are also.

Put $x=1$ in equation *

$\therefore 1=-A+0 \Rightarrow A=-1$

To find A, substitute $x=1$.

Put $x=1\frac{1}{2}$ in equation *

$\therefore 2=0+\frac{1}{2}B \Rightarrow B=4$

To find B, substitute $x=1\frac{1}{2}$.

So $\frac{2x-1}{(x-1)(2x-3)} \equiv \frac{-1}{x-1}+\frac{4}{2x-3}$

Solutionbank C4
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 2

Question:

It is given that $f(x) = \frac{3x+7}{(x+1)(x+2)(x+3)}$.

Express f(x) as the sum of three partial fractions. *E*

Solution:

$$\frac{3x+7}{(x+1)(x+2)(x+3)}$$

$$\equiv \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

$$\equiv \frac{A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)}{(x+1)(x+2)(x+3)}$$

Set $\frac{3x+7}{(x+1)(x+2)(x+3)}$ identical to $\frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$.

Use a common denominator and add the three fractions.

Compare numerators of fractions

$3x+7 \equiv A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$

Now put the numerators equal to each other.

Put $x = -2$ in equation
$1 = 0 - B + 0 \Rightarrow B = -1$

To find B, substitute $x = -2$.

Put $x = -3$ in equation
$-2 = 0 + 0 + 2C \Rightarrow C = -1$

To find C, substitute $x = -3$.

Put $x = -1$ in equation
$4 = 2A \Rightarrow A = 2$

To find A, substitute $x = -1$.

So $\frac{3x+7}{(x+1)(x+2)(x+3)} \equiv \frac{2}{(x+1)} - \frac{1}{(x+2)} - \frac{1}{(x+3)}$

Solutionbank C4
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 3

Question:

Given that $f(x) = \frac{2}{(2-x)(1+x)^2}$, express f(x) in the form

$\frac{A}{(2-x)} + \frac{B}{(1+x)} + \frac{C}{(1+x)^2}$. ***E***

Solution:

$$\frac{2}{(2-x)(1+x)^2} \equiv \frac{A}{2-x} + \frac{B}{(1+x)} + \frac{C}{(1+x)^2}$$

$$\equiv \frac{A(1+x)^2 + B(2-x)(1+x) + C(2-x)}{(2-x)(1+x)^2}$$

You need denominators of $(2-x), (1+x)$ and $(1+x)^2$.

Add the three fractions.

Compare numerators of fractions

$2 \equiv A(1+x)^2 + B(2-x)(1+x) + C(2-x)$

Set the numerators equal.

Put $x = 2$

$2 = A \times 9 + 0 + 0$

$\therefore A = \frac{2}{9}$

To find A substitute $x = 2$.

Put $x = -1$

$2 = 0 + 0 + 3C$

$\therefore C = \frac{2}{3}$

To find C substitute $x = -1$.

$\therefore 2 \equiv \frac{2}{9}(1+x)^2 + B(2-x)(1+x) + \frac{2}{3}(2-x)$

$2 \equiv \frac{2}{9} + \frac{4}{9}x + \frac{2}{9}x^2 + 2B + Bx - Bx^2 + \frac{4}{3} - \frac{2}{3}x$

Equate terms in x^2 on both sides

$0 = \frac{2}{9}x^2 - Bx^2 \quad \therefore B = \frac{2}{9}$

Equate terms in x^2 to find B.

$\therefore \frac{2}{(2-x)(1+x)^2} = \frac{2}{9(2-x)} + \frac{2}{9(1+x)} + \frac{2}{3(1+x)^2}$

Solutionbank C4
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 4

Question:

$\frac{14x^2+13x+2}{(x+1)(2x+1)^2} \equiv$
$\frac{A}{x+1}+\frac{B}{2x+1}+\frac{C}{(2x+1)^2}$.
Find the values of the constants A, B and C. *E*

Solution:

$$\frac{14x^2+13x+2}{(x+1)(2x+1)^2} = \frac{A}{x+1}+\frac{B}{2x+1}+\frac{C}{(2x+1)^2}$$

You need denominators of $(x+1), (2x+1)$ and $(2x+1)^2$.

$$= \frac{A(2x+1)^2+B(x+1)(2x+1)+C(x+1)}{(x+1)(2x+1)^2}$$

Add the three fractions.

Compare numerators of fractions

$$14x^2+13x+2 = A(2x+1)^2+B(x+1)(2x+1)+C(x+1)$$

Set the numerators equal.

Put $x=-1$

$\therefore 3 = A+0+0 \Rightarrow A=3$

To find A set $x=-1$.

Put $x=-\frac{1}{2}$

$\therefore \frac{14}{4}-\frac{13}{2}+2=\frac{1}{2}C \Rightarrow C=-2$

To find C set $x=-\frac{1}{2}$.

$\therefore 14x^2+13x+2 = 3(2x+1)^2+B(x+1)(2x+1)-2(x+1)$

Compare coefficients of x^2:

Equate terms in x^2.
$14x^2 = 3.2^2x^2+2Bx^2$

$14 = 12+2B \Rightarrow B=1$

Solve equation to find B.

Check constant term

$2 = 3+1-2$

$$\therefore \frac{14x^2+13x+2}{(x+1)(2x+1)^2} = \frac{3}{x+1}+\frac{1}{2x+1}-\frac{2}{(2x+1)^2}$$

Solutionbank C4
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 5

Question:

$$f(x)=\frac{x^2+6x+7}{(x+2)(x+3)}, x\in\mathbb{R}.$$

Given that $f(x)=A+\frac{B}{(x+2)}+\frac{C}{(x+3)}$ find the values of A, B and C. *E*

Solution:

$$f(x)=\frac{x^2+6x+7}{(x+2)(x+3)}=\frac{x^2+6x+7}{x^2+5x+6}$$

This is an improper fraction so multiply out the denominator.

$$\begin{array}{r} 1 \text{ rem}(x+1) \\ x^2+5x+6 \enclose{longdiv}{x^2+6x+7} \\ \underline{x^2+5x+6} \\ x+1 \end{array}$$

Then divide the denominator into the numerator.

It goes in 1 time with a remainder of $(x+1)$.

$$\therefore f(x) = 1+\frac{x+1}{(x+2)(x+3)}$$

Write f(x) as a mixed fraction.

$$= 1+\frac{B}{x+2}+\frac{C}{x+3}$$

The denominators must be $(x+2)$ and $(x+3)$.

$$= 1+\frac{B(x+3)+C(x+2)}{(x+2)(x+3)}$$

Add the two fractions.

$$\therefore A = 1$$

Compare numerators of remainder term

Set the numerators equal.

$x+1=B(x+3)+C(x+2)$

Put $x=-2$

$-1=B\Rightarrow B=-1$

Substitute $x=-2$ to find B.

Put $x=-3$

$-2 = -C\Rightarrow C=2$

Substitute $x=-3$ to find C.

$$\therefore f(x) = 1-\frac{1}{(x+2)}+\frac{2}{(x+3)}$$

Write out the full solution.

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Review Exercise
Exercise A, Question 6

Question:

Given that $\mathrm{f}(x)=\frac{11-5x^2}{(x+1)(2-x)}$, find constants A and B such that

$\mathrm{f}(x)=5+\frac{A}{(x+1)}+\frac{B}{(2-x)}$. ***E***

Solution:

6

$$\mathrm{f}(x)=\frac{11-5x^2}{(x+1)(2-x)}$$

This is an improper fraction so multiply out the denominator.

$$=\frac{11-5x^2}{2+x-x^2}$$

$$=\frac{10+5x-5x^2}{2+x-x^2}+\frac{1-5x}{2+x-x^2}$$

Either divide denominator into numerator to obtain 5 with $(1-5x)$ as remainder or split numerator, as shown.

$$=5+\frac{A}{(x+1)}+\frac{B}{(2-x)}$$

where

$$\frac{1-5x}{(x+1)(2-x)}\equiv\frac{A}{x+1}+\frac{B}{2-x}$$

Use partial fractions with denominators $(x+1)$ and $(2-x)$.

$$\equiv\frac{A(2-x)+B(x+1)}{(x+1)(2-x)}$$

Add the two fractions.

$\therefore 1-5x=A(2-x)+B(x+1)$

Set the numerators equal.

Put $x=2$

$-9=3B\Rightarrow B=-3$

Substitute $x=2$ to find B.

Put $x=-1$

$6=3A\Rightarrow A=2$

Substitute $x=1$ to find B.

$\therefore \mathrm{f}(x)=5+\frac{2}{x+1}-\frac{3}{2-x}$

Write out full solution.

Solutionbank C4
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 7

Question:

$f(x) = \frac{9 - 3x - 12x^2}{(1-x)(1+2x)}$.

Given that $f(x) = A + \frac{B}{(1-x)} + \frac{C}{(1+2x)}$, find the values of the constants A, B and C. ***E***

Solution:

$$f(x) = \frac{9 - 3x - 12x^2}{(1-x)(1+2x)}$$

Multiply out the denominator.

$$= \frac{9 - 3x - 12x^2}{1 + x - 2x^2}$$

Numerator is split into two parts. The first is a multiple of the denominator. The second is the remainder.

$$= \frac{6 + 6x - 12x^2}{1 + x - 2x^2} + \frac{3 - 9x}{1 + x - 2x^2}$$

This could be done by long division to give $6 + \frac{3 - 9x}{1 + x - 2x^2}$.

$$= 6 + \frac{B}{1-x} + \frac{C}{1+2x}$$

where

$$\frac{3 - 9x}{(1-x)(1+2x)} = \frac{B}{1-x} + \frac{C}{1+2x}$$

Use partial fractions with denominators $(1-x)$ and $(1+2x)$.

$$= \frac{B(1+2x) + C(1-x)}{(1-x)(1+2x)}$$

Add the two fractions.

$\therefore 3 - 9x \equiv B(1+2x) + C(1-x)$

Set the numerators equal.

Put $x = -\frac{1}{2}$

$\therefore 7\frac{1}{2} = 1\frac{1}{2}C \Rightarrow C = 5$

Substitute $x = -\frac{1}{2}$ to find C.

Put $x = 1$

$\therefore -6 = 3B \Rightarrow B = -2$.

Substitute $x = -1$ to find A.

So

$$f(x) = 6 - \frac{2}{1-x} + \frac{5}{1+2x}$$

Write out the full solution.

Solutionbank C4
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 8

Question:

Use the Binomial theorem to expand $\sqrt{(4-9x)}, |x|<\frac{4}{9}$, in ascending powers of x, as far as the term in x^3, simplifying each term. *E*

Solution:

$$\sqrt{(4-9x)} = (4-9x)^{\frac{1}{2}}$$

Write in index form.

$$= \left[4(1-\frac{9}{4}x)\right]^{\frac{1}{2}}$$

Take out a factor of 4.

$$= 4^{\frac{1}{2}}\left(1-\frac{9}{4}x\right)^{\frac{1}{2}}$$

Write $4^{\frac{1}{2}}$ as $\sqrt{4}=2$.

$$= 2\left(1-\frac{9}{4}x\right)^{\frac{1}{2}}$$

Expand $\left(1-\frac{9x}{4}\right)^{\frac{1}{2}}$ using the binomial expansion with $n=\frac{1}{2}$ and $x=\frac{-9x}{4}$.

$$= 2\left[1-\frac{9}{8}x+\frac{\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-9x}{4}\right)^2}{2!}+\frac{\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-9}{4}x\right)^3}{3!}+\cdots\right]$$

$$= 2\left[1-\frac{9}{8}x-\frac{81}{128}x^2-\frac{729}{1024}x^3\cdots\right]$$

Simplify coefficients.

$$= 2-\frac{9}{4}x-\frac{81}{64}x^2-\frac{729}{512}x^3\cdots$$

Multiply by the 2.

Solutionbank C4
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 9

Question:

$f(x) = (2-5x)^{-2}, |x| < \frac{2}{5}$.

Find the binomial expansion of f(x), in ascending powers of x, as far as the term in x^3, giving each coefficient as a simplified fraction. ***E***

Solution:

$$\begin{aligned} f(x) &= (2-5x)^{-2} \\ &= \left[2\left(1-\frac{5}{2}x\right)\right]^{-2} \\ &= 2^{-2}\left(1-\frac{5}{2}x\right)^{-2} \\ &= \frac{1}{4}\left(1-\frac{5}{2}x\right)^{-2} \end{aligned}$$

Take out a factor of 2.

Write 2^{-2} as $\frac{1}{4}$.

$$= \frac{1}{4}\left[1+(-2)\left(\frac{-5x}{2}\right)+\frac{(-2)(-3)}{2!}\left(\frac{-5x}{2}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{-5x}{2}\right)^3+\right]$$

$$= \frac{1}{4}\left[1+5x+\frac{75x^2}{4}+\frac{125x^3}{2}+\cdots\right]$$

$$= \frac{1}{4}+\frac{5}{4}x+\frac{75}{16}x^2+\frac{125}{8}x^3+\cdots$$

Expand $\left(1-\frac{5}{2}x\right)^{-2}$ using binomial expansion with $n=-2$ and $x=\frac{-5}{2}x$.

Simplify the coefficients.

Multiply by $\frac{1}{4}$.

Solutionbank C4
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 10

Question:

$f(x) = (3 + 2x)^{-3}, |x| < \frac{3}{2}$.

Find the binomial expansion of f(x), in ascending powers of x, as far as the term in x^3. Give each coefficient as a simplified fraction. *E*

Solution:

$f(x) = (3 + 2x)^{-3}$

$\left[3\left(1 + \frac{2x}{3}\right)\right]^{-3}$ ← Take out a factor of 3.

$= 3^{-3}\left(1 + \frac{2x}{3}\right)^{-3}$

Write 3^{-3} as $\frac{1}{27}$.

$= \frac{1}{27}\left(1 + (-3)\left(\frac{2x}{3}\right) + \frac{(-3)(-4)}{2!}\left(\frac{2x}{3}\right)^2 + \frac{(-3)(-4)(-5)}{3!}\left(\frac{2x}{3}\right)^3 + \cdots\right)$

Expand $\left(1 + \frac{2x}{3}\right)^{-3}$ using the binomial expansion with $n = -3$ and $x = \frac{2x}{3}$.

$= \frac{1}{27}\left(1 - 2x + \frac{48x^2}{18} - \frac{480x^3}{162} + \cdots\right)$ ← Simplify coefficients.

$= \frac{1}{27} - \frac{2x}{27} + \frac{8x^2}{81} - \frac{80x^3}{729} +$ ← Multiply by $\frac{1}{27}$.

Solutionbank C4
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 11

Question:

$$f(x)=\frac{1}{\sqrt{(1-x)}}-\sqrt{(1+x)}, -1<x<1$$

Find the series expansion of f(x) in ascending powers of x up to and including the term in x^3. ***E***

Solution:

$$f(x) = \frac{1}{\sqrt{(1-x)}}-\sqrt{(1+x)}$$

$$= (1-x)^{-\frac{1}{2}}-(1+x)^{\frac{1}{2}}$$

Write each expression in index form.

$$= \left[1+\left(\frac{-1}{2}\right)(-x)+\frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)(-x)^2}{2} + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)(-x)^3}{3!}\cdots\right]$$

Replace n by $\frac{1}{2}$ and x by $-x$ in binomial expansion.

$$-\left[1+\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{-1}{2}\right)x^2}{2}+\frac{\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)x^3}{3!}+\right]$$

Replace n by $-\frac{1}{2}$ in binomial expansion.

$$= \left[1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5x^3}{16}+\cdots\right] - \left[1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\cdots\right]$$

Simplify the terms in both series expansions.

$$= \frac{4}{8}x^2+\frac{4}{16}x^3+\cdots$$

Collect terms.

$$= \frac{1}{2}x^2+\frac{1}{4}x^3+\cdots$$

Simplify answer.

Solutionbank C4
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 12

Question:

Given that

$$\frac{3+5x}{(1+3x)(1-x)} \equiv \frac{A}{(1+3x)} + \frac{B}{(1-x)},$$

a find the values of the constants A and B.

b Hence or otherwise find the series expansion, in ascending powers of x, up to and including the term in x^2, of $\frac{3+5x}{(1+3x)(1-x)}$.

c State, with a reason, whether your series expansion in part **b** is valid for $x = \frac{1}{2}$. ***E***

Solution:

a

$$\frac{3+5x}{(1+3x)(1-x)} \equiv \frac{A}{1+3x}+\frac{B}{1-x}$$

The denominators must be $(1+3x)$ and $(1-x)$.

$$\equiv \frac{A(1-x)+B(1+3x)}{(1+3x)(1-x)}$$

Add the fractions.

$$\therefore 3+5x \equiv A(1-x)+B(1+3x)$$

Set the numerators equal.

Put $x=1$

$\therefore 8=4B \Rightarrow B=2$

Set $x=1$ to find B.

Put $x=-\frac{1}{3}$

$\therefore 3-\frac{5}{3}=\frac{4}{3}A \Rightarrow A=1$

Set $x=\frac{-1}{3}$ to find A.

b

$$\therefore \frac{3+5x}{(1+3x)(1-x)} = \frac{1}{1+3x}+\frac{2}{1-x}$$
$$= (1+3x)^{-1}+2(1-x)^{-1}$$

Write in index form.

Expand using binomial theorem:

$$= \left[1+(-1)(3x)+\frac{(-1)(-2)}{1\times 2}(3x)^2+\cdots\right]$$

Expand $(1+3x)^{-1}$ using the binomial expansion with $n=-1$ and $x=3x$.

$$+ \; 2\left[1+(-1)(-x)+\frac{(-1)(-2)(-x)^2}{1\times 2}+\cdots\right]$$

Expand $2(1-x)^{-1}$ using the binomial expansion with $n=-1$ and $x=(-x)$.

$$= \left[1-3x+9x^2 \quad +\cdots\right]+2\left(1+x+x^2+\right)$$

Simplify each expression.

$$= 3-x+11x^2-\cdots$$

Collect the terms.

c Not valid when $x=\frac{1}{2}$, as expansion of $(1+3x)^{-1}$ is valid for $|3x|<1$ only.

Terms are $(3x),(3x)^2\cdots$ and when $x=\frac{1}{2}, 3x>1$ and the terms get larger.

Solutionbank C4
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 13

Question:

$\mathrm{f}(x) = \dfrac{3x-1}{(1-2x)^2}, |x| < \dfrac{1}{2}$.

Given that, for $x \neq \dfrac{1}{2}, \dfrac{3x-1}{(1-2x)^2} \equiv \dfrac{A}{(1-2x)} + \dfrac{B}{(1-2x)^2}$, where A and B are constants,

a find the values of A and B.

b Hence or otherwise find the series expansion of $\mathrm{f}(x)$, in ascending powers of x, up to and including the term in x^3, simplifying each term. *E*

Solution:

a

$$\frac{3x-1}{(1-2x)^2} \equiv \frac{A}{(1-2x)}+\frac{B}{(1-2x)^2}$$

$$\equiv \frac{A(1-2x)+B}{(1-2x)^2}$$ ← Add the fractions.

$$\therefore 3x-1 \equiv A(1-2x)+B$$ ← Set the numerators equal.

Put $x=\frac{1}{2}$

$$\frac{1}{2} = B$$ ← Set $x=\frac{1}{2}$ to find B.

$$\therefore 3x-1 \equiv A(1-2x)+\frac{1}{2}$$

Compare coefficients of x

$$3=-2A \Rightarrow A=-\frac{3}{2}$$ ← As expressions are identical equate terms in x and put coefficients equal.

[check constant term $-1=-\frac{3}{2}+\frac{1}{2}$]

$$\therefore \frac{3x-1}{(1-2x)^2}=-\frac{3}{2}(1-2x)^{-1}+\frac{1}{2}(1-2x)^{-2}$$ ← Write in index form.

b Use binomial expansions:

$$= -\frac{3}{2}\left[1+(-1)(-2x)+\frac{(-1)(-2)}{2!}(-2x)^2+\frac{(-1)(-2)(-3)(-2x)^3}{3!}+\cdots\right]$$

$$+\frac{1}{2}\left[1+(-2)(-2x)+\frac{(-2)(-3)}{2!}(-2x)^2+\frac{(-2)(-3)(-4)(-2x)^3}{3!}+\cdots\right]$$

Expand $-\frac{3}{2}(1-2x)^{-1}$ using the binomial expansion with $n=-1$ and $x=-2x$.

