Solutionbank

Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise A, Question 1

Question:

Find $\frac{dy}{dx}$ for each of the following, leaving your answer in terms of the parameter t:

(a) $x = 2t$, $y = t^2 - 3t + 2$

(b) $x = 3t^2$, $y = 2t^3$

(c) $x = t + 3t^2$, $y = 4t$

(d) $x = t^2 - 2$, $y = 3t^5$

(e) $x = \frac{2}{t}$, $y = 3t^2 - 2$

(f) $x = \frac{1}{2t-1}$, $y = \frac{t^2}{2t-1}$

(g) $x = \frac{2t}{1+t^2}$, $y = \frac{1-t^2}{1+t^2}$

(h) $x = t^2 e^t$, $y = 2t$

(i) $x = 4\sin 3t$, $y = 3\cos 3t$

(j) $x = 2 + \sin t$, $y = 3 - 4\cos t$

(k) $x = \sec t$, $y = \tan t$

(l) $x = 2t - \sin 2t$, $y = 1 - \cos 2t$

Solution:

(a) $x = 2t$, $y = t^2 - 3t + 2$

$\frac{dx}{dt} = 2$, $\frac{dy}{dt} = 2t - 3$

Using the chain rule

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{2t - 3}{2}$$

(b) $x = 3t^2, y = 2t^3$

$\frac{dx}{dt} = 6t, \frac{dy}{dt} = 6t^2$

Using the chain rule

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{6t^2}{6t} = t$$

(c) $x = t + 3t^2, y = 4t$

$\frac{dx}{dt} = 1 + 6t, \frac{dy}{dt} = 4$

$\therefore \frac{dy}{dx} = \frac{4}{1 + 6t}$ (from the chain rule)

(d) $x = t^2 - 2, y = 3t^5$

$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 15t^4$

$\therefore \frac{dy}{dx} = \frac{15t^4}{2t} = \frac{15t^3}{2}$ (from the chain rule)

(e) $x = \frac{2}{t}, y = 3t^2 - 2$

$\frac{dx}{dt} = -2t^{-2}, \frac{dy}{dt} = 6t$

$\therefore \frac{dy}{dx} = \frac{6t}{-2t^{-2}} = -3t^3$ (from the chain rule)

(f) $x = \frac{1}{2t - 1}, y = \frac{t^2}{2t - 1}$

As $x = (2t - 1)^{-1}, \frac{dx}{dt} = -2(2t - 1)^{-2}$ (from the chain rule)

Use the quotient rule to give

$$\frac{dy}{dt} = \frac{(2t-1)(2t) - t^2(2)}{(2t-1)^2} = \frac{2t^2-2t}{(2t-1)^2} = \frac{2t(t-1)}{(2t-1)^2}$$

Hence $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$

$$= \frac{2t(t-1)}{(2t-1)^2} \div -2(2t-1)^{-2}$$

$$= \frac{2t(t-1)}{(2t-1)^2} \div \frac{-2}{(2t-1)^2}$$

$$= \frac{2t(t-1)}{(2t-1)^2} \times \frac{(2t-1)^2}{-2}$$

$= -t(t-1)$ or $t(1-t)$

(g) $x = \frac{2t}{1+t^2}, y = \frac{1-t^2}{1+t^2}$

$$\frac{dx}{dt} = \frac{(1+t^2)2 - 2t(2t)}{(1+t^2)^2} = \frac{2-2t^2}{(1+t^2)^2}$$

and

$$\frac{dy}{dt} = \frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2} = \frac{-4t}{(1+t^2)^2}$$

Hence

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{-4t}{(1+t^2)^2} \div \frac{2-2t^2}{(1+t^2)^2}$$

$$= \frac{-4t}{2(1-t^2)}$$

$= -\frac{2t}{(1-t^2)}$ or $\frac{2t}{t^2-1}$

(h) $x = t^2e^t, y = 2t$

$\frac{dx}{dt} = t^2e^t + e^t2t$ (from the product rule) and $\frac{dy}{dt} = 2$

$$\therefore \frac{dy}{dx} = \frac{2}{t^2e^t + 2te^t} = \frac{2}{te^t(t+2)} \quad \text{(from the chain rule)}$$

(i) $x = 4\sin 3t,\ y = 3\cos 3t$

$$\frac{dx}{dt} = 12\cos 3t,\ \frac{dy}{dt} = -9\sin 3t$$

$$\therefore \frac{dy}{dx} = \frac{-9\sin 3t}{12\cos 3t} = -\frac{3}{4}\tan 3t \quad \text{(from the chain rule)}$$

(j) $x = 2 + \sin t,\ y = 3 - 4\cos t$

$$\frac{dx}{dt} = \cos t,\ \frac{dy}{dt} = 4\sin t$$

$$\therefore \frac{dy}{dx} = \frac{4\sin t}{\cos t} = 4\tan t \quad \text{(from the chain rule)}$$

(k) $x = \sec t,\ y = \tan t$

$$\frac{dx}{dt} = \sec t\tan t,\ \frac{dy}{dt} = \sec^2 t$$

Hence $\frac{dy}{dx} = \frac{\sec^2 t}{\sec t\tan t}$

$$= \frac{\sec t}{\tan t}$$

$$= \frac{1}{\cos t} \times \frac{\cos t}{\sin t}$$

$$= \frac{1}{\sin t}$$

$$= \operatorname{cosec} t$$

(l) $x = 2t - \sin 2t,\ y = 1 - \cos 2t$

$$\frac{dx}{dt} = 2 - 2\cos 2t,\ \frac{dy}{dt} = 2\sin 2t$$

Hence $\frac{dy}{dx} = \frac{2\sin 2t}{2 - 2\cos 2t}$

$$= \frac{2 \times 2\sin t\cos t}{2 - 2(1 - 2\sin^2 t)} \quad \text{(using double angle formulae)}$$

$$= \frac{\sin t\cos t}{\sin^2 t}$$

$$= \frac{\cos t}{\sin t}$$

$$= \cot t$$

Solutionbank
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise A, Question 2

Question:

(a) Find the equation of the tangent to the curve with parametric equations $x = 3t - 2\sin t$, $y = t^2 + t\cos t$, at the point P, where $t = \frac{\pi}{2}$.

(b) Find the equation of the tangent to the curve with parametric equations $x = 9 - t^2$, $y = t^2 + 6t$, at the point P, where $t = 2$.

Solution:

(a) $x = 3t - 2\sin t$, $y = t^2 + t\cos t$

$$\frac{dx}{dt} = 3 - 2\cos t, \quad \frac{dy}{dt} = 2t + \left(-t\sin t + \cos t \right)$$

$$\therefore \frac{dy}{dx} = \frac{2t - t\sin t + \cos t}{3 - 2\cos t}$$

When $t = \frac{\pi}{2}$, $\frac{dy}{dx} = \frac{\left(\pi - \frac{\pi}{2} \right)}{3} = \frac{\pi}{6}$

$\therefore$ the tangent has gradient $\frac{\pi}{6}$.

When $t = \frac{\pi}{2}$, $x = \frac{3\pi}{2} - 2$ and $y = \frac{\pi^2}{4}$

$\therefore$ the tangent passes through the point $\left(\frac{3\pi}{2} - 2, \frac{\pi^2}{4} \right)$

The equation of the tangent is

$$y - \frac{\pi^2}{4} = \frac{\pi}{6}\left[x - \left(\frac{3\pi}{2} - 2 \right) \right]$$

$$\therefore y - \frac{\pi^2}{4} = \frac{\pi}{6}x - \frac{\pi^2}{4} + \frac{\pi}{3}$$

i.e. $y = \frac{\pi}{6}x + \frac{\pi}{3}$

(b) $x = 9 - t^2$, $y = t^2 + 6t$

$\frac{dx}{dt} = -2t, \frac{dy}{dt} = 2t + 6$

$\therefore \frac{dy}{dx} = \frac{2t + 6}{-2t}$

At the point where $t = 2$, $\frac{dy}{dx} = \frac{10}{-4} = \frac{-5}{2}$

Also at $t = 2$, $x = 5$ and $y = 16$.

$\therefore$ the tangent has equation

$y - 16 = \frac{-5}{2}\left(x - 5\right)$

$\therefore 2y - 32 = -5x + 25$

i.e. $2y + 5x = 57$

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Differentiation
Exercise A, Question 3

Question:

(a) Find the equation of the normal to the curve with parametric equations $x = e^t$, $y = e^t + e^{-t}$, at the point P, where $t = 0$.

(b) Find the equation of the normal to the curve with parametric equations $x = 1 - \cos 2t$, $y = \sin 2t$, at the point P, where $t = \frac{\pi}{6}$.