Expand $\frac{1}{2}(1-2x)^{-2}$ using the binomial expansion with $n=-2$ and $x=-2x$.

$$= -\frac{3}{2}\left[1+2x+4x^2+8x^3+\cdots\right]+\frac{1}{2}\left[1+4x+12x^2+32x^3+\cdots\right]$$ ← Simplify each expression.

$$= -1-x+0x^2+4x^3+\cdots$$ ← Collect the terms.

$$= -1-x+4x^3+\cdots$$

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 14

Question:

$$f(x) = \frac{3x^2 + 16}{(1-3x)(2+x)^2}$$

$$\equiv \frac{A}{(1-3x)} + \frac{B}{(2+x)} + \frac{C}{(2+x)^2}, |x| < \frac{1}{3}.$$

a Find the values of A and C and show that $B = 0$.

b Hence, or otherwise, find the series expansion of $f(x)$, in ascending powers of x, up to and including the term in x^3. Simplify each term. ***E***

Solution:

a

$$\frac{3x^2+16}{(1-3x)(2+x)^2} \equiv \frac{A}{(1-3x)}+\frac{B}{(2+x)}+\frac{C}{(2+x)^2}$$

$$\equiv \frac{A(2+x)^2+B(1-3x)(2+x)+C(1-3x)}{(1-3x)(2+x)^2}$$ ← Add the fractions.

$$\therefore 3x^2+16 \equiv A(2+x)^2+B(1-3x)(2+x)+C(1-3x)$$ ← Set the numerators equal.

Put $x=-2$

$28=7C \Rightarrow C=4$ ← Set $x=-2$ to find C.

Put $x=\frac{1}{3}$

$16\frac{1}{3}=\frac{49}{9}A \Rightarrow A=3$ ← Set $x=\frac{1}{3}$ to find A.

$\therefore 3x^2+16 \equiv 3(2+x)^2+B(1-3x)(2+x)+4(1-3x)$

Compare x^2 terms.

$3=3-3B \Rightarrow B=0.$

Compare constants. ← Equate coefficients of terms in x^2 or equate constant terms to find B.

$16=12+2B+4 \Rightarrow B=0$

b

$$\frac{3x^2+16}{(1-3x)(2+x)^2} \equiv \frac{3}{(1-3x)}+\frac{4}{(2+x)^2}$$

$$= 3(1-3x)^{-1}+4(2+x)^{-2}$$ ← Write in index form.

$$= 3(1-3x)^{-1}+4\times 2^{-2}\left(1+\frac{x}{2}\right)^{-2}$$ ← Take out a factor of 2 so $(2+x)^{-2}=\left[2\left(1+\frac{x}{2}\right)\right]^{-2}$.

$$= 3\left[1+(-1)(-3x)+\frac{(-1)(-2)(-3x)^2}{2!}+\frac{(-1)(-2)(-3)(-3x)^3}{3!}+\right]$$ ← Expand $3(1-3x)^{-1}$ using the binomial expansion with $n=-1$ and $x=-3x$.

$$+ \frac{4}{4}\left[1+(-2)\left(\frac{x}{2}\right)+\frac{(-2)(-3)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{x}{2}\right)^3+\cdots\right]$$

$$= 3\left[1+3x+9x^2+27x^3+\cdots\right]+\left[1-x+\frac{3x^2}{4}\frac{-4x^3}{8}+\cdots\right]$$ ← Expand $4\times 2^{-2}\left(1+\frac{x}{2}\right)^{-2}$ using binomial expansion with $n=-2$ and $x=\frac{x}{2}$.

$$= 4+8x+\frac{111x^2}{4}+\frac{161}{2}x^3+\cdots$$

Solutionbank C4
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 15

Question:

$$f(x)=\frac{25}{(3+2x)^2(1-x)}.$$

a Express f(x) as a sum of partial fractions.

b Find the series expansion of f(x) in ascending powers of x up to and including the term in x^2. Give each coefficient as a simplified fraction. *E*

Solution:

a

$$\frac{25}{(3+2x)^2(1-x)} \equiv \frac{A}{(1-x)}+\frac{B}{(3+2x)}+\frac{C}{(3+2x)^2}$$

The denominators must be $(1-x), (3+2x)$ and $(3+2x)^2$.

$$\equiv \frac{A(3+2x)^2+B(1-x)(3+2x)+C(1-x)}{(1-x)(3+2x)^2}$$

Add the fractions.

$$\therefore 25 \equiv A(3+2x)^2+B(1-x)(3+2x)+C(1-x) *$$

Set the numerators equal.

Put $x=1$ in *

$\therefore 25 = A\times 5^2+0+0$

$\therefore A = 1$

Set $x=1$ to find A.

Put $x=-\frac{3}{2}$ in *

$\therefore 25 = C\times\left(1-(-\frac{3}{2})\right)=C\times 2\frac{1}{2}$

$\therefore C = 10$

Set $x=-\frac{3}{2}$ to find C.

Compare coefficients of x^2 in *

$0=4A-2B$

As $A=1, B=2$

Equate coefficients of terms in x^2 to find B.

$$\therefore \frac{25}{(3+2x)^2(1-x)} \equiv \frac{1}{(1-x)}+\frac{2}{(3+2x)}+\frac{10}{(3+2x)^2}$$

b

RHS $= (1-x)^{-1}+2(3+2x)^{-1}+10(3+2x)^{-2}$

Write the right hand side in index form.

$$(1-x)^{-1} = 1+(-1)(-x)+\frac{(-1)(-2)(-x)^2}{2!}+\cdots$$

$$= 1+x+x^2+\cdots$$

Expand $(1-x)^{-1}$ using the binomial expansion with $n=-1$ and $x=(-x)$.

$$2(3+2x)^{-1} = 2\left[3\left(1+\frac{2x}{3}\right)\right]^{-1}$$

Take out a factor of 3.

$$= 2\times 3^{-1}\left(1+\frac{2x}{3}\right)^{-1}$$

Expand $2\times 3^{-1}\left(1+\frac{2x}{3}\right)^{-1}$ using the binomial expansion with $n=-1$ and $x=\left(\frac{2x}{3}\right)$.

$$= \frac{2}{3}\left(1+(-1)\left(\frac{2x}{3}\right)+\frac{(-1)(-2)}{2!}\left(\frac{2x}{3}\right)^2\right)$$

$$= \frac{2}{3}\left(1-\frac{2}{3}x+\frac{4}{9}x^2+\cdots\right)$$

$$= \frac{2}{3}-\frac{4}{9}x+\frac{8}{27}x^2\cdots$$

$$10(3+2x)^{-2} = 10\left[3\left(1+\frac{2x}{3}\right)\right]^{-2}$$

Take out a factor of 3.

$$= 10\times 3^{-2}\left(1+\frac{2x}{3}\right)^{-2}$$

$$= \frac{10}{9}\left(1+(-2)\left(\frac{2x}{3}\right)+\frac{(-2)(-3)}{2!}\left(\frac{2x}{3}\right)^2+\cdots\right)$$

$$= \frac{10}{9}\left(1-\frac{4x}{3}+\frac{4x^2}{3}\cdots\right)$$

$$= \frac{10}{9}-\frac{40x}{27}+\frac{40x^2}{27}\cdots$$

Expand $10\times 3^{-2}\left(1+\frac{2x}{3}\right)^{-2}$ using the binomial expansion with $n=-2$ and $x=\frac{2x}{3}$.

Adding these series expansions gives

$$\left(1+\frac{2}{3}+\frac{10}{9}\right)+\left(1-\frac{4}{9}-\frac{40}{27}\right)x+\left(1+\frac{8}{27}+\frac{40}{27}\right)x^2$$

Add the three series expansions and collect and simplify the coefficients.

$$=\frac{25}{9}+\frac{-25}{27}x+\frac{25}{9}x^2+\cdots$$

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Review Exercise
Exercise A, Question 16

Question:

When $(1+ax)^n$ is expanded as a series in ascending powers of x, the coefficients of x and x^2 are -6 and 45 respectively.

a Find the value of a and the value of n.

b Find the coefficient of x^3.

c Find the set of values of x for which the expansion is valid.

E[adapted]

Solution:

a $$(1+ax)^n \equiv 1 + nax + \frac{n(n-1)}{2}a^2x^2 + \cdots$$

$\therefore na = -6$ and ......(1)

$\frac{n(n-1)}{2}a^2 = 45$ ......(2)

Set coefficient of x, from binomial theorem, equal to 6 and set coefficient of x^2 equal to 45.

From (1) $a = \frac{-6}{n}$, substitute into equation (2).

$\therefore \frac{n(n-1)}{2} \times \frac{36}{n^2} = 45$

$\therefore 36n^2 - 36n = 90n^2$

$\therefore -36n = 54n^2$

Eliminate a from the simultaneous equations to obtain equation in one variable n.

$\Rightarrow n = 0$ or $n = \frac{-36}{54} = \frac{-2}{3}$

Solve to find non-zero value for n.

Substitute into equation (1) to give $a = 9$.

Check solutions in equation (2).

b Coefficient of $x^3 = \frac{n(n-1)(n-2)}{3!}a^3$

$$= \frac{-\frac{2}{3} \times -\frac{5}{3} \times -\frac{8}{3} \times 9^3}{3!}$$

$$= \frac{-80 \times 27}{6}$$

$$= -360$$

Substitute values found for n and a into the binomial expansion to give the coefficient of x^3.

c $|x| < \frac{1}{a}$, so $\frac{-1}{9} < x < \frac{1}{9}$

The terms in the expansion are $(9x)$, $(9x)^2$, $(9x)^3$ and so $|9x| < 1$.

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Review Exercise
Exercise A, Question 17

Question:

a Find the binomial expansion of $\sqrt{(1-x)}$, in ascending powers of x up to and including the term in x^3.

b By substituting a suitable value for x in this expansion, find an approximation to $\sqrt{0.9}$, giving your answer to 6 decimal places.

Solution:

a $\sqrt{(1-x)} = (1-x)^{\frac{1}{2}}$

Write the expression in index form.

$$= 1+\frac{1}{2}(-x)+\frac{\frac{1}{2}\left(\frac{-1}{2}\right)}{2!}(-x)^2+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{-3}{2}\right)}{3!}(-x)^3+\cdots$$

$$= 1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3\cdots$$

Replace n by $\frac{1}{2}$ and x by $-x$ in the binomial expansion. Simplify the terms.

b Let $(1-x)=0.9$ and solve

Put $x=0.1$ into expansion

This is valid as $|x|<1$.

$$\sqrt{0.9} = 1-0.05-\frac{1}{8}\times 0.01-\frac{1}{16}\times 0.001$$

$$= 1-0.05-0.00125-0.0000625$$

$$= 0.948688 \text{ (6 d.p.)}$$

This gives an estimate for $\sqrt{0.9}$. You would need to calculate further terms to give increased accuracy.

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Review Exercise
Exercise A, Question 18

Question:

In the binomial expansion, in ascending powers of x, of $(1+ax)^n$, where a and n are constants, the coefficient of x is 15. The coefficients of x^2 and of x^3 are equal.

a Find the value of a and the value of n.

b Find the coefficient of x^3.

Solution:

a

$$(1+ax)^n = 1+nax+\frac{n(n-1)}{2}a^2x^2+\cdots+\frac{n(n-1)(n-2)a^3x^3}{6}+\cdots$$

As coefficient of x is 15

$na = 15$ ...... (1)

Set the coefficient of x from the binomial theorem equal to 15 and set the coefficients of x^2 and x^3 as equal to each other.

As coefficient of x^2 and x^3 are equal:

$$\frac{n(n-1)}{2}a^2 = \frac{n(n-1)(n-2)a^3}{6}$$

and $\therefore (n-2)a = 3$ ...... (2)

Divide both sides of the equation by $\frac{n(n-1)}{6}a^2$.

Subtract equation on (2) from equation (1)

$2a = 12 \Rightarrow a = 6$

Substitute into equation (1)

Solve equations (1) and (2) as simultaneous equations and check your answer.

$\therefore n = \frac{15}{6} = \frac{5}{2}$.

b Coefficient of x^3 is

$$\frac{n(n-1)(n-2)}{6}a^3 = \frac{\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}}{6}\times 6^3 = \frac{15}{8}\times 36 = \frac{135}{2} = 67.5$$

Substitute the values you have found for a and n into the binomial expansion term for x^3.

[You could also check the term for x^2, which should be equal.]

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Review Exercise
Exercise A, Question 19

Question:

The vectors **u** and **v** are given by
$\mathbf{u} = 5\mathbf{i} - 4\mathbf{j} + s\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + t\mathbf{j} - 3\mathbf{k}$

a Given that the vectors **u** and **v** are perpendicular, find a relation between the scalars s and t.

b Given instead that the vectors **u** and **v** are parallel, find the values of the scalars s and t. *E*

Solution:

a $\mathbf{u}\cdot\mathbf{v} = \begin{pmatrix}5\\-4\\s\end{pmatrix}\cdot\begin{pmatrix}2\\t\\-3\end{pmatrix}$ ← Write in column matrix form.

$\therefore \mathbf{u}\cdot\mathbf{v} = 5\times2 + (-4)\times t + S\times -3$ ← Use $\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$.

Use the condition $\mathbf{u}\cdot\mathbf{v} = 0$ ← For perpendicular vectors the scalar product is zero.

$\therefore 10 - 4t - 3s = 0$
or $3s + 4t = 10$ ← Simplify your answers.

b As **u** and **v** are parallel
$\mathbf{v} = \lambda\mathbf{u}$ where λ is constant.
$\therefore 2\mathbf{i} + t\mathbf{j} - 3\mathbf{k} = \lambda(5\mathbf{i} - 4\mathbf{j} + s\mathbf{k})$ ← For parallel vectors one vector is a multiple of the other.

Compare coefficients of **i**, **j** and **k**.

$5\lambda = 2 \Rightarrow \lambda = \frac{2}{5}$ ← Equate the coefficients of x, y and z.

$t = -4\lambda \Rightarrow t = -\frac{8}{5} = -1.6$

$\lambda s = -3 \Rightarrow s = -3 \div \frac{2}{5}$

$= \frac{-15}{2} = -7.5$ ← Solve to find the values of s and t.

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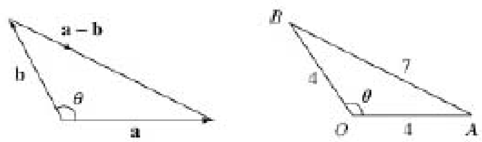
Review Exercise
Exercise A, Question 20

Question:

Find the angle between the vectors **a** and **b** given that $|\mathbf{a}|=4, |\mathbf{b}|=4$ and $|\mathbf{a}-\mathbf{b}|=7$. *E* **[adapted]**

Solution:



Use the triangle law and draw two triangles. One shows vectors. The other shows the magnitudes of the vectors.

Use the cosine rule on ΔOAB.

$$7^2 = 4^2+4^2-2\times4\times4\times\cos\theta$$

$$\therefore 49 = 16+16-32\cos\theta$$

use the cosine rule to find $\cos\theta$.

$$\Rightarrow 32\cos\theta = -17$$

$$\therefore \cos\theta = \frac{-17}{32}$$

$$\therefore \theta = 122^\circ (3\text{ s.f.})$$

The cosine is negative, so the angle is obtuse.

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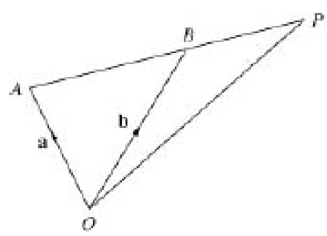
Review Exercise
Exercise A, Question 21

Question:

The position vectors of the points A and B relative to an origin O are $3\mathbf{i}+2\mathbf{j}-\mathbf{k}, 5\mathbf{i}+6\mathbf{j}+5\mathbf{k}$, respectively. Find the position vector of the point P which lies on AB produced such that $AP=3BP$. **E [adapted]**

Solution:



Draw a sketch to help you to see the triangles used later.

$$\overrightarrow{AB} = \mathbf{b}-\mathbf{a} \quad \ldots\ldots (1)$$
$$= 2\mathbf{i}+4\mathbf{j}+6\mathbf{k}$$

Use the triangle law to give $\overrightarrow{AB}$.

As $\overrightarrow{AP}=3\overrightarrow{BP}$
and $\overrightarrow{AP}=\overrightarrow{AB}+\overrightarrow{BP}$
$\therefore \overrightarrow{AB}+\overrightarrow{BP}=3\overrightarrow{BP}$
$\therefore \overrightarrow{AB}=2\overrightarrow{BP}$
$\therefore$ From (1) $\overrightarrow{BP}=\mathbf{i}+2\mathbf{j}+3\mathbf{k}$

As A, B and P are points on the same line, one vector is a multiple of the other.

$$\therefore \overrightarrow{OP} = \overrightarrow{OB}+\overrightarrow{BP}$$
$$= 6\mathbf{i}+8\mathbf{j}+8\mathbf{k}$$

Use the triangle law to give $\overrightarrow{OP}$.

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Review Exercise
Exercise A, Question 22

Question:

The points A and B have coordinates $(2t, 10, 1)$ and $(3t, 2t, 5)$ respectively.

a Find $|\overrightarrow{AB}|$.

b By differentiating $|\overrightarrow{AB}|^2$, find the value of t for which $|\overrightarrow{AB}|$ is a minimum.

c Find the minimum value of $|\overrightarrow{AB}|$.

Solution:

$\mathbf{a} = \begin{pmatrix} 2t \\ 10 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3t \\ 2t \\ 5 \end{pmatrix}$ ← Write down the position vectors of A and B.

a

$\therefore \overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} t \\ 2t-10 \\ 4 \end{pmatrix}$ ← Use $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

$\therefore |\overrightarrow{AB}| = \sqrt{t^2 + (2t-10)^2 + 4^2}$ ← Use the vector magnitude formula.

$= \sqrt{5t^2 - 40t + 116}$

b $|\overrightarrow{AB}|^2 = 5t^2 - 40t + 116$ ← Call this p and differentiate.

Differentiating with respect to t gives

$\frac{\mathrm{d}p}{\mathrm{d}t} = 10t - 40$

So

$10t - 40 = 0$

$t = 4$ ← Use the fact that $\frac{\mathrm{d}p}{\mathrm{d}t} = 0$ at a minimum.

$\frac{\mathrm{d}^2p}{\mathrm{d}t^2} = 10$, positive, $\therefore$ minimum ← Use the fact that if the second derivative is positive, the value is a minimum.

c

$|\overrightarrow{AB}| = \sqrt{5t^2 - 40t + 116}$ ← Substitute $t = 4$ back into $|\overrightarrow{AB}|$.

$= \sqrt{80 - 160 + 116}$

$= \sqrt{36}$

$= 6$

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Review Exercise
Exercise A, Question 23

Question:

The line l_1 has vector equation $\mathbf{r} = 11\mathbf{i} + 5\mathbf{j} + 6\mathbf{k} + \lambda(4\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})$ and the line l_2 has vector equation

$\mathbf{r} = 24\mathbf{i} + 4\mathbf{j} + 13\mathbf{k} + \mu(7\mathbf{i} + \mathbf{j} + 5\mathbf{k})$,

where λ and μ are parameters.

a Show that the lines l_1 and l_2 intersect.

b Find the coordinates of their point of intersection.

Given that θ is the acute angle between l_1 and l_2

c Find the value of $\cos\theta$. Give your answer in the form $k\sqrt{3}$, where k is a simplified fraction. ***E***

Solution:

a Assuming that the lines do intersect:

$$\begin{pmatrix}11+4\lambda\\5+2\lambda\\6+4\lambda\end{pmatrix}=\begin{pmatrix}24+7\mu\\4+\mu\\13+5\mu\end{pmatrix} \quad *$$

You can write the equations of the lines in column vector form and put them equal.

Rearranging gives:

$$\begin{aligned}4\lambda-7\mu &= 13 \quad (1)\\2\lambda-\mu &= -1 \quad (2)\\4\lambda-5\mu &= 7 \quad (3)\end{aligned}$$

Equate the x, y and z components.

Solve these simultaneous equations.
(1) − (3) gives

$$\begin{aligned}-2\mu &= 6\\ \therefore \mu &= -3\end{aligned}$$

Solve equations (1) and (3) simultaneously.