Solution:

(a) $x = e^t$, $y = e^t + e^{-t}$

$\frac{dx}{dt} = e^t$ and $\frac{dy}{dt} = e^t - e^{-t}$

$\therefore \frac{dy}{dx} = \frac{e^t - e^{-t}}{e^t}$

When $t = 0$, $\frac{dy}{dx} = 0$

$\therefore$ gradient of curve is 0

$\therefore$ normal is parallel to the y-axis.

When $t = 0$, $x = 1$ and $y = 2$

$\therefore$ equation of the normal is $x = 1$

(b) $x = 1 - \cos 2t$, $y = \sin 2t$

$\frac{dx}{dt} = 2 \sin 2t$ and $\frac{dy}{dt} = 2 \cos 2t$

$\therefore \frac{dy}{dx} = \frac{2 \cos 2t}{2 \sin 2t} = \cot 2t$

When $t = \frac{\pi}{6}$, $\frac{dy}{dx} = \frac{1}{\tan \frac{\pi}{3}} = \frac{1}{\sqrt{3}}$

$\therefore$ gradient of the normal is $-\sqrt{3}$

When $t = \frac{\pi}{6}$, $x = 1 - \cos \frac{\pi}{3} = \frac{1}{2}$ and $y = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\therefore$ equation of the normal is

$$y - \frac{\sqrt{3}}{2} = -\sqrt{3}\left(x - \frac{1}{2}\right)$$

i.e. $y - \frac{\sqrt{3}}{2} = -\sqrt{3}x + \frac{\sqrt{3}}{2}$

$\therefore y + \sqrt{3}x = \sqrt{3}$

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Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise A, Question 4

Question:

Find the points of zero gradient on the curve with parametric equations $x = \frac{t}{1-t}$, $y = \frac{t^2}{1-t}$, $t \neq 1$.

You do not need to establish whether they are maximum or minimum points.

Solution:

$x = \frac{t}{1-t}$, $y = \frac{t^2}{1-t}$

Use the quotient rule to give

$$\frac{dx}{dt} = \frac{(1-t) \times 1 - t(-1)}{(1-t)^2} = \frac{1}{(1-t)^2}$$

and

$$\frac{dy}{dt} = \frac{(1-t)2t - t^2(-1)}{(1-t)^2} = \frac{2t - t^2}{(1-t)^2}$$

$$\therefore \frac{dy}{dx} = \frac{2t - t^2}{(1-t)^2} \div \frac{1}{(1-t)^2} = t(2-t)$$

When $\frac{dy}{dx} = 0$, $t = 0$ or 2

When $t = 0$ then $x = 0$, $y = 0$
When $t = 2$ then $x = -2$, $y = -4$
$\therefore (0, 0)$ and $(-2, -4)$ are the points of zero gradient.

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Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise B, Question 1

Question:

Find an expression in terms of x and y for $\frac{dy}{dx}$, given that:

(a) $x^2 + y^3 = 2$

(b) $x^2 + 5y^2 = 14$

(c) $x^2 + 6x - 8y + 5y^2 = 13$

(d) $y^3 + 3x^2 y - 4x = 0$

(e) $3y^2 - 2y + 2xy = x^3$

(f) $x = \frac{2y}{x^2 - y}$

(g) $(x - y)^4 = x + y + 5$

(h) $e^x y = x e^y$

(i) $\sqrt{(xy)} + x + y^2 = 0$

Solution:

(a) $x^2 + y^3 = 2$

Differentiate with respect to x:

$$2x + 3y^2 \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{-2x}{3y^2}$$

(b) $x^2 + 5y^2 = 14$

$$2x + 10y \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{-2x}{10y} = -\frac{x}{5y}$$

(c) $x^2 + 6x - 8y + 5y^2 = 13$

$$2x + 6 - 8\frac{dy}{dx} + 10y\frac{dy}{dx} = 0$$

$$2x + 6 = \left(8 - 10y\right)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{2x+6}{8-10y} = \frac{x+3}{4-5y}$$

(d) $y^3 + 3x^2y - 4x = 0$

Differentiate with respect to x:

$$3y^2\frac{dy}{dx} + \left(3x^2\frac{dy}{dx} + y \times 6x\right) - 4 = 0$$

$$\frac{dy}{dx}\left(3y^2 + 3x^2\right) = 4 - 6xy$$

$$\therefore \frac{dy}{dx} = \frac{4-6xy}{3\left(x^2+y^2\right)}$$

(e) $3y^2 - 2y + 2xy - x^3 = 0$

$$6y\frac{dy}{dx} - 2\frac{dy}{dx} + \left(2x\frac{dy}{dx} + y \times 2\right) - 3x^2 = 0$$

$$\frac{dy}{dx}\left(6y - 2 + 2x\right) = 3x^2 - 2y$$

$$\therefore \frac{dy}{dx} = \frac{3x^2-2y}{2x+6y-2}$$

(f) $x = \dfrac{2y}{x^2 - y}$

$\therefore x^3 - xy = 2y$

i.e. $x^3 - xy - 2y = 0$

Differentiate with respect to x:

$$3x^2 - \left(x\frac{dy}{dx} + y \times 1\right) - 2\frac{dy}{dx} = 0$$

$$3x^2 - y = \frac{dy}{dx}\left(x + 2\right)$$

$$\therefore \frac{dy}{dx} = \frac{3x^2-y}{x+2}$$

(g) $(x-y)^4 = x + y + 5$

Differentiate with respect to x:

$4(x-y)^3\left(1-\frac{dy}{dx}\right) = 1 + \frac{dy}{dx}$ (The chain rule was used to differentiate the first term.)

$\therefore 4(x-y)^3 - 1 = \frac{dy}{dx}\left[1 + 4(x-y)^3\right]$

$\therefore \frac{dy}{dx} = \frac{4(x-y)^3 - 1}{1 + 4(x-y)^3}$

(h) $e^x y = xe^y$

Differentiate with respect to x:

$e^x\frac{dy}{dx} + ye^x = xe^y\frac{dy}{dx} + e^y \times 1$

$e^x\frac{dy}{dx} - xe^y\frac{dy}{dx} = e^y - ye^x$

$\frac{dy}{dx}\left(e^x - xe^y\right) = e^y - ye^x$

$\therefore \frac{dy}{dx} = \frac{e^y - ye^x}{e^x - xe^y}$

(i) $\sqrt{xy} + x + y^2 = 0$

Differentiate with respect to x:

$\frac{1}{2}(xy)^{-\frac{1}{2}}\left(x\frac{dy}{dx} + y \times 1\right) + 1 + 2y\frac{dy}{dx} = 0$

Multiply both sides by $2\sqrt{xy}$:

$\left(x\frac{dy}{dx} + y\right) + 2\sqrt{xy} + 4y\sqrt{xy}\frac{dy}{dx} = 0$

$\frac{dy}{dx}\left(x + 4y\sqrt{xy}\right) = -\left(2\sqrt{xy} + y\right)$

$\therefore \frac{dy}{dx} = \frac{-(2\sqrt{xy} + y)}{x + 4y\sqrt{xy}}$.

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Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise B, Question 2

Question:

Find the equation of the tangent to the curve with implicit equation $x^2 + 3xy^2 - y^3 = 9$ at the point (2, 1).

Solution:

$x^2 + 3xy^2 - y^3 = 9$

Differentiate with respect to x:

$$2x + \left[3x \left(2y \frac{dy}{dx} \right) + y^2 \times 3 \right] - 3y^2 \frac{dy}{dx} = 0$$

When $x = 2$ and $y = 1$

$$4 + \left(12 \frac{dy}{dx} + 3 \right) - 3 \frac{dy}{dx} = 0$$

$$\therefore 9 \frac{dy}{dx} = -7$$

i.e. $\frac{dy}{dx} = \frac{-7}{9}$

$\therefore$ the gradient of the tangent at (2, 1) is $\frac{-7}{9}$.

The equation of the tangent is

$$\left(y - 1 \right) = \frac{-7}{9} \left(x - 2 \right)$$

$$\therefore 9y - 9 = -7x + 14$$

$$\therefore 9y + 7x = 23$$

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Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise B, Question 3

Question:

Find the equation of the normal to the curve with implicit equation $(x + y) ^{3} = x^{2} + y$ at the point (1, 0).

Solution:

$(x + y) ^{3} = x^{2} + y$

Differentiate with respect to x:

$$3 (x + y) ^{2} \left(1 + \frac{dy}{dx} \right) = 2x + \frac{dy}{dx}$$

At the point (1, 0), $x = 1$ and $y = 0$

$$\therefore 3 \left(1 + \frac{dy}{dx} \right) = 2 + \frac{dy}{dx}$$

$$\therefore 2 \frac{dy}{dx} = -1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-1}{2}$$

$\therefore$ The gradient of the normal at (1, 0) is 2.