Substitute into (1) to give

$$4\lambda+21=13 \Rightarrow \lambda=-2$$

As this solution satisfies all three equations, the lines *do* meet.

y components must also be equal so $\mu=-3, \lambda=-2$ must

b Substituting into * gives the coordinates of the point of intersection

$$\begin{pmatrix}11-8\\5-4\\6-8\end{pmatrix}=\begin{pmatrix}24-21\\4-3\\13-15\end{pmatrix}=\begin{pmatrix}3\\1\\-2\end{pmatrix}$$

Substituting λ or μ will give the point of intersection.

$\therefore (3,1,-2)$ is point of intersection.

The directions of the lines are
$4\mathbf{i}+2\mathbf{j}+4\mathbf{k}$ and $7\mathbf{i}+\mathbf{j}+5\mathbf{k}$

c

$$\begin{aligned}\cos\theta &= \frac{(4\mathbf{i}+2\mathbf{j}+4\mathbf{k})\cdot(7\mathbf{i}+\mathbf{j}+5\mathbf{k})}{\sqrt{4^2+2^2+4^2}\sqrt{7^2+1^2+5^2}}\\ &= \frac{28+2+20}{\sqrt{36}\sqrt{75}}\\ &= \frac{50}{6\times5\sqrt{3}}\\ &= \frac{5}{3\sqrt{3}}\\ &= \frac{5}{9}\sqrt{3}\end{aligned}$$

Use $\cos\theta=\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$, where $\mathbf{a}$ and $\mathbf{b}$ are the direction vectors of the lines.

Simplify the surds.

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Review Exercise
Exercise A, Question 24

Question:

The line l_1 has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and the line l_2 has equation

$\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

a Show that l_1 and l_2 do not meet.

A is the point on l_1 where $\lambda = 1$ and B is the point on l_2 where $\mu = 2$.

b Find the cosine of the acute angle between AB and l_1. ***E***

Solution:

a Assume that the lines do meet:

$$\begin{pmatrix}1+\lambda\\0+\lambda\\-1\end{pmatrix}=\begin{pmatrix}1+2\mu\\3+\mu\\6-\mu\end{pmatrix}$$

Put the right hand sides of the equations of the two lines equal.

So
$$2\mu-\lambda = 0 \quad (1)$$
$$\mu-\lambda = -3 \quad (2)$$
$$-1 = 6-\mu \quad (3)$$

Equate the x, y and z components.

Solve equation (3) to give $\mu=7$ substitute into equation (1) to give $\lambda=14$.

Solve equations (1) and (3) simultaneously.

Check in equation (2) $7-14\neq-3$ to find a contradiction.

This implies that no values for λ and μ satisfy all three equations simultaneously
$\therefore$ The lines do not meet.

The values $\mu=7, \lambda=14$ do not satisfy equation (2) and so y components are not equal.

b A is the point $(2,1,-1)$ and
B is the point $(5, 5, 4)$

Substitute $\lambda=1$ into equation of line l_1.

Substitute $\mu=2$ into equation of line l_2.

So $\overrightarrow{AB}=\begin{pmatrix}3\\4\\5\end{pmatrix}$

Use $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}$.

Direction of l_1 is $\begin{pmatrix}1\\1\\0\end{pmatrix}$

Obtain the direction of the line l_1 from the equation of l_1.

$$\therefore \cos\theta = \frac{\begin{pmatrix}3\\4\\5\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}}{\sqrt{3^2+4^2+5^2}\sqrt{1^2+1^2+0^2}}=\frac{3+4+0}{\sqrt{50}\sqrt{2}}$$
$$=\frac{7}{10}$$

Use $\cos\theta=\dfrac{\mathbf{c}\cdot\mathbf{d}}{|\mathbf{c}||\mathbf{d}|}$ where $\mathbf{c}$ is the vector $\overrightarrow{AB}$ and $\mathbf{d}$ is the direction of the line l_1.

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Review Exercise
Exercise A, Question 25

Question:

The line l_1 has vector equation

$\mathbf{r} = 8\mathbf{i} + 12\mathbf{j} + 14\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} - \mathbf{k})$.

The points A, with coordinates $(4, 8, a)$, and B, with coordinates $(b, 13, 13)$, lie on this line.

a Find the values of a and b.

Given that the point O is the origin, and that the point P lies on l_1 such that OP is perpendicular to l_1,

b find the coordinates of P.

c Hence find the distance OP, giving your answer as a simplified surd. *E*

Solution:

a $\mathbf{r} = \begin{pmatrix} 8+\lambda \\ 12+\lambda \\ 14-\lambda \end{pmatrix}$

You can write the line equation in this form.

The position vector of A $\begin{pmatrix} 4 \\ 8 \\ a \end{pmatrix} = \begin{pmatrix} 8+\lambda \\ 12+\lambda \\ 14-\lambda \end{pmatrix}$

As A lies on the line equate position vectors.

use $4 = 8+\lambda$ or $8 = 12+\lambda$
$\therefore \lambda = -4$

Use the x or y coordinates to find λ.

Substitute to give $a = 14-\lambda = 18$

Find a using the value of λ.

The position vector of B $\begin{pmatrix} b \\ 13 \\ 13 \end{pmatrix} = \begin{pmatrix} 8+\lambda \\ 12+\lambda \\ 14-\lambda \end{pmatrix}$

Also B lies on the line.

Use $13 = 12+\lambda$ or $13 = 14-\lambda$

Use the y or z coordinates to find λ.

$\therefore \lambda = 1$
Substitute to give $b = 8+\lambda = 9$

Find b using this value of λ.

b Direction $\overrightarrow{OP}$ is $\begin{pmatrix} 8+\lambda \\ 12+\lambda \\ 14-\lambda \end{pmatrix}$

and direction of l_1 is $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

This is obtained from the equation of l_1.

These are perpendicular

$\therefore \begin{pmatrix} 8+\lambda \\ 12+\lambda \\ 14-\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0$

Use the condition for perpendicular lines, $\mathbf{c} \cdot \mathbf{d} = 0$.

$\therefore 8+\lambda+12+\lambda-(14-\lambda) = 0$
$\therefore 3\lambda+6 = 0$
$\Rightarrow \lambda = -2$
$\therefore$ Point P is at (6, 10, 16)

Substitute the value of λ into the line equation to give the coordinates of P.

c

Distance $OP = \sqrt{6^2+10^2+16^2}$
$= \sqrt{392}$
$= 14\sqrt{2}$

Use the formula for magnitude of a vector.

Simplify the surd using $\sqrt{392} = \sqrt{196}\sqrt{2}$.

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Review Exercise
Exercise A, Question 26

Question:

The line l_1 has equation

$\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$ and the line l_2 has equation $\mathbf{r} = \begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

Find, by calculation,

a the coordinates of B, the point of intersection of l_1 and l_2,

b the value of $\cos\theta$, where θ is the acute angle between l_1 and l_2. (Give your answer as a simplified fraction.)

The point A, which lies on l_1 has position vector $\mathbf{a} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. The point C, which lies on l_2, has position vector $\mathbf{c} = 5\mathbf{i} - \mathbf{j} - 2\mathbf{k}$. The point D lies in the plane ABC and $ABCD$ is a parallelogram.

c Show that $|\overrightarrow{AB}| = |\overrightarrow{BC}|$.

d Find the position vector of the point D. ***E***

Solution:

a As the two lines meet:

$$\begin{pmatrix}3+\lambda\\1-\lambda\\2+4\lambda\end{pmatrix}=\begin{pmatrix}\mu\\4-\mu\\-2\end{pmatrix}$$

You can write the equations in this form.

$$\therefore \mu-\lambda = 3 \quad (1)$$
$$-\mu+\lambda = -3 \quad (2)$$
$$4\lambda = -4 \quad (3)$$

Put the x, y and z components equal and rearrange.

Use equation (3) to find λ.

Solve equation (3) to give $\lambda=-1$
Substitute this value into equation (1)
Then $\mu=2$

Substitute λ into equation (1) or (2) to find μ.

$\therefore$ Point of intersection is $(2,2,-2)$

Use λ or μ in the line equations to find the coordinates of B.

b The directions of the lines are $\begin{pmatrix}1\\-1\\4\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\0\end{pmatrix}$

The directions of the lines are from the line equation.

$$\therefore \cos\theta = \frac{1\times1+(-1)\times(-1)+4\times0}{\sqrt{1^2+(-1)^2+4^2}\sqrt{1^2+(-1)^2}}$$
$$= \frac{2}{\sqrt{18}\sqrt{2}}$$
$$= \frac{1}{3}$$

Use $\cos\theta=\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}||\mathbf{y}|}$ where $\mathbf{x}$ and $\mathbf{y}$ are the directions of the lines.

c

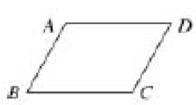
$$\overrightarrow{AB} = \mathbf{b}-\mathbf{a}=-\mathbf{i}+\mathbf{j}-4\mathbf{k}$$
$$\overrightarrow{BC} = \mathbf{c}-\mathbf{b}=3\mathbf{i}-3\mathbf{j}$$

Use the triangle law to find $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

$$|\overrightarrow{AB}| = \sqrt{((-1)^2+1^2+(-4)^2)}=\sqrt{18}=3\sqrt{2}$$
$$|\overrightarrow{BC}| = \sqrt{(3^2+(-3)^2)}=\sqrt{18}=3\sqrt{2}$$
$$\therefore |\overrightarrow{AB}|=|\overrightarrow{BC}|$$

Use the formula for the magnitude of a vector.

d



Draw a diagram and label the vertices of the parallelogram as A, B, C and D in a cyclic order.

$$\overrightarrow{BA} = \overrightarrow{CD}$$
$$\overrightarrow{BA} = \mathbf{i}-\mathbf{j}+4\mathbf{k}=\overrightarrow{CD}$$

Use the fact that opposite sides are equal and parallel.

$$\overrightarrow{OD} = \overrightarrow{OC}+\overrightarrow{CD}$$

Also $= (5\mathbf{i}-\mathbf{j}-2\mathbf{k})+(\mathbf{i}-\mathbf{j}+4\mathbf{k})$
$= (6\mathbf{i}-2\mathbf{j}+2\mathbf{k})$

Use the triangle law.

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Review Exercise
Exercise A, Question 27

Question:

The points A and B have position vectors $5\mathbf{j}+11\mathbf{k}$ and $c\mathbf{i}+d\mathbf{j}+21\mathbf{k}$ respectively, where c and d are constants.
The line AB has vector equation
$\mathbf{r}=5\mathbf{j}+11\mathbf{k}+\lambda(2\mathbf{i}+\mathbf{j}+5\mathbf{k})$.

a Find the value of c and the value of d.

The point P lies on the line AB, and $\overrightarrow{OP}$ is perpendicular to the line AB, where O is the origin.

b Find the position vector of P.

c Find the area of triangle OAB, giving your answer to 3 significant figures. *E*

Solution:

$$\mathbf{r} = \begin{pmatrix} 2\lambda \\ \lambda+5 \\ 5\lambda+11 \end{pmatrix}$$

You can write the line equation in this form.

a As B lies on the line

$2\lambda = c, (\lambda+5) = d, 5\lambda+11 = 21$

$\therefore$ Solving $5\lambda+11=21, \lambda=2$

and substituting into other equations gives $c=4, d=7$.

Use the z coordinate to find the value of λ.

Find c and d using the value of λ.

b $\begin{pmatrix} 2\lambda \\ \lambda+5 \\ 5\lambda+11 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} = 0$

use $\overrightarrow{OP} \cdot \text{y} = 0$ where y is the direction of the line and is obtained from the equation of the line.

$\therefore 2(2\lambda)+1(\lambda+5)+5(5\lambda+11)=0$

$\therefore 30\lambda+60=0$

$\therefore \lambda=-2$

$\therefore P$ has position vector $\begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}$

Substitute $\lambda=-2$ into the equation of the line.

c Area of $\Delta OAB = \frac{1}{2}|\overrightarrow{OA}|\cdot|\overrightarrow{OB}|\sin B\hat{O}A$

and $\cos B\hat{O}A = \dfrac{\overrightarrow{OA}\cdot\overrightarrow{OB}}{|\overrightarrow{OA}|\cdot|\overrightarrow{OB}|}$

Use area of triangle is $\frac{1}{2}$ $ab\sin C$.

$\overrightarrow{OA} = \begin{pmatrix} 0 \\ 5 \\ 11 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 4 \\ 7 \\ 21 \end{pmatrix}$

$$\therefore \cos B\hat{O}A = \frac{0\times4+5\times7+11\times21}{\sqrt{(0^2+5^2+11^2)}\sqrt{(4^2+7^2+21^2)}}$$

$$= \frac{266}{\sqrt{146}\sqrt{506}}$$

$$= 0.9787 \text{ (4 s.f.)}$$

Use the scalar product to find the angle between $\overline{OA}$ and $\overline{OB}$.

$\therefore B\hat{O}A = 11.86$ (4 s.f.)

$\therefore$ Area $= 27.9$ (3 s.f.)

Substitute $\sqrt{146}, \sqrt{506}$ and angle $11.86°$ into the formula for area of a triangle.

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Review Exercise
Exercise A, Question 28

Question:

The points A and B have position vectors $\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and $4\mathbf{i}+3\mathbf{j}-2\mathbf{k}$ respectively.

a Find $|\overrightarrow{AB}|$.

b Find a vector equation for the line l_1 which passes through the points A and B.

A second line l_2 has vector equation

$\mathbf{r}=6\mathbf{i}+4\mathbf{j}-3\mathbf{k}+\mu(2\mathbf{i}+\mathbf{j}-\mathbf{k})$.

c Show that the line l_2 also passes through B.

d Find the size of the acute angle between l_1 and l_2.

e Hence, or otherwise, find the shortest distance from A to l_2. *E*

Solution:

a $\mathbf{a} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}, \mathbf{b} = 4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

$$\therefore \overrightarrow{AB} = \mathbf{b} - \mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$$

Use the triangle law.

$$\therefore |\overrightarrow{AB}| = \sqrt{3^2 + 4^2 + (-5)^2} = \sqrt{50} \text{ or } 5\sqrt{2} \text{ or } 7.07$$

Use the formula for the magnitude of a vector.

b $\mathbf{r} = \mathbf{i} - \mathbf{j} + 3\mathbf{k} + \lambda(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$

or

$\mathbf{r} = 4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} + \mu(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$

There are other forms of this equation, but these two are the simplest.

c If $\mathbf{r} = 6\mathbf{i} + 4\mathbf{j} - 3\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} - \mathbf{k})$ passes through $4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

then
$$6 + 2\mu = 4$$
$$4 + \mu = 3$$
$$-3 - \mu = -2$$

Equate *x*, *y* and *z* components.

As $\mu = -1$ satisfies all three equations, the line passes through B as required.

Solve for μ and check that μ satisfies *all three* equations.

d The lines have directions $3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ and $2\mathbf{i} + \mathbf{j} - \mathbf{k}$

If the angle between the lines is θ then

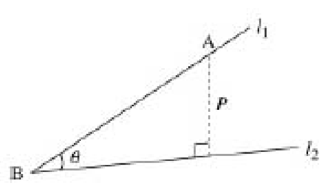
Use $\cos\theta = \dfrac{\mathbf{c} \cdot \mathbf{d}}{|\mathbf{c}||\mathbf{d}|}$ where $\mathbf{c}$ and $\mathbf{d}$ are the directions of the lines.

$$\cos\theta = \frac{(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} - \mathbf{k})}{|3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}||2\mathbf{i} + \mathbf{j} - \mathbf{k}|}$$
$$= \frac{3 \times 2 + 4 \times 1 + (-5) \times (-1)}{\sqrt{50}\sqrt{2^2 + 1^2 + (-1)^2}}$$
$$= \frac{15}{\sqrt{50}\sqrt{6}}$$

$$\therefore \cos\theta = \frac{\sqrt{3}}{2}$$
$$\text{and } \theta = 30^\circ$$

This answer is acute. If your answer is obtuse, subtract it from 180°.

e



Draw a diagram showing l_1, l_2 with common point B.

The shortest distance from point A to the line l_2 is

The shortest distance is the perpendicular distance.

$$|\overrightarrow{AB}|\sin\theta = 5\sqrt{2} \times \frac{1}{2}$$

Use trigonometry $\sin\theta = \dfrac{P}{|\overrightarrow{AB}|}$.

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Review Exercise
Exercise A, Question 29

Question:

The point A, with coordinates $(0, a, b)$ lies on the line l_1, which has equation
$\mathbf{r} = 6\mathbf{i} + 19\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$.

a Find the values of a and b.

The point P lies on l_1 and is such that OP is perpendicular to l_1, where O is the origin.

b Find the position vector of point P.

Given that B has coordinates $(5, 15, 1)$,

c show that the points A, P and B are collinear and find the ratio $AP:PB$. ***E***

Solution:

a $\mathbf{r} = \begin{pmatrix} 6+\lambda \\ 19+4\lambda \\ -1-2\lambda \end{pmatrix}$

You can write the equation in the form.

$\therefore 6+\lambda = 0 \quad (1)$
$19+4\lambda = a \quad (2)$
$-1-2\lambda = b \quad (3)$

Equate x, y and z coordinates of A to those of the line.

From equation (1) $\lambda = -6$

Find λ from equation (1).

Substituting this value for λ into equation (2) gives $a = -5$
Substituting $\lambda = -6$ into equation (3) gives $b = 11$

Find a and b using this value of λ.

b $\overrightarrow{OP} = \begin{pmatrix} 6+\lambda \\ 19+4\lambda \\ -1-2\lambda \end{pmatrix}$ and l_1 is in direction $\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$

The directions of $\overrightarrow{OP}$ and of l_1 are obtained from the equation of the line.

As $\overrightarrow{OP}$ is perpendicular to l_1,

$\begin{pmatrix} 6+\lambda \\ 19+4\lambda \\ -1-2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = 0$

Use condition for perpendicular lines, $\mathbf{c}\cdot\mathbf{d} = 0$.

$\therefore 6+\lambda+4(19+4\lambda)-2(-1-2\lambda) = 0$
i.e. $84+21\lambda = 0 \Rightarrow \lambda = -4$
$\therefore \overrightarrow{OP} = 2\mathbf{i}+3\mathbf{j}+7\mathbf{k}$

Substitute $\lambda = -4$ back into vector $\overrightarrow{OP}$.

c $\overrightarrow{OA} = -5\mathbf{j}+11\mathbf{k}$
$\therefore \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = 2\mathbf{i}+8\mathbf{j}-4\mathbf{k}$

Find $\overrightarrow{AP}$ using triangle law.

$\overrightarrow{OB} = 5\mathbf{i}+15\mathbf{j}+\mathbf{k}$
$\therefore \overrightarrow{PB} = \overrightarrow{OB} - \overrightarrow{OP} = 3\mathbf{i}+12\mathbf{j}-6\mathbf{k}$

Find $\overrightarrow{PB}$ using triangle law.

$\therefore \overrightarrow{AP} = \frac{2}{3}\overrightarrow{PB} \Rightarrow$ vectors are in the same direction, and as they have a point in mmon they are collinear.

Ratio

$$\overrightarrow{AP} : \overrightarrow{PB} = \frac{2}{3}\overrightarrow{PB} : \overrightarrow{PB} = \frac{2}{3} : 1 = 2 : 3$$

Note that each of these vectors is a multiple of $\mathbf{i}+4\mathbf{j}-2\mathbf{k}$ and so one is a multiple of the other.

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Review Exercise
Exercise A, Question 30

Question:

The point A has position vector $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and the point B has position vector $\mathbf{b} = \mathbf{i} + \mathbf{j} - 4\mathbf{k}$, relative to an origin O.

a Find the position vector of the point C, with position vector $\mathbf{c}$, given by $\mathbf{c} = \mathbf{a} + \mathbf{b}$.

b Show that $OACB$ is a rectangle, and find its exact area.

The diagonals of the rectangle, AB and OC meet at the point D.

c Write down the position vector of the point D.

d Find the size of the angle ADC. *E*

Solution:

a

$$\begin{aligned}\mathbf{c} &= \mathbf{a}+\mathbf{b}\\ &= (2\mathbf{i}+2\mathbf{j}+\mathbf{k})+(\mathbf{i}+\mathbf{j}-4\mathbf{k})\\ &= 3\mathbf{i}+3\mathbf{j}-3\mathbf{k}\end{aligned}$$

b As $\overline{OA}=\overline{BC}$ and $\overline{OB}=\overline{AC}$ $OACB$ is a parallelogram.

Opposite sides are equal and parallel.

As $\mathbf{a}\cdot\mathbf{b}=2+2-4=0$

a is perpendicular to **b**

Adjacent sides are perpendicular.

$\therefore OACB$ is a parallelogram with all of its angles right angles i.e., it is a rectangle

$$\begin{aligned}\text{Its area} &= |\mathbf{a}|\times|\mathbf{b}|\\ &= \sqrt{2^2+2^2+1^2}\times\sqrt{1^2+1^2+(-4)^2}\\ &= 3\times3\sqrt{2}\\ &= 9\sqrt{2}\end{aligned}$$

Use the formula for magnitude of a vector.

c The diagonals bisect each other.

D is the mid-point of OC, and of AB.

$$\therefore \mathbf{d}=\frac{3}{2}\mathbf{i}+\frac{3}{2}\mathbf{j}-\frac{3}{2}\mathbf{k}$$

d

$$\begin{aligned}\overrightarrow{AD} &= \mathbf{d}-\mathbf{a}=-\frac{1}{2}\mathbf{i}-\frac{1}{2}\mathbf{j}-\frac{5}{2}\mathbf{k}\\ \overrightarrow{CD} &= \mathbf{d}-\mathbf{c}=-\frac{3}{2}\mathbf{i}-\frac{3}{2}\mathbf{j}+\frac{3}{2}\mathbf{k}\end{aligned}$$

Use the triangle law to find $\overrightarrow{AD}$ and $\overrightarrow{CD}$, or $\overrightarrow{DA}$ and $\overrightarrow{DC}$.

$$\begin{aligned}\therefore \cos A\hat{D}C &= \frac{\overrightarrow{AD}\cdot\overrightarrow{CD}}{|\overrightarrow{AD}|\cdot|\overrightarrow{CD}|}\\ &= \frac{\frac{3}{4}+\frac{3}{4}-\frac{15}{4}}{\sqrt{\frac{1}{4}+\frac{1}{4}+\frac{25}{4}}\sqrt{\frac{9}{4}+\frac{9}{4}+\frac{9}{4}}}\\ &= \frac{-\frac{9}{4}}{\frac{27}{4}}\\ &= \frac{-1}{3}\end{aligned}$$

Use the formula $\cos\theta=\dfrac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}||\mathbf{y}|}$ with $x=\overrightarrow{AD}$ and $y=\overrightarrow{CD}$.

As $\cos A\hat{D}C$ is negative, angle ADC is obtuse.

$$\therefore A\hat{D}C = 109.5^\circ\ (1\text{ d.p.})$$

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Review Exercise
Exercise A, Question 31

Question:

Relative to a fixed origin O, the point A has position vector $5\mathbf{j}+5\mathbf{k}$ and the point B has position vector $3\mathbf{i}+2\mathbf{j}-\mathbf{k}$.

a Find a vector equation of the line L which passes through A and B.

The point C lies on the line L and OC is perpendicular to L.

b Find the position vector of C.

The points O, B and A together with the point D lie at the vertices of parallelogram $OBAD$.

c Find the position vector of D.

d Find the area of the parallelogram $OBAD$. ***E***

Solution:

a $\mathbf{a} = \begin{pmatrix}0\\5\\5\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}3\\2\\-1\end{pmatrix}$

The position vectors can be written in this form.

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a},$

Using $\overrightarrow{AB} = \begin{pmatrix}3\\2\\-1\end{pmatrix} - \begin{pmatrix}0\\5\\5\end{pmatrix}$

Use the triangle law.

$= \begin{pmatrix}3\\-3\\-6\end{pmatrix}$

The direction of the line is any multiple of $\begin{pmatrix}1\\-1\\-2\end{pmatrix}$

You might have found $\overrightarrow{BA}$ and used $\begin{pmatrix}-1\\1\\2\end{pmatrix}$.

An equation of the line is

$\mathbf{r} = \begin{pmatrix}0\\5\\5\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\-2\end{pmatrix}$ or $\mathbf{r} = \begin{pmatrix}3\\2\\-1\end{pmatrix} + \mu\begin{pmatrix}1\\-1\\-2\end{pmatrix}$

You only need *one* form of the equation.

b C lies on the line

$\therefore OC = \begin{pmatrix}\lambda\\5-\lambda\\5-2\lambda\end{pmatrix}$ or $\begin{pmatrix}3+\mu\\2-\mu\\-1-2\mu\end{pmatrix}$

You obtain your directions from the equation of line l.

The direction of L is $\begin{pmatrix}1\\-1\\-2\end{pmatrix}$

Use the condition for perpendicular vectors $\mathbf{x}\cdot\mathbf{y} = 0$.

As OC is perpendicular to L

$\begin{pmatrix}\lambda\\5-\lambda\\5-2\lambda\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\-2\end{pmatrix} = 0$

You could have used $\begin{pmatrix}3+\mu\\2-\mu\\-1-2\mu\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\-2\end{pmatrix}$ to obtain $\mu = \frac{-1}{2}$.

$\therefore \lambda - (5-\lambda) - 2(5-2\lambda) = 0$

i.e.,

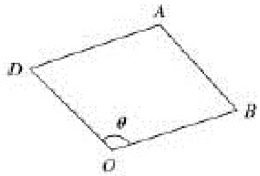
$6\lambda - 15 = 0$

$\therefore \lambda = \frac{15}{6} = \frac{5}{2}$

$\therefore \overrightarrow{OC} = \begin{pmatrix}\frac{5}{2}\\\frac{5}{2}\\0\end{pmatrix}$

Substitute your value of λ (or μ) to obtain the answer.

c



Draw a sketch of parallelogram *OBAD*, labelling vertices in order.

$\overrightarrow{OD} = \overrightarrow{BA}$

$= -\overrightarrow{AB}$ (found in a)

$= -3\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$

Opposite sides of a parallelogram are equal and parallel.

This is the position vector of *D*.

d Area of parallelogram *OBAD*

$= 2 \times \text{Area of} \Delta\ OBD$

$= |\overrightarrow{OB}| \times |\overrightarrow{OD}| \times \sin\theta$ *

where θ is angle between $\overrightarrow{OB}$ and $\overrightarrow{OD}$.

use $\cos\theta = \dfrac{\mathbf{b}\cdot\mathbf{d}}{|\mathbf{b}||\mathbf{d}|}$ to find θ.

Think through the method that you will use before you begin.

$$\mathbf{b}\cdot\mathbf{d} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 3 \\ 6 \end{pmatrix}$$

Use the formula for scalar product.

$= -3 \times 3 + 2 \times 3 + (-1) \times 6$

$= -9$

$|\mathbf{b}| = \sqrt{(3^2 + 2^2 + (-1)^2)}$

$= \sqrt{14}$

Use the formula for magnitude of a vector.

$|\mathbf{d}| = \sqrt{(-3)^2 + 3^2 + 6^2}$

$= \sqrt{54}$

$\therefore \cos\theta = \dfrac{-9}{\sqrt{14}\sqrt{54}} (\neq -0.327\cdots)$

You could use a calculator in this question.

$\therefore \theta = 109.1^\circ (4\text{ s.f.})$

$\therefore \text{Area} = \sqrt{14} \times \sqrt{54} \times \sin 109.1^\circ$

$= 26.0\ (3\text{ s.f.})$

Give the answer to 3 significant figures if you use a calculator.
The exact answer is $15\sqrt{3}$ and is easier to obtain using further mathematics techniques i.e. vector product.

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Review Exercise
Exercise A, Question 32

Question:

Find the gradient of the curve
$3x^3 - 2x^2y + y^3 = 17$
at the point with coordinates (2, 1). ***E***

Solution:

$3x^3 - 2x^2y + y^3 = 17$
Differentiate with respect to x:

$9x^2 - \left[2x^2\frac{dy}{dx} + 4xy\right]$
$+3y^2\frac{dy}{dx} = 0$

Substitute $x = 2, y = 1$

$36 - \left[8\frac{dy}{dx} + 8\right] + 3\frac{dy}{dx} = 0$

$\therefore 28 - 5\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{28}{5}$

$= 5\frac{3}{5}$

This is an implicit differentiation as you cannot make y the subject of the formula.

Use the product rule to differentiate the $2x^2y$ term.

Use the chain rule to differentiate y^3.

Differentiate 17 to give 0.

You can make $\frac{dy}{dx}$ the subject of the formula before substituting $x = 2, y = 1$ but the algebra is more difficult.
You would get $\frac{dy}{dx} = \frac{9x^2 - 4xy}{2x^2 - 3y^2}$.

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Review Exercise
Exercise A, Question 33

Question:

A curve has equation
$x^2 + 2xy - 3y^2 + 16 = 0$.

Find the coordinates of the points on the curve where $\frac{dy}{dx} = 0$. ***E***

Solution:

$x^2 + 2xy - 3y^2 + 16 = 0$ ...... (1)

Differentiate with respect to x. ← Use implicit differentiation as it is awkward to make y the subject of the formula.

$2x + \left[2x\frac{dy}{dx} + 2y\right] - 6y\frac{dy}{dx} + 0 = 0$

← Use the product rule to differentiate the $2xy$ term.

Use the chain rule to differentiate $-3y^2$.

Differentiate 16 to give 0.

Put $\frac{dy}{dx} = 0$

$\therefore 2x + 0 + 2y - 0 = 0$
i.e. $2(x + y) = 0$ ← Find the relationship between x and y when $\frac{dy}{dx} = 0$.

$\therefore x = -y$ ...... (2)

Substitute this into equation (1)

$y^2 - 2y^2 - 3y^2 + 16 = 0$ ← Solve equations (1) and (2) as simultaneous equation.

$\therefore 4y^2 = 16$
$\therefore y = \pm 2$
$\therefore x = \mp 2$

The points at which $\frac{dy}{dx} = 0$ are $(-2, 2)$ and $(2, -2)$. ← Match corresponding values for x and y to give the required coordinates.

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Review Exercise
Exercise A, Question 34

Question:

A curve C is described by the equation
$3x^2 - 2y^2 + 2x - 3y + 5 = 0$.
Find an equation of the normal to C at the point (0, 1), giving your answer in the form $ax + by + c = 0$, where a, b and c are integers. ***E***

Solution:

$3x^2 - 2y^2 + 2x - 3y + 5 = 0$
Differentiate with respect to x
Then

Use implicit differentiation.

$6x - 4y\frac{dy}{dx} + 2 - 3\frac{dy}{dx} + 0 = 0$

Use the chain rule to differentiate $-2y^2$ and $-3y$.

Substitute $x = 0, y = 1$ then

$-4\frac{dy}{dx} + 2 - 3\frac{dy}{dx} = 0$

$\therefore 7\frac{dy}{dx} = 2$

i.e. $\frac{dy}{dx} = \frac{2}{7}$

You could make $\frac{dy}{dx}$ the subject of the formula before substituting $x = 0, y = 1$.
In this case $\frac{dy}{dx} = \frac{6x + 2}{3 + 4y}$.

The gradient of the normal to C at (0,1) is $\frac{-7}{2}$

Use the result that $mm^1 = -1$ for perpendicular lines.

$\therefore$ Equation of the normal is $y - 1 = \frac{-7}{2}(x - 0)$

i.e. $y = \frac{-7}{2}x + 1$

This could be obtained directly from $y = mx + c$.

or $7x + 2y - 2 = 0$

give the answer in the form required by the question.

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Review Exercise
Exercise A, Question 35

Question:

A curve C is described by the equation

$3x^2 + 4y^2 - 2x + 6xy - 5 = 0$.

Find an equation of the tangent to C at the point $(1, -2)$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers. ***E***

Solution:

$3x^2 + 4y^2 - 2x + 6xy - 5 = 0$

Use implicit differentiation.

Differentiate with respect to x

$6x + 8y\frac{dy}{dx} - 2 + \left[6x\frac{dy}{dx} + 6y\right] - 0 = 0$

Use the chain rule to differentiate $4y^2$.

Use the product rule to differentiate $6xy$.

Substitute $x = 1$, $y = -2$

Then

$6 - 16\frac{dy}{dx} - 2 + 6\frac{dy}{dx} - 12 = 0$

If you rearranged you would get $\frac{dy}{dx} = \frac{2 - 6x - 6y}{8y + 6x}$.

$\therefore -8 - 10\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{-8}{10}$

Gradient of the tangent at $(1, -2)$ is $-\frac{8}{10}$.

$\therefore$ Equation of the tangent is

$(y + 2) = \frac{-8}{10}(x - 1)$

Use the equation on $y - y_1 = m(x - x_1)$.

$\therefore y + 2 = \frac{-8}{10}x + \frac{8}{10}$

$\therefore 10y + 8x + 12 = 0$

i.e. $4x + 5y + 6 = 0$

Multiply by 10 and collect the terms as required by the question.

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Review Exercise
Exercise A, Question 36

Question:

A set of curves is given by the equation $\sin x + \cos y = 0.5$.

a Use implicit differentiation to find an expression for $\frac{dy}{dx}$.

For $-\pi < x < \pi$ and $-\pi < y < \pi$.

b find the coordinates of the points where $\frac{dy}{dx} = 0$. ***E***

Solution:

a $\sin x + \cos y = 0.5$ *

Differentiate with respect to x:

$$\cos x - \sin y \frac{dy}{dx} = 0$$

Use the chain rule to differentiate $\cos y$.

$$\therefore \frac{dy}{dx} = \frac{\cos x}{\sin y}$$

Make $\frac{dy}{dx}$ the subject of the formula.

b When $\frac{dy}{dx} = 0$,

$$\cos x = 0$$
$$\therefore x = \pm\frac{\pi}{2}$$

Give answers in the range $-\pi < x < \pi$.

when $x = \frac{\pi}{2}$ substitute into *

$$1 + \cos y = 0.5$$
$$\therefore \cos y = -0.5$$
$$\therefore y = \frac{2\pi}{3} \text{ or } \frac{-2\pi}{3}.$$

Give answers in the range $-\pi < y < \pi$.

when $x = -\frac{\pi}{2}$ substitute into *

$$-1 + \cos y = 0.5$$
$$\therefore \cos y = 1.5$$

As $\cos y$ cannot be greater than 1 this equation has no solutions.

$\therefore$ Stationary points at $(\frac{\pi}{2}, \frac{2\pi}{3})$ and $(\frac{\pi}{2}, \frac{-2\pi}{3})$ only in the given range.

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Review Exercise
Exercise A, Question 37

Question:

a Given that $y = 2^x$, and using the result $2^x = e^{x \ln 2}$, or otherwise, show that $\frac{dy}{dx} = 2^x \ln 2$.

b Find the gradient of the curve with equation $y = 2^{x^2}$ at the point with coordinates (2, 16). ***E***

Solution:

a

$y = 2^x = e^{x \ln 2}$ ← You use the result that if $y = e^{kx}, \frac{dy}{dx} = ke^{kx}$.

$\frac{dy}{dx} = \ln 2 e^{x \ln 2}$

$= \ln 2 (e^{\ln 2})^x$

$= \ln 2 \times 2^x$ ← Note that $e^{\ln 2} = 2$.

or

$y = 2^x$

$\ln y = \ln 2^x$

$= x \ln 2$ ← You could use a different method, by taking logs of both sides and using implicit differentiation.

Differentiate with respect to x:

$\frac{1}{y}\frac{dy}{dx} = \ln 2$

$\therefore \frac{dy}{dx} = y \ln 2$

$= 2^x \ln 2$

b

$y = 2^{x^2}$ ← Use the chain rule. If $y = 2^{f(x)}, \frac{dy}{dx} = f'(x) 2^{f(x)} \ln 2$.

$\frac{dy}{dx} = (2x) 2^{x^2} \ln 2$

When $x = 2$

$\frac{dy}{dx} = 4 \times 2^4 \ln 2$

$= 64 \ln 2$ ← Substitute $x = 2$ into your expression.

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Review Exercise
Exercise A, Question 38

Question:

Find the coordinates of the minimum point on the curve with equation $y = x2^x$. ***E***

Solution:

$y = x2^x \quad *$

$\frac{dy}{dx} = x \cdot 2^x \ln 2 + 2^x \times 1$

$= 2^x(x \ln 2 + 1)$

Use the product rule.

At a minimum point,

$\frac{dy}{dx} = 0$

Put $\frac{dy}{dx} = 0$ and solve.

$\therefore x \ln 2 + 1 = 0$

i.e. $x = \frac{-1}{\ln 2}$

Substitute into $*$ to give:

$y = \frac{-1}{\ln 2} \times 2^{\frac{-1}{\ln 2}}$

$= \frac{-1}{\ln 2} \times \frac{1}{2^{\frac{1}{\ln 2}}} \dagger$

Substitute x value into the equation given, to find y.

Let $2^{\frac{1}{\ln 2}} = u$

Take lns of both sides

You may simplify $2^{\frac{1}{\ln 2}} = e$.

$\ln 2^{\frac{1}{\ln 2}} = \ln u$

$\therefore \frac{1}{\ln 2} \times \ln 2 = \ln u$

i.e. $\ln u = 1 \Rightarrow u = e$.

Substitute back into $\dagger$

$y = \frac{-1}{e \ln 2}$

$\therefore$ minimum point is at $\left(\frac{-1}{\ln 2}, \frac{-1}{e \ln 2}\right)$.

To check that this is indeed a minimum point you would need to find

$\frac{d^2y}{dx^2} = 2^x \ln 2(x \ln 2 + 2)$

As this is positive at $x = \frac{-1}{\ln 2}$ the turning point is a minimum.

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Review Exercise
Exercise A, Question 39

Question:

The value £V of a car t years after the 1st January 2001 is given by the formula $V = 10000 \times (1.5)^{-t}$.

a Find the value of the car on 1st January 2005.

b Find the value of $\frac{dV}{dt}$ when $t = 4$. ***E***

Solution:

a $V = 10000 \times (1.5)^{-t}$

On 1st January 2005, $t = 4$

$\therefore V = 10\,000 \times (1.5)^{-4}$

$= £1975.31 (2 \text{ d.p.})$

Give your answer to a suitable accuracy.

b

$\frac{dV}{dt} = -10000 \times (1.5)^{-t} \times \ln 1.5$

$= -800.92 \; (2 \text{ d.p.})$

Differentiate and substitute $t = 4$.

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Review Exercise
Exercise A, Question 40

Question:

A spherical balloon is being inflated in such a way that the rate of increase of its volume, V cm^3, with respect to time t seconds is given by

$\frac{dV}{dt} = \frac{k}{V}$, where k is a positive constant.

Given that the radius of the balloon is r cm, and that $V = \frac{4}{3}\pi r^3$,

a prove that r satisfies the differential equation
$\frac{dr}{dt} = \frac{B}{r^5}$, where B is a constant.

b Find a general solution of the differential equation obtained in part **a**. ***E***

Solution:

a

$$V = \frac{4}{3}\pi r^3$$

$$\therefore \frac{dV}{dr} = 4\pi r^2 \quad *$$

Use the chain rule:

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

Substitute $\frac{dV}{dt} = \frac{k}{V}$ (given) and $\frac{dV}{dr} = 4\pi r^2$ (from $*$) into the chain rule:

$$\therefore \frac{k}{V} = 4\pi r^2 \times \frac{dr}{dt}$$

$$\therefore \frac{dr}{dt} = \frac{k}{V} \div 4\pi r^2$$

$$= \frac{k}{\frac{4}{3}\pi r^3} \times \frac{1}{4\pi r^2}$$

$$= \frac{3k}{16\pi^2 r^5}.$$

You need to find $\frac{dV}{dr}$ in order to connect $\frac{dr}{dt}$ and $\frac{dV}{dt}$, using the chain rule.

Substitute $V = \frac{4}{3}\pi r^3$ and note that $\div \frac{4\pi r^2}{1}$ is the same as $\times$ by $\frac{1}{4\pi r^2}$.

b Separate the variables.

$$\int r^5 \, dr = \int \frac{3k}{16\pi^2} \, dt$$

$$\therefore \frac{r^6}{6} = \frac{3k}{16\pi^2} t + A$$

$$\therefore r = \left[\frac{9k}{8\pi^2} t + A'\right]^{\frac{1}{6}}$$

Integrate each side and include constant of integration.

Multiply by 6 and take the sixth root to give r.

$A' = 6A$.