$\therefore$ the equation of the normal is

$y - 0 = 2 (x - 1)$

i.e. $y = 2x - 2$

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Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise B, Question 4

Question:

Find the coordinates of the points of zero gradient on the curve with implicit equation $x^2 + 4y^2 - 6x - 16y + 21 = 0$.

Solution:

$x^2 + 4y^2 - 6x - 16y + 21 = 0$ ①

Differentiate with respect to x:

$$2x + 8y \frac{dy}{dx} - 6 - 16 \frac{dy}{dx} = 0$$

$$8y \frac{dy}{dx} - 16 \frac{dy}{dx} = 6 - 2x$$

$$\left(8y - 16 \right) \frac{dy}{dx} = 6 - 2x$$

$$\therefore \frac{dy}{dx} = \frac{6 - 2x}{8y - 16}$$

For zero gradient $\frac{dy}{dx} = 0 \Rightarrow 6 - 2x = 0 \Rightarrow x = 3$

Substitute $x = 3$ into ① to give

$9 + 4y^2 - 18 - 16y + 21 = 0$

$\Rightarrow 4y^2 - 16y + 12 = 0 \; [\div 4]$

$\Rightarrow y^2 - 4y + 3 = 0$

$\Rightarrow (y - 1)(y - 3) = 0$

$\Rightarrow y = 1$ or 3

$\therefore$ the coordinates of the points of zero gradient are (3, 1) and (3, 3).

Solutionbank
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise C, Question 1

Question:

Find $\frac{dy}{dx}$ for each of the following:

(a) $y = 3^x$

(b) $y = \left(\frac{1}{2} \right)^x$

(c) $y = xa^x$

(d) $y = \frac{2^x}{x}$

Solution:

(a) $y = 3^x$

$\frac{dy}{dx} = 3^x \ln 3$

(b) $y = \left(\frac{1}{2} \right)^x$

$\frac{dy}{dx} = \left(\frac{1}{2} \right)^x \ln \frac{1}{2}$

(c) $y = xa^x$

Use the product rule to give

$\frac{dy}{dx} = xa^x \ln a + a^x \times 1 = a^x \left(x \ln a + 1 \right)$

(d) $y = \frac{2^x}{x}$

Use the quotient rule to give

$\frac{dy}{dx} = \frac{x \times 2^x \ln 2 - 2^x \times 1}{x^2} = \frac{2^x (x \ln 2 - 1)}{x^2}$

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Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise C, Question 2

Question:

Find the equation of the tangent to the curve $y = 2^x + 2^{-x}$ at the point $\left(2 , 4 \frac{1}{4} \right)$.

Solution:

$y = 2^x + 2^{-x}$

$\frac{dy}{dx} = 2^x \ln 2 - 2^{-x} \ln 2$

When $x = 2$, $\frac{dy}{dx} = 4 \ln 2 - \frac{1}{4} \ln 2 = \frac{15}{4} \ln 2$

$\therefore$ the equation of the tangent at $\left(2 , 4 \frac{1}{4} \right)$ is

$y - 4 \frac{1}{4} = \frac{15}{4} \ln 2 \left(x - 2 \right)$

$\therefore 4y = (15 \ln 2) x + (17 - 30 \ln 2)$

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Edexcel AS and A Level Modular Mathematics

Differentiation

Exercise C, Question 3

Question:

A particular radioactive isotope has an activity R millicuries at time t days given by the equation $R = 200 \left(0.9 \right)^{t}$. Find the value of $\frac{dR}{dt}$, when $t = 8$.

Solution:

$R = 200 \left(0.9 \right)^{t}$

$\frac{dR}{dt} = 200 \times \ln 0.9 \times \left(0.9 \right)^{t}$

Substitute $t = 8$ to give

$\frac{dR}{dt} = - 9.07$ (3 s.f.)

Solutionbank
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise C, Question 4

Question:

The population of Cambridge was 37 000 in 1900 and was about 109 000 in 2000. Find an equation of the form $P = P_0 k^t$ to model this data, where t is measured as years since 1900. Evaluate $\frac{dP}{dt}$ in the year 2000. What does this value represent?