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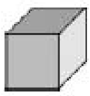
Review Exercise
Exercise A, Question 41

Question:



At time t seconds the length of the side of a cube is x cm, the surface area of the cube is S cm^2, and the volume of the cube is V cm^3.

The surface area of the cube is increasing at a constant rate of 8 cm^2 s^{-1}.

Show that

a $\quad \dfrac{dx}{dt} = \dfrac{k}{x}$, where k is a constant to be found,

b $\quad \dfrac{dV}{dt} = 2V^{\frac{1}{3}}$.

Given that $V = 8$ when $t = 0$,

c $\quad$ solve the differential equation in part **b**, and find the value of t when $V = 16\sqrt{2}$. ***E***

Solution:

Let S be the surface area

a

$S = 6x^2$ — The surface is made up of six squares each of area x^2.

$\therefore \frac{dS}{dx} = 12x$ — Differentiate to give $\frac{dS}{dx}$.

Given that $\frac{dS}{dt} = 8$ — This comes from the information that the surface area is increasing at a constant rate of $8\text{ cm}^2 s^{-1}$.

Use $\frac{dS}{dt} = \frac{dS}{dx} \times \frac{dx}{dt}$ — Use the chain rule to connect $\frac{dS}{dt}, \frac{dS}{dx}$ and $\frac{dx}{dt}$.

$\therefore 8 = 12x \times \frac{dx}{dt}$

$\therefore \frac{dx}{dt} = \frac{8}{12x}$ — Make $\frac{dx}{dt}$ the subject of the formula

$= \frac{k}{x}$ where $k = \frac{8}{12} = \frac{2}{3}$

b

$V = x^3$ — The volume of a cube is x^3.

$\therefore \frac{dV}{dx} = 3x^2$ — Differentiate to give $\frac{dV}{dx}$.

Use

$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$ — Use the chain rule to correct $\frac{dV}{dt}, \frac{dV}{dx}$ and $\frac{dx}{dt}$.

$= 3x^2 \frac{k}{x}$

$= 3kx = 2x$

$= 2V^{\frac{1}{3}}.$ — As $V = x^3$, so $x = V^{1/3}$

c Separate the variables

$\int \frac{dV}{V^{\frac{1}{3}}} = \int 2\, dt$

$\therefore \int V^{-\frac{1}{3}}\, dV = \int 2\, dt$ — Integrate and include an arbitrary constant A.

i.e. $\frac{3}{2}V^{\frac{2}{3}} + A = 2t$

But $V = 8$ when $t = 0$ — Use the initial condition to find A.

$\therefore \frac{3}{2} \times 4 + A = 0$

i.e. $A = -6$

$\therefore \frac{3}{2}V^{\frac{2}{3}} - 6 = 2t \quad *$

when

$V = 16\sqrt{2}$

Substitute $V = 16\sqrt{2}$ into the solution of the differential equation to find t.

$V = 2^{4\frac{1}{2}}$

$\therefore V^{\frac{2}{3}} = (2^{\frac{9}{2}})^{\frac{2}{3}}$

$= 2^3$

Substitute into *

$\therefore \frac{3}{2} \times 8 - 6 = 2t$

i.e. $t = 3$

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Review Exercise
Exercise A, Question 42

Question:

Liquid is poured into a container at a constant rate of 30 cm^3 s^{-1}. At time t seconds liquid is leaking from the container at a rate of $\frac{2}{15}V$ cm^3 s^{-1}, where V cm^3 is the volume of liquid in the container at that time.

a Show that

$$-15\frac{dV}{dt} = 2V - 450.$$

Given that $V = 1000$ when $t = 0$,

b find the solution of the differential equation, in the form $V = f(t)$.

c Find the limiting value of V as $t \to \infty$. *E*

Solution:

a $\frac{dV}{dt} = 30 - \frac{2}{15}V$

This is liquid being poured in and increases the volume in the container.

This is denoting the liquid leaking out, so decreases the volume in the container. So you need a minus sign.

Multiply this equation by -15

$\therefore -15\frac{dV}{dt} = 2V = 450$

This answer was given in the question.

b Separate the variables:

$$\int \frac{-15\, dV}{2V - 450} = \int dt.$$

$$\therefore -\frac{15}{2}\ln|2V - 450| = t + c$$

Integrate and include a constant on one side of your equation.

Given that $V = 1000$ when $t = 0$

$$\therefore \frac{-15}{2}\ln 1550 = c$$

Use the initial condition to find c.

$$\therefore \frac{-15}{2}\ln\frac{2V - 450}{1550} = t$$

Collect the two ln terms together $\frac{-15}{2}\left[\ln(2V - 450) - \ln 1550\right] = -\frac{15}{2}\ln\frac{2V - 450}{1550}$.

$$\therefore \ln\frac{V - 225}{775} = \frac{-2}{15}t$$

$$\therefore \frac{V - 225}{775} = e^{-\frac{2}{15}t}$$

Take exponentials of both sides.

$$\therefore V = 225 + 775e^{-\frac{2}{15}t}$$

Make V the subject of the formula.

c As $t \to \infty, e^{-\frac{2}{15}t} \to 0$

$\therefore$ limiting value of V is 225

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Review Exercise
Exercise A, Question 43

Question:

Liquid is pouring into a container at a constant rate of $20\text{ cm}^3\text{ s}^{-1}$ and is leaking out at a rate proportional to the volume of the liquid already in the container.

a Explain why, at time t seconds, the volume, $V\text{ cm}^3$, of liquid in the container satisfies the differential equation

$$\frac{dV}{dt} = 20 - kV,$$

where k is a positive constant.
The container is initially empty.

b By solving the differential equation, show that

$$V = A + Be^{-kt},$$

giving the values of A and B in terms of k.
Given also that $\frac{dV}{dt} = 10$ when $t = 5$,

c find the volume of liquid in the container at 10 s after the start. ***E***

Solution:

a Rate of change of volume is $\frac{dV}{dt}$ cm^3 s^{-1}

Increase is 20 cm^3 s^{-1}

Decrease is kV cm^3 s^{-1}, where k is constant of proportionality.

Explain the minus sign and the function of the constant k.

$\therefore \frac{dV}{dt} = 20 - kV$

b Separate the variables:

$$\int \frac{dV}{20-kV} = \int dt$$

$$\therefore -\frac{1}{k}\ln|20-kV| = t + c$$

You need to include a constant of integration c.

When $t = 0, V = 0$

You were told that the container was initially empty i.e. $V = 0$ when $t = 0$.

$$\therefore -\frac{1}{k}\ln 20 = c$$

Use this to find c.

$$\therefore -\frac{1}{k}\ln\frac{20-kV}{20} = t$$

Combine the two in terms together as

$$-\frac{1}{k}(\ln(20-kV) - \ln 20) = -\frac{1}{k}\ln\frac{20-kV}{20}$$

Multiply both sides by $-k$

$$\therefore \ln\frac{20-kV}{20} = -kt$$

$$\therefore \frac{20-kV}{20} = e^{-kt}$$

Take exponentials of each side.

$$\therefore kV = 20 - 20e^{-kt}$$

$$\therefore V = \frac{20}{k} - \frac{20}{k}e^{-kt} *$$

Rearrange to give V as the subject of the formula.

i.e. $A = \frac{20}{k}$ and $B = -\frac{20}{k}$

Differentiate the equation *

c

$$\frac{dV}{dt} = 20e^{-kt}$$

Differentiate to give $\frac{dV}{dt}$.

Substitute $\frac{dV}{dt} = 10$ when $t = 5$

$$\therefore 10 = 20\,e^{-5k}$$

use the given information to find k.

$$\therefore e^{-5k} = \frac{1}{2}$$

Taking lns:

$$-5k = \ln\frac{1}{2} \text{ or } 5k = \ln 2$$

$$\therefore k = \frac{1}{5}\ln 2 \text{ or } 0.1386 \text{ (4 d.p.)}$$

Substitute into equation *

$$V = \frac{100}{\ln 2} - \frac{100}{\ln 2}\left(\frac{1}{2}\right)^{\frac{t}{5}}$$

This is the particular solution of the differential equation.

When $t = 10$

$$V = \frac{100}{\ln 2} - \frac{100}{\ln 2} \times \frac{1}{4}$$

$$= \frac{75}{\ln 2}$$

$$= 108.2 \text{ (1 d.p.)}$$

Give V to a suitable accuracy.

or Volume $= 108\text{ cm}^3$ (3 s.f.)

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Review Exercise
Exercise A, Question 44

Question:

a Express $\frac{2x-1}{(x-1)(2x-3)}$ in partial fractions.

b Given that $x \geq 2$, find the general solution of the differential equation

$$(2x-3)(x-1)\frac{dy}{dx} = (2x-1)y.$$

c Hence find the particular solution of this differential equation that satisfies $y = 10$ at $x = 2$, giving your answer in the form $y = f(x)$.
E

Solution:

a

$$\frac{2x-1}{(x-1)(2x-3)} \equiv \frac{A}{(x-1)} + \frac{B}{(2x-3)}$$

Use denominators $(x-1)$ and $(2x-3)$.

$A = -1$ and $B = 4$

Compare numerators to find A and B (see earlier question 1).

$$\therefore \frac{2x-1}{(x-1)(2x-3)} \equiv \frac{-1}{x-1} + \frac{4}{2x-3} \quad *$$

b $(2x-3)(x-1)\frac{dy}{dx} = (2x-1)y$

Separating the variables.

$$\int \frac{dy}{y} = \int \frac{(2x-1)dx}{(2x-3)(x-1)}$$

Use the partial fractions from part a to split this fraction.

$$\therefore \ln y = \int \frac{-1}{x-1}dx + \int \frac{4}{2x-3}dx$$

These fractions can be integrated to give ln functions.

$$= -\ln|x-1| + 2\ln|2x-3| + c$$

$$\therefore \ln y = -\ln|x-1| + \ln(2x-3)^2 + \ln A$$

Express the constant as $\ln A$.

$$\ln y = \ln A\frac{(2x-3)^2}{(x-1)}$$

Combine the ln terms using the law for combining logs.

$$\therefore y = \frac{A(2x-3)^2}{(x-1)}$$

Make y the subject of the formula.

c Given $y = 10$ when $x = 2$

$\therefore 10 = A$

Use the given coordinates to find the value of the constant.

$\therefore$ Particular solution is

$$y = \frac{10(2x-3)^2}{(x-1)}.$$

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Review Exercise
Exercise A, Question 45

Question:

The rate of decrease of the concentration of a drug in the blood stream is proportional to the concentration C of that drug which is present at that time. The time t is measured in hours from the administration of the drug and C is measured in micrograms per litre.

a Show that this process is described by the differential equation $\frac{dC}{dt} = -kC$, explaining why k is a positive constant.

b Find the general solution of the differential equation, in the form $C = f(t)$.

After 4 hours, the concentration of the drug in the blood stream is reduced to 10% of its starting value C_0.

c Find the exact value of k. *E*

Solution:

a $\frac{dC}{dt}$ is the rate of change of concentration. ← Explain the term $\frac{dC}{dt}$

$\frac{dC}{dt} = -kC$, because k is the constant of proportionality. ← Explain the nature of k.

The negative sign and $k > 0$ indicates rate of decrease. ← Explain the negative sign.

b Separate the variables.

$$\int \frac{dC}{C} = -\int k \, dt$$

$\therefore \ln C = -kt + \ln A,$ ← Give the constant of integration as $\ln A$.

where $\ln A$ is a constant.

$\therefore \ln \frac{C}{A} = -kt$ ← Combine $\ln C - \ln A = \ln \frac{C}{A}$.

$\therefore \frac{C}{A} = e^{-kt}$ ← Take exponentials of each side of the equation.

$\therefore C = Ae^{-kt}$

c When $t = 0, C = C_0$ ← The starting concentration is C_0.

$\Rightarrow C_0 = A$

$\therefore C = C_0 e^{-kt}$

when $t = 4, C = \frac{1}{10} C_0$ ← The concentration is 10% of its starting value after 4 hours.

$\therefore \frac{1}{10} C_0 = C_0 e^{-4k}$

$\therefore e^{4k} = 10$

i.e. $4k = \ln 10$

$\therefore k = \frac{1}{4} \ln 10$ ← As an exact value of k is required – give this answer.

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Review Exercise
Exercise A, Question 46

Question:

A radioactive isotope decays in such a way that the rate of change of the number, N, of radioactive atoms present after t days, is proportional to N.

a Write down a differential equation relating N and t.

b Show that the general solution may be written as $N = Ae^{-kt}$, where A and k are positive constants.

Initially the number of radioactive atoms present is 7×10^{18} and 8 days later the number present is 3×10^{17}.

c Find the value of k.

d Find the number of radioactive atoms present after a further 8 days.

E

Solution:

a $\frac{dN}{dt} = -kN$, where k is a positive constant.

$\frac{dN}{dt}$ is rate of change of N.

k is the constant of proportionality.

$-$ sign because 'decays' implies that N is decreasing.

b Separate the variables

$$\therefore \int \frac{dN}{N} = -\int k \, dt$$

$$\therefore \ln N = -kt + \ln A$$

Put your arbitrary constant as $\ln A$.

$$\therefore \ln \frac{N}{A} = -kt$$

Collect the ln terms, as $\ln N - \ln A = \ln \frac{N}{A}$.

$$\therefore \frac{N}{A} = e^{-kt}$$

$$\therefore N = Ae^{-kt} \quad *$$

Take exponentials to give the required answer.

c When $t = 8, N = 7 \times 10^{18}$

$\therefore 7 \times 10^{18} = A$

Initially – means when $t = 0$ substitute into equation *

when $t = 8, N = 3 \times 10^{17}$

$$\therefore 3 \times 10^{17} = 7 \times 10^{18} e^{-8k}$$

Substitute $t = 8, N = 3 \times 10^{17}$ and $A = 7 \times 10^{18}$ into equation *.

$$\therefore e^{8k} = \frac{7 \times 10^{18}}{3 \times 10^{17}} = \frac{70}{3}$$

$$\therefore 8k = \ln\left[\frac{70}{3}\right]$$

Take lns of both sides of the equation.

$$\therefore k = \frac{1}{8}\ln\left[\frac{70}{3}\right] = 0.394 \text{ (3 s.f.)}$$

d When $t = 16$

After a further 8 days means that $t = 16$.

$$N = 7 \times 10^{18} e^{-2\ln\frac{70}{3}}$$

Substitute $A = 7 \times 10^{18}, kt = 2\ln\frac{70}{3}$ into equation *.

$$= 7 \times 10^{18} \div e^{\ln\left(\frac{70}{3}\right)^2}$$

$$= 7 \times 10^{18} \times \frac{9}{4900}$$

$$= 1.286 \times 10^{16} \text{ (4 s.f.)}$$

Give your answer to an appropriate accuracy.

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Review Exercise
Exercise A, Question 47

Question:

The volume of a spherical balloon of radius r cm is $V \text{ cm}^3$, where $V = \frac{4}{3}\pi r^3$.

a Find $\frac{dV}{dr}$.

The volume of the balloon increases with time t seconds according to the formula

$$\frac{dV}{dt} = \frac{1000}{(2t+1)^2},\ t \geq 0.$$

b Using the chain rule, or otherwise, find an expression in terms of r and t for $\frac{dr}{dt}$.

c Given that $V = 0$ when $t = 0$, solve the differential equation $\frac{dV}{dt} = \frac{1000}{(2t+1)^2}$ to obtain V in terms of t.

d Hence, at time $t = 5$,

i find the radius of the balloon, giving your answer to 3 significant figures,

ii show that the rate of increase of the radius of the balloon is approximately $2.90 \times 10^{-2} \text{ cm s}^{-1}$. *E*

Solution:

a

$$V = \frac{4}{3}\pi r^3$$

$$\therefore \frac{dV}{dr} = 4\pi r^2$$

b From the chain rule:-

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

Use the chain rule to connect $\frac{dV}{dt}, \frac{dV}{dr}$ and $\frac{dr}{dt}$.

As

$$\frac{dV}{dt} = \frac{1000}{(2t+1)^2}$$

$$\therefore \frac{1000}{(2t+1)^2} = 4\pi r^2 \times \frac{dr}{dt}$$

$$\therefore \frac{dr}{dt} = \frac{250}{\pi(2t+1)^2 r^2} \quad *$$

Make $\frac{dr}{dt}$ the subject of the formula.

c $\frac{dV}{dt} = \frac{1000}{(2t+1)^2}$

Separating the variables.

$$\int dV = \int \frac{1000}{(2t+1)^2} dt$$

This integration is reverse of the chain rule.

$$\therefore V = -500(2t+1)^{-1} + c$$

Do not forget the constant of integration, c.

But $V = 0$ when $t = 0$

$$\therefore 0 = -500 + c$$

$$\text{i.e. } c = 500$$

Use initial conditions to find c.

$$\therefore V = 500 - \frac{500}{(2t+1)}$$

d **(i)** When $t = 5$,

$$V = 500 - \frac{500}{11}$$

$$= 454.5\cdots$$

Find volume and then use $V = \frac{4}{3}\pi r^3$ to find the radius $r = \sqrt[3]{\frac{V}{\frac{4}{3}\pi}}$.

Using

$$V = \frac{4}{3}\pi r^3 = 454.5\ldots$$

$$\therefore r = 4.77 \text{ (3 s.f.)}$$

(ii) Substitute $r = 4.77$, $t = 5$ into $*$

Use the answer to part **b**.

$$\therefore \frac{dr}{dt} = 0.0289\ldots \approx 2.90 \times 10^{-2}$$

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Review Exercise
Exercise A, Question 48

Question:

A population growth is modelled by the differential equation $\frac{dP}{dt} = kP$, where P is the population, t is the time measured in days and k is a positive constant.

Given that the initial population is P_0,

a solve the differential equation, giving P in terms of P_0, k and t.

Given also that $k = 2.5$,

b find the time taken, to the nearest minute, for the population to reach $2P_0$.

In an improved model the differential equation is given as $\frac{dP}{dt} = \lambda P \cos \lambda t$, where P is the population, t is the time measured in days and λ is a positive constant.

Given, again, that the initial population is P_0 and that time is measured in days,

c solve the second differential equation, giving P in terms of P_0, λ and t.

Given also that $\lambda = 2.5$,

d find the time taken, to the nearest minute, for the population to reach $2P_0$ for the first time, using the improved model. ***E***

Solution:

a

$\frac{dP}{dt} = kP$

Separate the variables.

$$\int \frac{dP}{P} = \int k\,dt$$

$$\therefore \ln P = kt + \ln P_0$$

(as $P = P_0$ when $t = 0$)

$\ln P_0$ is the arbitrary constant which is found from the initial condition.

$$\therefore \ln P - \ln P_0 = kt$$

$$\ln \frac{P}{P_0} = kt$$

i.e. $$\therefore \frac{P}{P_0} = e^{kt}$$

$$\therefore P = P_0\,e^{kt}$$

Collect the two ln terms and use the law that $\ln P - \ln P_0 = \ln \frac{P}{P_0}$.

Take exponentials and make P the subject of the formula.

b Substitute $k = 2.5$ and $P = 2P_0$

$$\therefore 2P_0 = P_0 e^{2.5t}$$

$$\therefore e^{2.5t} = 2$$

$$\therefore 2.5t = \ln 2$$

$$t = \frac{1}{2.5}\ln 2$$

$$= 0.277\ldots \text{days}$$

$$= 6.65\text{ h}$$

$$= 6\text{ h}39\text{ minutes}$$

Take lns and make t the subject of the formula.

The units are days and need to be converted to minutes, so multiply by 24 then by 60.

c $\frac{dP}{dt} = \lambda P \cos \lambda t$

Separate the variables.

$$\int \frac{dP}{P} = \int \lambda \cos \lambda t\,dt$$

$$\therefore \ln P = \sin \lambda t + \ln P_0$$

$$\therefore \ln \frac{P}{P_0} = \sin \lambda t$$

$$\therefore P = P_0 e^{\sin \lambda t}$$

The method is similar to that used in part **a**.

d Substitute $P = 2P_0$ and $\lambda = 2.5$

$\therefore e^{\sin 2.5t} = 2$

$\therefore \sin 2.5t = \ln 2$

$\therefore 2.5t = \sin^{-1}(\ln 2)$

$\therefore t = 0.306$ days

$= 7.35$ h

$= 441$ mins or

7 h 21 min

Use radians to calculate $\sin^{-1}(\ln 2)$.

Again change the time from days to minutes.

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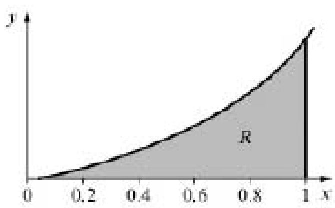
Review Exercise
Exercise A, Question 49

Question:



The diagram shows the graph of the curve with equation

$y = xe^{2x}, x \geq 0.$

The finite region R bounded by the lines $x = 1$, the x-axis and the curve is shown shaded in the diagram.

a Use integration to find the exact value of the area for R.

b Complete the table with the values of y corresponding to $x = 0.4$ and 0.8.

x	0	0.2	0.4	0.6	0.8	1
$y = xe^{2x}$	0	0.29836		1.99207		7.38906

c Use the trapezium rule with all the values in the table to find an approximate value of this area, giving your answer to 4 significant figures. ***E***

Solution:

a Let $I = \int_0^1 x\,e^{2x}\,dx$

Let $u = x$ and $\frac{dv}{dx} = e^{2x}$.

$u = x \Rightarrow \frac{du}{dx} = 1$

$v = \frac{1}{2}e^{2x} \Leftarrow \frac{dv}{dx} = e^{2x}$

Complete the table for u, v, $\frac{du}{dx}$ and $\frac{dv}{dx}$.
Take care to differentiate u but integrate $\frac{dv}{dx}$.

Use the integration by parts formula

$$I = \left[\frac{1}{2}x\,e^{2x}\right]_0^1 - \int_0^1 1\times\frac{1}{2}e^{2x}\,dx$$

$$= \left[\frac{1}{2}xe^{2x}\right]_0^1 - \left[\frac{1}{4}e^{2x}\right]_0^1$$

$$= \frac{1}{2}e^2 - \left[\frac{1}{4}e^2 - \frac{1}{4}\right]$$

$$= \frac{1}{4}e^2 + \frac{1}{4}$$

Notice that $\int \not\equiv \frac{du}{dx}dx$ is a simpler integral than $\int u\frac{dv}{dx}dx$.

Apply the limits on the uv term and to the integral term.

b

x	0	0.2	0.4	0.6	0.8	1
y	0	0.29836	0.89022	1.99207	3.96243	7.38906

Complete the table to find the values of y.

c

$$I = \frac{1}{2}\times 0.2\left[0 + 2(0.29836 + 0.89022 + 1.99207 + 3.96243) + 7.38906\right]$$

$$= 0.1[21.67522]$$

$$= 2.168\ (4\text{ s.f.})$$

Solutionbank C4
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 50

Question:

a Given that $y = \sec x$, complete the table with the values of y corresponding to $x = \frac{\pi}{16}, \frac{\pi}{8}$ and $\frac{\pi}{4}$.

x	0	$\frac{\pi}{16}$	$\frac{\pi}{8}$	$\frac{3\pi}{16}$	$\frac{\pi}{4}$
y	1			1.20269	

b Use the trapezium rule, with all the values for y in the completed table, to obtain an estimate for $\int_0^{\frac{\pi}{4}} \sec x \, dx$.

Show all the steps of your working and give your answer to 4 decimal places.

The exact value of $\int_0^{\frac{\pi}{4}} \sec x \, dx$ is $\ln(1+\sqrt{2})$.

c Calculate the % error in using the estimate you obtained in part **b**.

E

Solution:

a

x	0	$\frac{\pi}{16}$	$\frac{\pi}{8}$	$\frac{3\pi}{16}$	$\frac{\pi}{4}$
y	1	1.01959	1.08239	1.20269	1.41421

Complete the table. Ensure that your calculator is set to use radians and use $\left(\sec\frac{\pi}{16}\right) = \left[\cos\left(\frac{\pi}{16}\right)\right]^{-1}$ or $\frac{1}{\cos\frac{\pi}{16}}$.

b

$$I = \frac{1}{2} \cdot \frac{\pi}{16}\left[1+2(1.01959+1.08239+1.20269)+1.41421\right]$$

$$= \frac{\pi}{32} \times 9.02355$$

$$= 0.88588\ldots = 0.8859 \text{ (4 d.p.)}$$

c Percentage error is

$$\frac{\left(0.8859-\ln(1+\sqrt{2})\right)}{\ln\left(1+\sqrt{2}\right)} \times 100 =$$

0.5136% (4 d.p.)

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 51

Question:

$I = \int_0^5 e^{\sqrt{(3x+1)}}\, dx$.

a Given that $y = e^{\sqrt{(3x+1)}}$, complete the table with the values of y corresponding to $x = 2$, 3 and 4.

x	0	1	2	3	4	5
y	e^1	e^2				e^4

b Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate for the original integral I, giving your answer to 4 significant figures.

c Use the substitution $t = \sqrt{(3x+1)}$ to show that I may be expressed as $\int_a^b kte^t\, dt$, giving the values of a, b and k.

d Use integration by parts to evaluate this integral, and hence find the value of I correct to 4 significant figures, showing all the steps in your working. ***E***

Solution:

a

x	0	1	2	3	4	5
y	e^1	e^2	14.094	23.624	36.802	e^4

You could complete the table with $e^{\sqrt{7}}$, $e^{\sqrt{10}}$ and $e^{\sqrt{13}}$.

b

$$I = \frac{1}{2}\times 1\left[e^1 + 2\left(e^2 + 14.094 + 23.624 + 36.802\right) + e^4\right]$$

$$= \frac{1}{2}\times 221.1\ldots$$

$$= 110.6 \text{ (4 s.f.)}$$

c $\quad I = \int_0^5 e^{\sqrt{(3x+1)}}\,dx$

You need to replace each x term with a corresponding t term. First replace dx with a term in dt.

Let

$$t = \sqrt{(3x+1)}$$

$$\frac{dt}{dx} = \frac{3}{2}(3x+1)^{-\frac{1}{2}} = \frac{3}{2t}$$

Replace dx with $\frac{2t}{3}dt$

Use $t = \sqrt{(3x+1)}$ to change the limits. When $x = 0, t = 1$ and when $x = 5, t = 4$.

x	t
0	1
5	4

So $I = \int_1^4 e^t \cdot \frac{2t}{3}dt = \int_1^4 \frac{2}{3}te^t\,dt$

i.e. $a = 1, b = 4$ and $k = \frac{2}{3}$.

d

$$u = \frac{2}{3}t \Rightarrow \frac{du}{dt} = \frac{2}{3}$$

Let $u = \frac{2}{3}t$ and $\frac{dv}{dt} = e^t$

$$v = e^t \Leftarrow \frac{dv}{dt} = e^t$$

Complete the table for $u, v, \frac{du}{dt}$ and $\frac{dv}{dt}$.

$$\therefore I = \left[\frac{2}{3}t\,e^t\right]_1^4 - \int_1^4 \frac{2}{3}e^t\,dt$$

$$= \frac{8}{3}e^4 - \frac{2}{3}e - \left[\frac{2}{3}e^t\right]_1^4$$

$$= \frac{8}{3}e^4 - \frac{2}{3}e - \frac{2}{3}e^4 + \frac{2}{3}e$$

Apply the limits to both terms.

$$= 2e^4$$

$$= 109.2 \text{ (4 s.f.)}$$

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 52

Question:

The following is a table of values for $y=\sqrt{(1+\sin x)}$, where x is in radians.

x	0	0.5	1	1.5	2
y	1	1.216	p	1.413	q

a Find the value of p and the value of q.

b Use the trapezium rule and all the values of y in the completed table to obtain an estimate of I, where

$$I=\int_0^2 \sqrt{(1+\sin x)}\,dx.$$ ***E***

Solution:

a

$p = 1.357$ (3 d.p.)

$q = 1.382$ (3 d.p.)

Your calculator should be in radian mode.

b Using the trapezium rule

$$I = \frac{1}{2}\times 0.5[1+2(1.216+1.357+1.413)+1.382]$$
$$= 0.25\times 10.354$$
$$= 2.5885$$
$$= 2.589 \text{ (4 s.f.)}$$

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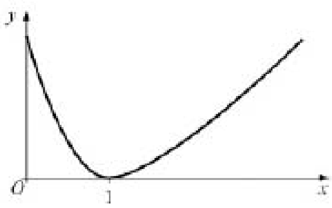
Review Exercise
Exercise A, Question 53

Question:



The figure shows a sketch of the curve with equation $y=(x-1)\ln x, x>0$.

a Copy and complete the table with the values of y corresponding to $x=1.5$ and $x=2.5$.

x	1	1.5	2	2.5	3
y	0		ln 2		2 ln 3

Given that $I=\int_1^3 (x-1)\ln x \, dx$,

b Use the trapezium rule

i with values of y at $x=1, 2$ and 3 to find an approximate value for I to 4 significant figures,

ii with values of y at $x=1, 1.5, 2, 2.5$ and 3 to find another approximate value for I to 4 significant figures.

c Explain, with reference to the figure, why an increase in the number of values improves the accuracy of the approximation.

d Show, by integration, that the exact value of $\int_1^3 (x-1)\ln x \, dx$ is $\frac{3}{2}\ln 3$. ***E***

Solution:

a

x	1	1.5	2	2.5	3
y	0	0.5 ln 1.5	ln 2	1.5 ln 2.5	2 ln 3

You may leave your answers in terms of ln at this stage.

b **i** Trapezium rule with 2 strips

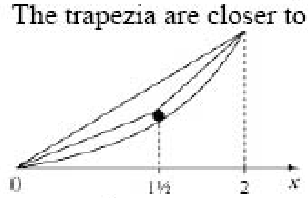
$$I = \frac{1}{2}\times 1[0+2\times\ln 2+2\ln 3]$$
$$= \frac{1}{2}\times 3.5835\ldots$$
$$= 1.792 \text{ (4 s.f.)}$$

ii Trapezium rule with 4 strips:

$$I = \frac{1}{2}\times 0.5[0+2(0.5\ln 1.5+\ln 2+1.5\ln 2.5)+2\ln 3]$$
$$= 0.25\times 6.737856\cdots$$
$$= 1.684 \text{ (4 s.f.)}$$

Show all your working.

c The trapezia are closer to the required area when there are more strips.



A diagram can help you to explain.

d Let $I = \int_1^3 (x-1)\ln x \, dx$.

Let $u = \ln x$ and $\frac{dv}{dx} = x-1$.

$$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$$
$$v = \frac{x^2}{2} - x \Leftarrow \frac{dv}{dx} = x-1$$

Complete the table for $u, v, \frac{du}{dx}$ and $\frac{dv}{dx}$.

$$\therefore I = \left[\left(\frac{x^2}{2}-x\right)\ln x\right]_1^3 - \int_1^3 \frac{1}{x}\cdot\left(\frac{x^2}{2}-x\right)dx$$
$$= \frac{3}{2}\ln 3 - \int_1^3\left(\frac{x}{2}-1\right)dx$$
$$= \frac{3}{2}\ln 3 - \left[\frac{x^2}{4}-x\right]_1^3$$
$$= \frac{3}{2}\ln 3 - \left[\left(\frac{9}{4}-3\right)-\left(\frac{1}{4}-1\right)\right]$$
$$= \frac{3}{2}\ln 3$$

Apply the limits to the uv term and to the integral term.

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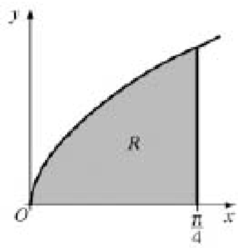
Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 54

Question:



The figure shows part of the curve with equation $\sqrt{(\tan x)}$. The finite region R, which is bounded by the curve, the x-axis and the line $x = \frac{\pi}{4}$, is shown shaded in the figure.

a Given that $y = \sqrt{(\tan x)}$, copy and complete the table with the values of y corresponding to $x = \frac{\pi}{16}, \frac{\pi}{8}$ and $\frac{3\pi}{16}$, giving your answers to 5 decimal places.

x	0	$\frac{\pi}{16}$	$\frac{\pi}{8}$	$\frac{3\pi}{16}$	$\frac{\pi}{4}$
y	0				1

b Use the trapezium rule with all the values of y in the completed table to obtain an estimate for the area of the shaded region R, giving your answer to 4 decimal places.

The region R is rotated through 2π radians around the x-axis to generate a solid of revolution.

c Use integration to find an exact value for the volume of the solid generated. ***E***

Solution:

a

x	0	$\frac{\pi}{16}$	$\frac{\pi}{8}$	$\frac{3\pi}{16}$	$\frac{\pi}{4}$
y	0	0.44600	0.64360	0.81742	1

Ensure that your calculator is in radian mode.

b From the trapezium rule

$$\text{Area} \approx \frac{1}{2}\times\frac{\pi}{16}[0+2(0.44600+0.64360+0.81742)+1]$$

$$\approx \frac{\pi}{32}\times 4.81404$$

$$= 0.4726$$

c Volume

Use the formula $v = \pi\int y^2 \, dx$.

$$= \pi\int_0^{\frac{\pi}{4}} (\sqrt{\tan x})^2 \, dx$$

$$= \pi\int_0^{\frac{\pi}{4}} \tan x \, dx$$

$\int \tan x \, dx = \ln|\cos x|$ - is given in your formula book.

$$= \pi\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} \, dx$$

or $\pi[\ln\sec x]_0^{\frac{\pi}{4}}$

$$= \pi[-\ln\cos x]_0^{\frac{\pi}{4}}$$

$$= \pi\left[-\ln\frac{1}{\sqrt{2}}\right]$$

$$= \pi\ln\sqrt{2} \text{ or } \frac{1}{2}\pi\ln 2$$

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Review Exercise
Exercise A, Question 55

Question:

Using the substitution $u^2 = 2x - 1$, or otherwise, find the exact value of

$\int_1^5 \frac{3x}{\sqrt{(2x-1)}}\,dx.$ ***E***

Solution:

Let $I = \int_1^5 \frac{3x}{\sqrt{(2x-1)}}\,dx$

Let $u^2 = 2x - 1$

$2u\frac{du}{dx} = 2$

So replace dx with $u\,du$ and $x = \frac{u^2+1}{2}$

Also $\frac{1}{\sqrt{(2x-1)}} = \frac{1}{u}$

Replace each term in x with a term in u.

x	u
1	1
5	3

Change the limits. When $x = 1$, $u^2 = 1$ and when $x = 5, u^2 = 9$.

So

$$I = \int_1^3 \frac{\frac{3(u^2+1)}{2}}{u} \times u\,du$$

$$= \int_1^3 \left(\frac{3}{2}u^2 + \frac{3}{2}\right) du.$$

Simplify and integrate.

$$= \left[\frac{1}{2}u^3 + \frac{3}{2}u\right]_1^3$$

$$= \left(\frac{27}{2} + \frac{9}{2}\right) - \left(\frac{1}{2} + \frac{3}{2}\right)$$

Evaluate the integral using the new u limits.

$$= 18 - 2$$

$$= 16$$

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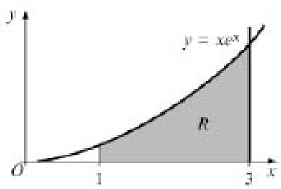
Review Exercise
Exercise A, Question 56

Question:



The figure shows the finite region R, which is bounded by the curve $y = xe^x$, the line $x = 1$, the line $x = 3$ and the x-axis.
The region R is rotated through 360 degrees about the x-axis.
Use integration by parts to find an exact value for the volume of the solid generated. ***E***

Solution:

Use

$$V = \pi \int y^2 \mathrm{d}x$$

$$= \pi \int_1^3 x^2 \mathrm{e}^{2x} \mathrm{d}x$$

Let $u = x^2$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x}$

$$u = x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$$

$$v = \frac{1}{2}\mathrm{e}^{2x} \Leftarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x}$$

Complete the table for u, v, $\frac{\mathrm{d}u}{\mathrm{d}x}$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$. Take care to differentiate u but integrate $\frac{\mathrm{d}v}{\mathrm{d}x}$.

$$\therefore V = \pi\left[x^2 \cdot \frac{1}{2}\mathrm{e}^{2x}\right]_1^3 - \pi\int_1^3 \frac{1}{2}\mathrm{e}^{2x} \cdot 2x \, \mathrm{d}x.$$

$$\text{i.e. } V = \pi\left[\frac{9}{2}\mathrm{e}^6 - \frac{1}{2}\mathrm{e}^2\right] - \pi\int_1^3 x\mathrm{e}^{2x} \, \mathrm{d}x.$$

This integral is simpler than V but still not one you can write down. Use integration by parts again with $u = x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x}$.

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$

$$v = \frac{1}{2}\mathrm{e}^{2x} \Leftarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x}$$

Complete a new table for the new u, v, $\frac{\mathrm{d}u}{\mathrm{d}x}$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$.

$$\therefore V = \pi\left[\frac{9}{2}\mathrm{e}^6 - \frac{1}{2}\mathrm{e}^2\right] - \pi\left[x \cdot \frac{1}{2}\mathrm{e}^{2x}\right]_1^3 + \pi\int_1^3 \frac{1}{2}\mathrm{e}^{2x} \cdot 1 \, \mathrm{d}x$$

$$= \pi\left[\frac{9}{2}\mathrm{e}^6 - \frac{1}{2}\mathrm{e}^2\right] - \pi\left[\frac{3}{2}\mathrm{e}^6 - \frac{1}{2}\mathrm{e}^2\right] + \pi\left[\frac{1}{4}\mathrm{e}^{2x}\right]_1^3$$

$$= \frac{13}{4}\pi\mathrm{e}^6 - \frac{\pi}{4}\mathrm{e}^2$$

Apply the integration by parts formula a second time.

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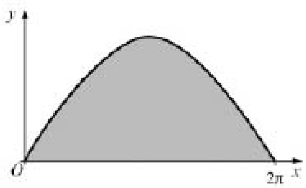
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 57

Question:



The curve with equation $y = 3\sin\frac{x}{2}, 0 \le x \le 2\pi$, is shown in the figure.

The finite region enclosed by the curve and the x-axis is shaded.

a Find, by integration, the area of the shaded region.

This region is rotated through 2π radians about the x-axis.

b Find the volume of the solid generated.

Solution:

a

$$\text{Area} = \int_0^{2\pi} 3\sin\frac{x}{2}\,dx$$

Recall (5) in the introduction to integration. Integrating a sin function gives a change of sign and a cos function.

$$= \left[3 \times -2\cos\frac{x}{2}\right]_0^{2\pi}$$

The 2 here is obtained from dividing by $\frac{1}{2}$ which arises from the chain rule.

$$= \left[-6\cos\frac{x}{2}\right]_0^{2\pi}$$

$$= [6 - (-6)]$$

$$\text{Area} = 12$$

b

$$\text{Volume} = \pi\int_0^{2\pi} 9\sin^2\frac{x}{2}\,dx$$

You cannot integrate $\sin^2\frac{x}{2}$, but you can write this in terms of $\cos x$.

Recall $\cos 2A = 1 - 2\sin^2 A$

So $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$

So $\sin^2\frac{x}{2} = \frac{1}{2}(1 - \cos x)$

$$\therefore \text{Volume} = \frac{9\pi}{2}\int_0^{2\pi}(1 - \cos x)\,dx$$

You can now integrate each term directly.

$$= \frac{9\pi}{2}[x + \sin x]_0^{2\pi}$$

$$= \frac{9\pi}{2} \times (2\pi - 0)$$

$$\text{volume} = 9\pi^2$$

Solutionbank C4
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 58

Question:

Use integration by parts to find the exact value of $\int_1^3 x^2 \ln x \, dx$. **E**

Solution:

Let $I = \int_1^3 x^2 \ln x \, dx$

$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$v = \frac{x^3}{3} \Leftarrow \frac{dv}{dx} = x^2$

Since there is a ln x term Let $u = \ln x$ and $\frac{dv}{dx} = x^2$.

Complete the table for $u, v, \frac{du}{dx}$ and $\frac{dv}{dx}$. Take care to differentiate u but integrate $\frac{dv}{dx}$.