Solution:

$P = P_0 k^t$

When $t = 0$, $P = 37\ 000$

$\therefore 37\ 000 = P_0 \times k^0 = P_0 \times 1$

$\therefore P_0 = 37\ 000$

$\therefore P = 37\ 000 (k)^t$

When $t = 100$, $P = 109\ 000$

$\therefore 109\ 000 = 37\ 000 (k)^{100}$

$\therefore k^{100} = \frac{109\ 000}{37\ 000}$

$\therefore k = \sqrt[100]{\frac{109}{37}} \approx 1.01$

$\frac{dP}{dt} = 37\ 000 k^t \ln k$

When $t = 100$

$\frac{dP}{dt} = 37\ 000 \times \left(\frac{109}{37} \right) \times \ln k = 1000 \times 109 \times \frac{1}{100} \ln \frac{109}{37}$

$= 1178$ people per year

Rate of increase of the population during the year 2000.

Solutionbank
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise D, Question 1

Question:

Given that $V = \frac{1}{3}\pi r^3$ and that $\frac{dV}{dt} = 8$, find $\frac{dr}{dt}$ when $r = 3$.

Solution:

$V = \frac{1}{3}\pi r^3$

$\therefore \frac{dV}{dr} = \pi r^2$

Using the chain rule

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$\therefore 8 = \pi r^2 \times \frac{dr}{dt}$

$\therefore \frac{dr}{dt} = \frac{8}{\pi r^2}$

When $r = 3$, $\frac{dr}{dt} = \frac{8}{9\pi}$

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Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise D, Question 2

Question:

Given that $A = \frac{1}{4}\pi r^2$ and that $\frac{\mathrm{d}r}{\mathrm{d}t} = 6$, find $\frac{\mathrm{d}A}{\mathrm{d}t}$ when $r = 2$.

Solution:

$A = \frac{1}{4}\pi r^2$

$\frac{\mathrm{d}A}{\mathrm{d}r} = \frac{1}{2}\pi r$

Using the chain rule

$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{2}\pi r \times 6 = 3\pi r$

When $r = 2$, $\frac{\mathrm{d}A}{\mathrm{d}t} = 6\pi$

Solutionbank
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise D, Question 3

Question:

Given that $y = x\mathrm{e}^{x}$ and that $\frac{\mathrm{d}x}{\mathrm{d}t} = 5$, find $\frac{\mathrm{d}y}{\mathrm{d}t}$ when $x = 2$.

Solution:

$y = x\mathrm{e}^{x}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = x\mathrm{e}^{x} + \mathrm{e}^{x} \times 1$

Using the chain rule

$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{e}^{x}\left(x + 1 \right) \times 5$

When $x = 2$, $\frac{\mathrm{d}y}{\mathrm{d}t} = 15\mathrm{e}^{2}$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Differentiation

Exercise D, Question 4

Question:

Given that $r = 1 + 3\cos\theta$ and that $\frac{d\theta}{dt} = 3$, find $\frac{dr}{dt}$ when $\theta = \frac{\pi}{6}$.

Solution:

$r = 1 + 3\cos\theta$

$\frac{dr}{d\theta} = -3\sin\theta$

Using the chain rule

$\frac{dr}{dt} = \frac{dr}{d\theta} \times \frac{d\theta}{dt} = -3\sin\theta \times 3 = -9\sin\theta$

When $\theta = \frac{\pi}{6}$, $\frac{dr}{dt} = \frac{-9}{2}$

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Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise E, Question 1

Question:

In a study of the water loss of picked leaves the mass M grams of a single leaf was measured at times t days after the leaf was picked. It was found that the rate of loss of mass was proportional to the mass M of the leaf.
Write down a differential equation for the rate of change of mass of the leaf.

Solution:

$\frac{dM}{dt}$ represents rate of change of mass.

$\therefore \frac{dM}{dt} \propto -M$, as rate of *loss* indicates a negative quantity.

$\therefore \frac{dM}{dt} = -kM$, where k is the positive constant of proportionality.

Solutionbank
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise E, Question 2

Question:

A curve C has equation $y = \mathrm{f}(x)$, $y > 0$. At any point P on the curve, the gradient of C is proportional to the product of the x and the y coordinates of P.

The point A with coordinates (4, 2) is on C and the gradient of C at A is $\frac{1}{2}$.

Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{xy}{16}$.

Solution:

The gradient of the curve is given by $\frac{\mathrm{d}y}{\mathrm{d}x}$.

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} \propto xy$ (which is the *product* of x and y)

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = kxy$, where k is a constant of proportion.