Using the integration by parts formula.

$$I = \left[\frac{x^3}{3}\ln x\right]_1^3 - \int_1^3 \frac{x^3}{3} \cdot \frac{1}{x} \, dx$$

Apply the integration by parts formula.

$$= 9\ln 3 - \int_1^3 \frac{x^2}{3} \, dx$$

Simplify the $v\frac{du}{dx}$ term.

$$= 9\ln 3 - \left[\frac{x^3}{9}\right]_1^3$$

$$= 9\ln 3 - \left[3 - \frac{1}{9}\right]$$

$$= 9\ln 3 - \frac{26}{9}.$$

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 59

Question:

Use the substitution $u = 1 - x^2$ to find the exact value of

$$\int_0^{\frac{1}{2}} \frac{x^3}{(1-x^2)^{\frac{1}{2}}}\,dx.$$

Solution:

Let $u = 1 - x^2$

Then $\frac{du}{dx} = -2x$

and $x^2 = 1 - u$

so $\int \frac{x^3}{(1-x^2)^{\frac{1}{2}}}dx = \int \frac{x^2}{(1-x^2)^{\frac{1}{2}}}x\,dx = \int \frac{1-u}{u^{\frac{1}{2}}}(-\frac{du}{2})$

$= -\frac{1}{2}\int \frac{1-u}{u^{\frac{1}{2}}}du = -\frac{1}{2}\int u^{-\frac{1}{2}} - u^{\frac{1}{2}}du = \left[-u^{\frac{1}{2}} + \frac{1}{3}u^{\frac{3}{2}}\right]$

This implies that $x\,dx = -\frac{du}{2}$.

Use $x^3 = x^2 x$ and $x^2 = 1 - u$ with $x\,dx = -\frac{du}{2}$.

Simplify and integrate

As limits for x were 0 and $\frac{1}{2}$, limits for u are 1 and $\frac{3}{4}$

So evaluate $\left[-u^{\frac{1}{2}} + \frac{1}{3}u^{\frac{3}{2}}\right]_1^{\frac{3}{4}} = (-\frac{\sqrt{3}}{2} + \times\frac{3\sqrt{3}}{3\times 4\sqrt{4}}) - (-1 + \frac{1}{3})$

$= (-\frac{3\sqrt{3}}{8}) - (-\frac{2}{3})$

$= \frac{2}{3} - \frac{3\sqrt{3}}{8}$

As the variable has changed, so must the limits. So use $u = 1 - x^2$ to find the new limits.

Use the new limits to evaluate the answer.

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Review Exercise
Exercise A, Question 60

Question:

a Express $\frac{5x+3}{(2x-3)(x+2)}$ in partial fractions.

b Hence find the exact value of $\int_2^6 \frac{5x+3}{(2x-3)(x+2)}\,dx$, giving your answer as a single logarithm. ***E***

Solution:

a

$$\frac{5x+3}{(2x-3)(x+2)} \equiv \frac{A}{(2x-3)} + \frac{B}{(x+2)}$$

Use denominators $(2x-3)$ and $(x+2)$.

$$\equiv \frac{A(x+2)+B(2x-3)}{(2x-3)(x+2)}$$

$\therefore 5x+3 \equiv A(x+2)+B(2x-3)$

Equate numerators.

Put $x=-2$, then $-7=0-7B \Rightarrow B=1$

Put $x=\frac{3}{2}$, then $\frac{21}{2}=\frac{7}{2}A \Rightarrow A=3$

$\therefore \frac{5x+3}{(2x-3)(x+2)} \equiv \frac{3}{2x-3}+\frac{1}{x+2}$

b

$$\int_2^6 \frac{5x+3}{(2x-3)(x+2)}\,dx = \int_2^6 \frac{3}{2x-3}\,dx + \int_2^6 \frac{1}{x+2}\,dx$$

Rewrite the integral using partial fractions.

$$= \left[\frac{3}{2}\ln(2x-3)+\ln(x+2)\right]_2^6$$

Integrate and do not forget to divide by 2.

$$= \frac{3}{2}\ln 9+\ln 8-\ln 4$$

Substitute the limits noting $\ln 1 = 0$.

$$= \ln 9^{\frac{3}{2}}+\ln\frac{8}{4}$$

$$= \ln 27+\ln 2$$

$$= \ln 54$$

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Review Exercise
Exercise A, Question 61

Question:

a Use integration by parts to find

$\int x\cos 2x\,\mathrm{d}x$.

b Prove that the answer to part a may be expressed as

$\frac{1}{2}\sin x(2x\cos x - \sin x) + C,$

where C is an arbitrary constant. ***E***

Solution:

a Let $I = \int x\cos 2x\mathrm{d}x$

Let $u = x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos 2x$.

$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$

$v = \frac{1}{2}\sin 2x \Leftarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \cos 2x$

Complete the table for $u, v, \frac{\mathrm{d}u}{\mathrm{d}x}$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$.

$\therefore I = \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x \cdot 1\,\mathrm{d}x$

$= \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c$

This integral can now be integrated directly.

b

$\therefore I = \frac{1}{2}x \cdot 2\sin x\cos x + \frac{1}{4}\left(1 - 2\sin^2 x\right) + c$

$= \frac{1}{2}\sin x\left(2x\cos x - \sin x\right) + \frac{1}{4} + c$

$= \frac{1}{2}\sin x\left(2x\cos x - \sin x\right) + c'$

Use double angle formulae:
$\sin 2x = 2\sin x\cos x$
and $\cos 2x = 1 - 2\sin^2 x$.

Where $c' = \frac{1}{4} + c$.

Solutionbank C4

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 62

Question:

Use the substitution $u = 2^x$ to find the exact value of

$$\int_0^1 \frac{2^x}{(2^x+1)}\,dx.$$ ***E***

Solution:

Let $I = \int_0^1 \frac{2^x}{2^x+1}\,dx$.

Let $u = 2^x$

$\frac{du}{dx} = 2^x \cdot \ln 2$

You need to replace each `$x$' term with a corresponding `u' term.

Replace $2^x\,dx$ by $\frac{1}{\ln 2}\,du$.

x	u
0	1
1	2

Change the limits:
when $x = 0, u = 2^0 = 1$
$x = 1, u = 2^1 = 2$.

Then $I = \int_1^2 \frac{1}{u+1} \cdot \frac{1}{\ln 2}\,du$.

$= \frac{1}{\ln 2}\left[\ln(u+1)\right]_1^2$

$= \frac{1}{\ln 2}\left[\ln 3 - \ln 2\right]$

Use the limits for u to evaluate the integral.

$= \frac{1}{\ln 2}\ln\frac{3}{2}.$

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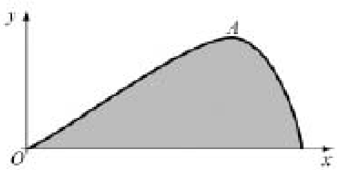
Review Exercise
Exercise A, Question 63

Question:



The figure shows a graph of $y = x\sqrt{\sin x}, 0 < x < \pi$.
The finite region enclosed by the curve and the x-axis is shaded as shown in the figure. A solid body S is generated by rotating this region through 2π radians about the x-axis. Find the exact value of the volume of S.
***E*(adapted)**

Solution:

$$\text{Volume} = \pi\int_0^{\pi}(x\sqrt{\sin x})^2\,dx$$

$$= \pi\int_0^{\pi}x^2\sin x\,dx$$

Use $v = \pi\int y^2 dx$.

Use integration by parts.

Let $u = x^2$ and $\frac{dv}{dx} = \sin x$.

$u = x^2 \Rightarrow \frac{du}{dx} = 2x$

$v = -\cos x \Leftarrow \frac{dv}{dx} = \sin x$

Complete the table for $u, v, \frac{du}{dx}$ and $\frac{dv}{dx}$.

$$\therefore \text{Volume} = \pi\left[[-x^2\cos x]_0^{\pi} - \int_0^{\pi} -2x\cos x\,dx\right]$$

$$= \pi\left(\pi^2 + \int_0^{\pi}2x\cos x\,dx\right)$$

This integral is simpler than the original one but you will need to use integration by parts again, with $u = 2x$ and $\frac{dv}{dx} = \cos x$.

$u = 2x \Rightarrow \frac{du}{dx} = 2$

$v = \sin x \Leftarrow \frac{dv}{dx} = \cos x$

$$\therefore \text{Volume} = \pi\left[\pi^2 + [2x\sin x]_0^{\pi} - \int_0^{\pi}2\sin x\,dx\right]$$

This term becomes zero as $2\pi\sin\pi - 0 = 0$.

$$= \pi\left(\pi^2 + [2\cos x]_0^{\pi}\right)$$

$$= \pi\left(\pi^2 + [-2-2]\right)$$

$$= \pi\left(\pi^2 - 4\right)$$

$$= \pi^3 - 4\pi$$

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Review Exercise
Exercise A, Question 64

Question:

a Find $\int x\cos 2x \, dx$.

b Hence, using the identity $\cos 2x = 2\cos^2 x - 1$, deduce $\int x\cos^2 x \, dx$. ***E***

Solution:

a Let $I = \int x\cos 2x \, dx$ ← use integration by parts and let $u = x$ and $\frac{dv}{dx} = \cos 2x$.

$u = x \Rightarrow \frac{du}{dx} = 1$

$\therefore v = \frac{1}{2}\sin 2x \Leftarrow \frac{dv}{dx} = \cos 2x$ ← Complete the table for $u, v, \frac{du}{dx}$ and $\frac{dv}{dx}$.

$\therefore I = \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x.1\,dx$

$= \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c$ ← Do not forget to add the constant.

b $\therefore \int x(2\cos^2 x - 1)dx = I$

So $\int x\cos^2 x \, dx = \frac{1}{2}I + \frac{1}{2}\int x \, dx$.

$= \left(\frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x\right) + \frac{1}{4}x^2 + c$

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Review Exercise
Exercise A, Question 65

Question:

$$\frac{2(4x^2+1)}{(2x+1)(2x-1)} \equiv A + \frac{B}{(2x+1)} + \frac{C}{(2x-1)}.$$

a Find the values of the constants A, B and C.

b Hence show that the exact value of

$$\int_1^2 \frac{2(4x^2+1)}{(2x+1)(2x-1)}\,dx \text{ is } 2 + \ln k,$$

giving the value of the constant k. ***E***

Solution:

a Let

$$f(x) = \frac{2(4x^2+1)}{(2x+1)(2x-1)}$$
$$= \frac{8x^2+2}{4x^2-1}$$

$$\begin{array}{r} 2\ r\ 4 \\ 4x^2-1\ \overline{)8x^2+2} \\ \underline{8x^2-2} \\ 4 \end{array}$$

Divide the denominator into the numerator.

$$\therefore f(x) \equiv 2+\frac{4}{(2x+1)(2x-1)}$$
$$\equiv 2+\frac{A}{2x+1}+\frac{B}{2x-1}$$

Express as partial fractions, using denominators $2x+1$ and $2x-1$.

where $\frac{4}{(2x+1)(2x-1)} \equiv \frac{A(2x-1)+B(2x+1)}{(2x+1)(2x-1)}$

Equate numerators

$4 \equiv A(2x-1)+B(2x+1)$

Put

$x = \frac{1}{2};\ 4 = 2B \Rightarrow B = 2$

$x = -\frac{1}{2};\ 4 = -2A \Rightarrow A = -2$

$\therefore f(x) \equiv 2-\frac{2}{(2x+1)}+\frac{2}{(2x-1)}$

or $A = 2, B = -2, C = 2$

b

$$\therefore \int_1^2 f(x)\,dx = \int_1^2 2-\frac{2}{2x+1}+\frac{2}{2x-1}\,dx$$
$$= \left[2x-\ln|2x+1|+\ln|2x-1|\right]_1^2$$
$$= 4-\ln 5+\ln 3-(2-\ln 3)$$
$$= 2-\ln 5+2\ln 3$$
$$= 2+\ln 9-\ln 5$$
$$= 2+\ln\frac{9}{5}.$$

Use the partial fractions from part **a**.

Integrate each term using $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|$.

Use the laws of lns to combine the log terms, noting that $2\ln 3 = \ln 3^2 = \ln 9$.

i.e. $k = \frac{9}{5}$ or 1.8.

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Review Exercise
Exercise A, Question 66

Question:

$f(x) = (x^2 + 1)\ln x$.

Find the exact value of $\int_1^e f(x)\,dx$. ***E***

Solution:

Let $I = \int_1^e (x^2 + 1)\ln x$

Use integration by parts with $u = \ln x$ and so $\frac{dv}{dx} = x^2 + 1$.

Let

$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$v = \frac{x^3}{3} + x \Leftarrow \frac{dv}{dx} = (x^2 + 1)$

Complete the table for $u, v, \frac{du}{dx}$ and $\frac{dv}{dx}$.

Using integration by parts:

$$\therefore I = \left[\left(\frac{x^3}{3} + x\right)\ln x\right]_1^e - \int_1^e \frac{1}{x}\left(\frac{x^3}{3} + x\right) dx$$

Apply the limits to the uv term and to $\int v\frac{du}{dx}\,dx$.

$$= \left(\frac{e^3}{3} + e\right) - \int_1^e \left(\frac{x^2}{3} + 1\right) dx$$

Evaluate the limits on uv and remember $\ln 1 = 0$.

$$= \frac{e^3}{3} + e - \left[\frac{x^3}{9} + x\right]_1^e$$

$$= \frac{e^3}{3} + e - \left[\frac{e^3}{9} + e - \frac{1}{9} - 1\right]$$

$$= \frac{2e^3}{9} + \frac{10}{9}$$

$$= \frac{1}{9}\left(2e^3 + 10\right)$$

This is an exact answer.

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Review Exercise
Exercise A, Question 67

Question:

The curve C is described by the parametric equations

$x = 3\cos t, y = \cos 2t, 0 \le t \le \pi.$

Find a Cartesian equation of the curve C. ***E***

Solution:

$x = 3\cos t$

$\therefore \cos t = \frac{x}{3}$

Rearrange to make cos t the subject of the formula.

$$y = \cos 2t$$

$$= 2\cos^2 t - 1$$

Use the double angle formula.

$$y = 2\left(\frac{x}{3}\right)^2 - 1$$

Eliminate t to give a Cartesian equation.

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Review Exercise
Exercise A, Question 68

Question:

The point $P(a, 4)$ lies on a curve C. C has parametric equations $x = 3t \sin t, y = 2 \sec t, 0 \le t < \frac{\pi}{2}$. Find the exact value of a. ***E***

Solution:

Put $y = 2 \sec t = 4$ ← As point P has y coordinate 4.

Then $\sec t = 2$ ← Now solve the resulting trigonometric equation.

So $\cos t = \frac{1}{2}$

$\therefore t = \frac{\pi}{3}$ ← Give your answer in radians as $0 \le t < \frac{\pi}{2}$.

$\therefore x = 3\frac{\pi}{3}\sin\frac{\pi}{3}$ ← Substitute the value of t into $x = 3t \sin t$.

i.e. $x = \pi\frac{\sqrt{3}}{2}$ ← This is the x coordinate of P and so is equal to a.

So $a = \frac{\pi\sqrt{3}}{2}$

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Review Exercise
Exercise A, Question 69

Question:

A curve has parametric equations

$x = 2\cot t, y = 2\sin^2 t, 0 < t \leq \frac{\pi}{2}$.

a Find an expression for $\frac{dy}{dx}$ in terms of the parameter t.

b Find an equation of the tangent to the curve at the point where $t = \frac{\pi}{4}$.

c Find a Cartesian equation of the curve in the form $y = f(x)$.
State the domain on which the curve is defined. *E*

Solution:

a

$$x = 2\cot t, y = 2\sin^2 t$$

Use the chain rule to differentiate $2\sin^2 t$.

$$\frac{dx}{dt} = -2\text{cosec}^2 t, \frac{dy}{dt} = 4\sin t\cos t$$

Use $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$.

$$\therefore \frac{dy}{dx} = \frac{4\sin t\cos t}{-2\text{cosec}^2 t}$$

$$= -2\sin^3 t\cos t$$

Simplify using $\text{cosec}\, t = \frac{1}{\sin t}$.

b At $t = \frac{\pi}{4}$, gradient $= -2 \times \left(\frac{1}{\sqrt{2}}\right)^3 \times \left(\frac{1}{\sqrt{2}}\right)$

$$= -\frac{1}{2}.$$

Find the value of the gradient of the curve at $t = \frac{\pi}{4}$.

The coordinates of the point where $t = \frac{\pi}{4}$ are:

$$x = 2, y = 2 \times \left(\frac{1}{\sqrt{2}}\right)^2 = 1$$

$\therefore$ The equation of the tangent is

$$(y-1) = -\frac{1}{2}(x-2)$$

The tangent has the same gradient as the curve.

$$\therefore y = \frac{-1}{2}x + 2.$$

c As $x = 2\cot t, \cot t = \frac{x}{2}$.

Rearrange to make $\cot t$ and $\text{cosec}^2 t$ the subjects of the formulae.

Also as $y = 2\sin^2 t, \sin^2 t = \frac{y}{2}$ and

$$\text{cosec}^2 t = \frac{2}{y}$$

use $1 + \cot^2 t = \text{cosec}^2 t$

Write down the relationship between $\cot^2 t$ and $\text{cosec}^2 t$.

then $1 + \left(\frac{x}{2}\right)^2 = \left(\frac{2}{y}\right)$

Eliminate t to give a Cartesian equation.

$$\therefore \left(\frac{2}{y}\right) = \frac{4+x^2}{4}$$

Take the reciprocal of each side of the equation.

$$\left(\frac{y}{2}\right) = \frac{4}{4+x^2}$$

$$y = \frac{8}{4+x^2}$$

As $0 < t \le \frac{\pi}{2}, \cot t \ge 0$

As $x = 2\cot t, x \ge 0$

This is the domain of the function.

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Review Exercise
Exercise A, Question 70

Question:

A curve has parametric equations
$x = 7\cos t - \cos 7t$, $y = 7\sin t - \sin 7t$.

$$\frac{\pi}{8} < t < \frac{\pi}{3}.$$

a Find an expression for $\frac{dy}{dx}$ in terms of t.
You need not simplify your answer.

b Find an equation of the normal to the curve at the point where $t = \frac{\pi}{6}$. Give your answer in its simplest exact form. ***E***

Solution:

a

$x = 7\cos t - \cos 7t; y = 7\sin t - \sin 7t$

$\frac{dx}{dt} = -7\sin t + 7\sin 7t; \frac{dy}{dt} = 7\cos t - 7\cos 7t$

using the chain rule :

$\frac{dy}{dx} = \frac{7\cos t - 7\cos 7t}{-7\sin t + 7\sin 7t}$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

b When $t = \frac{\pi}{6}$

$\frac{dy}{dx} = \frac{7\times\frac{\sqrt{3}}{2} + 7\times\frac{\sqrt{3}}{2}}{-7\times\frac{1}{2} - 7\times\frac{1}{2}}$

$= \frac{7\sqrt{3}}{-7}$

$= -\sqrt{3}$

Substitute $t = \frac{\pi}{6}$ to find the gradient of the curve.

$\therefore$ Gradient of the normal at the point where $t = \frac{\pi}{6}$ is $\frac{1}{\sqrt{3}}$.

Use $mm^1 = -1$, the condition for perpendicular lines to find the gradient of the normal.

When $t = \frac{\pi}{6}$,

$x = 7\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = 4\sqrt{3}$

$y = 7\times\frac{1}{2} + \frac{1}{2} = 4$

Find the co-ordinates of the point on the curve when $t = \frac{\pi}{6}$.

$\therefore$ Equation of the normal is

$y - 4 = \frac{1}{\sqrt{3}}(x - 4\sqrt{3})$

$\therefore y\sqrt{3} = x$

Use $y - y_1 = m(x - x_1)$ for the equation of a straight line.

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Review Exercise
Exercise A, Question 71

Question:

A curve has parametric equations

$x = \tan^2 t, y = \sin t, 0 < t < \frac{\pi}{2}$.

a Find an expression for $\frac{dy}{dx}$ in terms of t.
You need not simplify your answer.

b Find an equation of the tangent to the curve at the point where $t = \frac{\pi}{4}$.
Give your answer in the form $y = ax + b$, where a and b are constants to be determined.

c Find a Cartesian equation of the curve in the form $y^2 = f(x)$. *E*

Solution:

a

$x = \tan^2 t,\ y = \sin t$

$\frac{dx}{dt} = 2\tan t\sec^2 t,\ \frac{dy}{dt} = \cos t$

Use the chain rule to differentiate $\tan^2 t$.

using the chain rule:

$\frac{dy}{dx} = \frac{\cos t}{2\tan t\sec^2 t}$

Use $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$.

b When $t = \frac{\pi}{4}$

$\frac{dy}{dx} = \frac{\frac{1}{\sqrt{2}}}{2\times 1\times(\sqrt{2})^2}$

$= \frac{1}{4\sqrt{2}}$

Substitute $t = \frac{\pi}{4}$ to find the value of the gradient of the curve.

$\therefore$ Gradient of the tangent where $t = \frac{\pi}{4}$ is $\frac{1}{4\sqrt{2}}$.

At $t = \frac{\pi}{4}, x = 1, y = \frac{1}{\sqrt{2}}$.

Find the co-ordinates of the point on the curve where $t = \pi/4$.

The equation of the tangent is:

$y - \frac{1}{\sqrt{2}} = \frac{1}{4\sqrt{2}}(x-1)$

$\therefore y = \frac{1}{4\sqrt{2}}x + \frac{3}{4\sqrt{2}}$ or $y = \frac{\sqrt{2}}{8}x + \frac{3\sqrt{2}}{8}$

The tangent has the same gradient as the curve at the point $\left(1, \frac{1}{\sqrt{2}}\right)$.

Use $y - y_1 = m(x - x_1)$.

c

$y^2 = \sin^2 t$

$= 1 - \cos^2 t$

Use $\cos^2 t + \sin^2 t = 1$.

$= 1 - \frac{1}{\sec^2 t}$

Use $\sec t = \frac{1}{\cos t}$.

$= 1 - \frac{1}{1+\tan^2 t}$

Use $1 + \tan^2 t = \sec^2 t$.

$= 1 - \frac{1}{1+x}$ or $y^2 = \frac{x}{1+x}$

Eliminate t by using $\tan^2 t = x$.

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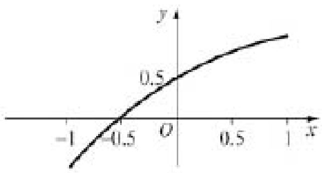
Review Exercise
Exercise A, Question 72

Question:



The curve shown in the figure has parametric equations

$$x = \sin t, y = \sin\left(t + \frac{\pi}{6}\right), -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

a Find an equation of the tangent to the curve at the point where $t = \frac{\pi}{6}$.

b Show that a Cartesian equation of the curve is

$$y = \frac{\sqrt{3}}{2}x + \frac{1}{2}\sqrt{(1 - x^2)}, -1 < x < 1.$$ ***E***

Solution:

a

$x = \sin t, y = \sin\left(t+\frac{\pi}{6}\right)$

$\frac{dx}{dt} = \cos t, \frac{dy}{dt} = \cos\left(t+\frac{\pi}{6}\right)$

using the chain rule:

$\frac{dy}{dx} = \frac{\cos(t+\frac{\pi}{6})}{\cos t}$

Use $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$.

At the point where $t = \frac{\pi}{6}$,

$\frac{dy}{dx} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$.

Substitute $t = \frac{\pi}{6}$ to find the gradient of the curve which is also the gradient of the tangent.

$\therefore$ Gradient of the tangent at $t = \frac{\pi}{6}$ is $\frac{1}{\sqrt{3}}$.

Also at $t = \frac{\pi}{6}, x = \frac{1}{2}, y = \frac{\sqrt{3}}{2}$

Find the values of x and y when $t = \frac{\pi}{6}$.

Equation of the tangent is:

$y - \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}\left(x - \frac{1}{2}\right)$

Use $y - y_1 = m(x - x_1)$ for the equation of a straight line.

$\therefore y = \frac{1}{\sqrt{3}}x - \frac{1}{2\sqrt{3}} + \frac{\sqrt{3}}{2}$

i.e. $y = \frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}$

b

$y = \sin(t+\frac{\pi}{6})$

$= \sin t \cos\frac{\pi}{6} + \cos t \sin\frac{\pi}{6}$.

$= \frac{\sqrt{3}}{2}\sin t + \frac{1}{2}\cos t$

Expand, using addition formula.

Replace $\cos\frac{\pi}{6}$ by $\frac{\sqrt{3}}{2}$ and $\sin\frac{\pi}{6}$ by $\frac{1}{2}$.

As $x = \sin t$, using $\cos^2 t = 1 - \sin^2 t$

means that $\cos t = \sqrt{1-x^2}$

Eliminate t by using $\sin t = x$ and $\cos t = \sqrt{(1-x^2)}$.

$\therefore y = \frac{\sqrt{3}}{2}x + \frac{1}{2}\sqrt{(1-x^2)}$

As $-1 < \sin t < 1 \Rightarrow -1 < x < 1$.

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Review Exercise
Exercise A, Question 73

Question:

The curve C has parametric equations

$x = \frac{1}{1+t}, y = \frac{1}{1-t}, -1 < t < 1.$

The line l is a tangent to C at the point where $t = \frac{1}{2}$.

a Find an equation for the line l.

b Show that a Cartesian equation for the curve C is $y = \frac{x}{2x-1}$. ***E***

Solution:

a

$$x = \frac{1}{1+t}, y = \frac{1}{1-t}$$

Differentiate $(1+t)^{-1}$ and $(1-t)^{-1}$ using the chain rule.

$$\frac{dx}{dt} = \frac{-1}{(1+t)^2}, \frac{dy}{dt} = \frac{1}{(1-t)^2}$$

$$\therefore \frac{dy}{dx} = -\frac{(1+t)^2}{(1-t)^2}$$

Use $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$.

At the point where $t = \frac{1}{2}$,

$$\frac{dy}{dx} = -\frac{\frac{9}{4}}{\frac{1}{4}} = -9$$

Substitute $t = \frac{1}{2}$ to find the gradient of the curve and thus the tangent.

$\therefore$ Gradient of tangent, where $t = \frac{1}{2}$, is -9.

Also $x = \frac{2}{3}$ and $y = 2$ where $t = \frac{1}{2}$.

Find the values of x and y when $t = \frac{1}{2}$.

$\therefore$ Equation of tangent is

$$y - 2 = -9\left(x - \frac{2}{3}\right)$$

Use $y - y_1 = m(x - x_1)$.

i.e. $y = -9x + 8$.

b As $x = \frac{1}{1+t}$

$$1 + t = \frac{1}{x}$$

$$\therefore t = \frac{1}{x} - 1$$

Rearrange to make t the subject of the formula.

Substitute into $y = \frac{1}{1-t}$

$$\therefore y = \frac{1}{1 - \left(\frac{1}{x} - 1\right)} = \frac{1}{2 - \frac{1}{x}} = \frac{x}{2x - 1}$$

Eliminate t and simplify the fraction multiplying numerator and denominator by x.

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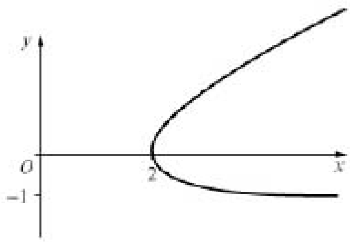
Review Exercise
Exercise A, Question 74

Question:



The curve shown has parametric equations

$x = t + \frac{1}{t}$, $y = t - 1$ for $t > 0$.

a Find the value of the parameter t at each of the points where $x = 2\frac{1}{2}$.

b Find the gradient of the curve at each of these points.

c Find the area of the finite region enclosed between the curve and the line $x = 2\frac{1}{2}$. ***E***

Solution:

a $x = t + \frac{1}{t}, y = t - 1$ for $t > 0$

As $x = 2\frac{1}{2}, t + \frac{1}{t} = 2\frac{1}{2}$

$$\therefore t^2 - 2\frac{1}{2}t + 1 = 0$$
$$\text{i.e. } 2t^2 - 5t + 2 = 0$$
$$\therefore (2t-1)(t-2) = 0$$
$$\Rightarrow t = \frac{1}{2} \text{ or } 2$$

Multiply both sides of this equation by t and collect the terms to give a quadratic equation.

b

$$\frac{dx}{dt} = 1 - \frac{1}{t^2}, \frac{dy}{dt} = 1$$
$$\therefore \frac{dy}{dx} = 1 - \frac{1}{t^3} = \frac{t^2}{t^2 - 1}$$

Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ and use the chain rule $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

When $t = \frac{1}{2}$, gradient $= \frac{\frac{1}{4}}{\frac{1}{4} - 1} = \frac{-1}{3}$

$t = 2$, gradient $= \frac{4}{4-1} = \frac{4}{3}$.

Substitute the values of t found in part **a**.

c

$$\text{Area} = \int y \frac{dx}{dt}\, dt$$
$$= \int_{\frac{1}{2}}^{2} (t-1)\left(1 - \frac{1}{t^2}\right) dt$$
$$= \int_{\frac{1}{2}}^{2} t - 1 - \frac{1}{t} + \frac{1}{t^2}\, dt$$
$$= \left[\frac{t^2}{2} - t - \ln t - \frac{1}{t}\right]_{\frac{1}{2}}^{2}$$

Substitute $y = t - 1$ and $\frac{dx}{dt} = 1 - \frac{1}{t^2}$.

Expand the brackets.

Integrate each term.

use of limits $t = 2$ and $t = \frac{1}{2}$ to give

$$\text{area} = -\ln 2 - \frac{1}{2} - \left(\frac{1}{8} - \frac{1}{2} - \ln\frac{1}{2} - 2\right)$$
$$= 1\frac{7}{8} - 2\ln 2 = 0.4887$$

Substitute $t = 2$ and $t = \frac{1}{2}$ then subtract.

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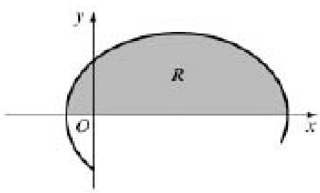
Review Exercise
Exercise A, Question 75

Question:



The curve shown in the figure has parametric equations

$x = t - 2\sin t, y = 1 - 2\cos t,$

$0 \le t \le 2\pi.$

a Show that the curve crosses the x-axis where $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$.

The finite region R is enclosed by the curve and the x-axis, as shown shaded in the figure

b Show that the area R is given by the integral

$$\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 2\cos t)^2 \, dt.$$

c Use this integral to find the exact value of the shaded area. ***E***

Solution:

a The curve crosses the x-axis when $y=0$.

As $y=1-2\cos t$, when $y=0$

$\cos t=\frac{1}{2}$

$\therefore t=\frac{\pi}{3}$ or $\frac{5\pi}{3}$.

b Area of R is given by $\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} y\frac{dx}{dt}\,dt$

As $x=t-2\sin t$,

$\frac{dx}{dt}=1-2\cos t.$

$$\therefore \text{Area} = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}}(1-2\cos t)(1-2\cos t)\,dt$$

Substitute $y=1-2\cos t$ and $\frac{dx}{dt}=1-2\cos t$ into the integral.

$$= \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}}(1-2\cos t)^2\,dt.$$

c

$$\therefore \text{Area} = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}}(1-4\cos t+4\cos^2 t)\,dt$$

Expand the bracket.

$$= \left[t-4\sin t\right]_{\frac{\pi}{3}}^{\frac{5\pi}{3}}+2\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}}(\cos 2t+1)\,dt$$

Integrate $1-4\cos t$ directly.

Use double angle formula $\cos 2t=2\cos^2 t-1$ to replace $4\cos^2 t$ with $2(\cos 2t+1)$.

$$= \left[t-4\sin t\right]_{\frac{\pi}{3}}^{\frac{5\pi}{3}}+\left[\sin 2t+2t\right]_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$$

Now integrate $(2\cos 2t+2)$.

$$= \left[3t-4\sin t+\sin 2t\right]_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$$

Collect the terms.

$$= \left(5\pi+2\sqrt{3}-\frac{\sqrt{3}}{2}\right)-\left(\pi-2\sqrt{3}+\frac{\sqrt{3}}{2}\right)$$

$$= 4\pi+4\sqrt{3}-\sqrt{3}$$

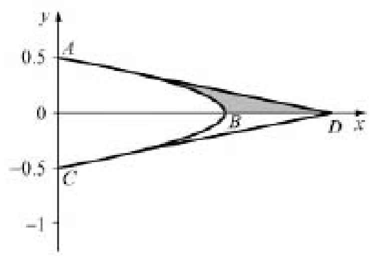
$$= 4\pi+3\sqrt{3}$$

Use the limits to find an exact answer.

Solutionbank C4
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 76

Question:



The curve shown in the figure has parametric equations

$x = a\cos 3t, y = a\sin t, -\frac{\pi}{6} \le t \le \frac{\pi}{6}.$

The curve meets the axes at points A, B and C, as shown.
The straight lines shown are tangents to the curve at the points A and C and meet the x-axis at point D. Find, in terms of a

a the equation of the tangent to A,

b the area of the finite region between the curve, the tangent at A and the x-axis, shown shaded in the figure.

Given that the total area of the finite region between the two tangents and the curve is 10 cm^2

c find the value of a. ***E***

Solution:

a At point A, $x = 0$

$\therefore a\cos 3t = 0 \Rightarrow 3t = \frac{\pi}{2}$

Find the co-ordinates of the point A.

$\therefore t = \frac{\pi}{6}$.

But $y = a\sin t$

At $t = \frac{\pi}{6}, y = \frac{a}{2}$.

$\therefore$ A is the point $(0, \frac{a}{2})$

$x = a\cos 3t, y = a\sin t$

$\frac{dx}{dt} = -3a\sin 3t, \frac{dy}{dt} = a\cos t$

$\therefore \frac{dy}{dx} = -\frac{\cos t}{3\sin 3t}$

Use $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

when $t = \frac{\pi}{6}, \frac{dy}{dx} = -\frac{\frac{\sqrt{3}}{2}}{3} = -\frac{\sqrt{3}}{6}$.

Find the gradient at the point A.

$\therefore$ Equation of the tangent at A is $y - \frac{a}{2} = -\frac{\sqrt{3}}{6}(x - 0)$

Use $y - y_1 = m(x - x_1)$.

$\therefore y = \frac{-\sqrt{3}}{6}x + \frac{a}{2}$

b This tangent meets the x-axis when $y = 0$, at the point D.

$\therefore \frac{\sqrt{3}}{6}x = \frac{a}{2}$

$\therefore x = \sqrt{3}a$

Find the point where the tangent meets the x-axis.

Area of triangle AOD is $\frac{1}{2} \times \sqrt{3}a \times \frac{a}{2}$

$= \frac{1}{4}\sqrt{3}a^2$

Use area of triangle $= \frac{1}{2}$ base $\times$ height i.e. $\frac{1}{2}OD \times OA$.

At the point B, $t = 0$

$\therefore$ Area of region required $= \frac{1}{4}\sqrt{3}a^2 - \int y\frac{dx}{dt}\,dt$

$\therefore \text{Area} = \frac{1}{4}\sqrt{3}a^2 - \int_{\pi/6}^{0} a\sin t(-3a\sin 3t)\,dt$

area = area of triangle − area beneath the curve.

$= \frac{1}{4}\sqrt{3}a^2 + \frac{3a^2}{2}\int_{\frac{\pi}{6}}^{0} \cos 2t - \cos 4t\,dt$

Use $2\sin t\sin 3t = \cos 2t - \cos 4t$
This is from the trigonometric 'factor formulae' – see C3.

$= \frac{1}{4}\sqrt{3}a^2 + \frac{3a^2}{2}\left[\frac{1}{2}\sin 2t - \frac{1}{4}\sin 4t\right]_{\frac{\pi}{6}}^{0}$

The $\frac{\pi}{6}$ limit corresponds to point A and the 0 limit corresponds to point B.

$= \frac{1}{4}\sqrt{3}a^2 + \frac{3a^2}{2}\left[0 - \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{8}\right]$

Use the limits and the result $\sin\frac{\pi}{3} = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$ to give an exact answer.

$= \frac{1}{4}\sqrt{3}a^2 - \frac{3}{16}\sqrt{3}a^2$

$= \frac{1}{16}\sqrt{3}a^2$

c Total area is $2 \times \frac{1}{16}\sqrt{3}a^2 = 10$

The total area is twice the area found in part **b**.

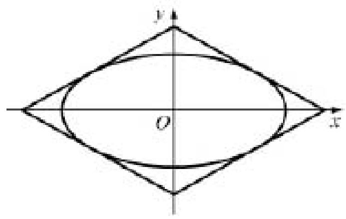
$\therefore a^2 = \frac{80}{\sqrt{3}}$

$\therefore a = 6.796$ (4 s.f.)

Solutionbank C4
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 77

Question:



A table top, in the shape of a parallelogram, is made from two types of wood. The design is shown in the figure. The area inside the ellipse is made from one type of wood, and the surrounding area is made from a second type of wood.

The ellipse has parametric equations,

$$x = 5\cos\theta, y = 4\sin\theta, 0 \le \theta \le 2\pi.$$

The parallelogram consists of four line segments, which are tangents to the ellipse at the points where

$\theta = \alpha,\ \theta = -\alpha,\ \theta = \pi - \alpha,\ \theta = -\pi + \alpha.$

a Find an equation of the tangent to the ellipse at $(5\cos\alpha, 4\sin\alpha)$, and show that it can be written in the form
$5y\sin\alpha + 4x\cos\alpha = 20.$

b Find by integration the area enclosed by the ellipse.

c Hence show that the area enclosed between the ellipse and the parallelogram is

$$\frac{80}{\sin 2\alpha} - 20\pi. \qquad \textit{E}$$

Solution:

a

$$x = 5\cos\theta, y = 4\sin\theta$$

$$\frac{dx}{d\theta} = -5\sin\theta, \frac{dy}{d\theta} = 4\cos\theta$$

From the chain rule

$$\frac{dy}{dx} = \frac{4\cos\theta}{-5\sin\theta} = -\frac{4}{5}\cot\theta$$

Use $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$.

The gradient of the tangent at $(5\cos\alpha, 4\sin\alpha) = \frac{-4}{5}\cot\alpha$

Substitute $\theta = \alpha$ to give gradient at particular point.

$\therefore$ Equation of the tangent is

$$y - 4\sin\alpha = -\frac{4}{5}\cot\alpha(x - 5\cos\alpha)$$

Use $y - y_1 = m(x - x_1)$.

i.e. $5y\sin\alpha - 20\sin^2\alpha = -4\cos\alpha \times x + 20\cos^2\alpha$

Multiply both sides of the equation by sin α.

$$\therefore 5y\sin\alpha + 4x\cos\alpha = 20(\cos^2\alpha + \sin^2\alpha) = 20 \times 1 = 20$$

Collect terms using $\cos^2\alpha + \sin^2\alpha = 1$.

b

$$\text{Area} = \int_{2\pi}^{0} y\frac{dx}{d\theta}\,d\theta$$

$$= \int_{2\pi}^{0} 4\sin\theta(-5\sin\theta)\,d\theta$$

Substitute $y = 4\sin\theta$ and $\frac{dx}{d\theta} = -5\sin\theta$ into integral.

$$= -10\int_{2\pi}^{0} 2\sin^2\theta\,d\theta$$

$$= 10\int_{2\pi}^{0} \cos 2\theta - 1\,d\theta$$

Use double angle formula $\cos 2\theta = 1 - 2\sin^2\theta$.

$$= 10\left[\frac{1}{2}\sin 2\theta - \theta\right]_{2\pi}^{\theta}$$

Integrate and use appropriate limits.

Use limits 0 and 2π to obtain area $= 20\pi$

c Area of triangle formed by tangent at $(5\cos\alpha, 4\sin\alpha)$ and the coordinate axes:

Tangent meets x-axis at $x = \dfrac{5}{\cos\alpha}$

Tangent meets y-axis at $y = \dfrac{4}{\sin\alpha}$

> Find the points where the tangent crosses the x- and y-axis.

$$\therefore \text{Area of triangle} = \frac{1}{2} \times \frac{5}{\cos\alpha} \times \frac{4}{\sin\alpha}$$
$$= \frac{10}{\sin\alpha\cos\alpha}$$
$$= \frac{20}{\sin 2\alpha}.$$

> Use area of triangle $= \frac{1}{2}$ base $\times$ height.

Parallelogram is made up of four such triangles.

$\therefore$ Area of parallelogram $= \dfrac{80}{\sin 2\alpha}$

> From symmetry the area of the parallelogram is $4 \times$ area of triangle.

$\therefore$ Enclosed area $= \dfrac{80}{\sin 2\alpha} - 20\pi$.

> Area required = area of parallelogram − Area enclosed by ellipse.