When $x = 4$, $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}$

$\therefore \frac{1}{2} = k \times 4 \times 2$

$\therefore k = \frac{1}{16}$

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{xy}{16}$

Solutionbank
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise E, Question 3

Question:

Liquid is pouring into a container at a constant rate of 30 $\text{cm}^3\ \text{s}^{-1}$. At time t seconds liquid is leaking from the container at a rate of $\frac{2}{15}V$ cm^3s^{-1}, where V cm^3 is the volume of liquid in the container at that time.

Show that $-15\frac{dV}{dt}=2V-450$

Solution:

Let the rate of increase of the volume of liquid be $\frac{dV}{dt}$.

Then $\frac{dV}{dt}=30-\frac{2}{15}V$

Multiply both sides by -15:

$-15\frac{dV}{dt}=2V-450$

Solutionbank
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise E, Question 4

Question:

An electrically charged body loses its charge Q coulombs at a rate, measured in coulombs per second, proportional to the charge Q.
Write down a differential equation in terms of Q and t where t is the time in seconds since the body started to lose its charge.

Solution:

The rate of change of the charge is $\frac{\mathrm{d}Q}{\mathrm{d}t}$.

$\therefore \frac{\mathrm{d}Q}{\mathrm{d}t} \propto -Q$, as the body is *losing* charge the negative sign is required.

$\therefore \frac{\mathrm{d}Q}{\mathrm{d}t} = -kQ$, where k is the positive constant of proportion.

Solutionbank
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Differentiation
Exercise E, Question 5

Question:

The ice on a pond has a thickness *x* mm at a time *t* hours after the start of freezing. The rate of increase of *x* is inversely proportional to the square of *x*. Write down a differential equation in terms of *x* and *t*.

Solution:

The rate of increase of *x* is $\frac{dx}{dt}$.

$\therefore \frac{dx}{dt} \propto \frac{1}{x^2}$, as there is an *inverse* proportion.

$\therefore \frac{dx}{dt} = \frac{k}{x^2}$, where *k* is the constant of proportion.

Solutionbank

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Differentiation
Exercise E, Question 6

Question:

In another pond the amount of pondweed (P) grows at a rate proportional to the amount of pondweed already present in the pond. Pondweed is also removed by fish eating it at a constant rate of Q per unit of time.
Write down a differential equation relating P and t, where t is the time which has elapsed since the start of the observation.

Solution:

The rate of increase of pondweed is $\frac{\mathrm{d}P}{\mathrm{d}t}$.

This is proportional to P.

$\therefore \frac{\mathrm{d}P}{\mathrm{d}t} \propto P$

$\therefore \frac{\mathrm{d}P}{\mathrm{d}t} = kP$, where k is a constant.

But also pondweed is removed at a rate Q

$\therefore \frac{\mathrm{d}P}{\mathrm{d}t} = kP - Q$

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Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise E, Question 7

Question:

A circular patch of oil on the surface of some water has radius r and the radius increases over time at a rate inversely proportional to the radius.
Write down a differential equation relating r and t, where t is the time which has elapsed since the start of the observation.

Solution:

The rate of increase of the radius is $\frac{dr}{dt}$.

$\therefore \frac{dr}{dt} \propto \frac{1}{r}$, as it is *inversely* proportional to the radius.

$\therefore \frac{dr}{dt} = \frac{k}{r}$, where k is the constant of proportion.

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Differentiation
Exercise E, Question 8

Question:

A metal bar is heated to a certain temperature, then allowed to cool down and it is noted that, at time t, the rate of loss of temperature is proportional to the difference in temperature between the metal bar, θ, and the temperature of its surroundings θ_0.
Write down a differential equation relating θ and t.

Solution:

The rate of change of temperature is $\frac{d\theta}{dt}$.

$\therefore \frac{d\theta}{dt} \propto -\left(\theta - \theta_0\right)$ The rate of *loss* indicates the negative sign.

$\therefore \frac{d\theta}{dt} = -k\left(\theta - \theta_0\right)$, where k is the positive constant of proportion.

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Differentiation
Exercise E, Question 9

Question:

Fluid flows out of a cylindrical tank with constant cross section. At time t minutes, $t > 0$, the volume of fluid remaining in the tank is $V \text{ m}^3$. The rate at which the fluid flows in $\text{m}^3 \text{ min}^{-1}$ is proportional to the square root of V. Show that the depth h metres of fluid in the tank satisfies the differential equation $\frac{dh}{dt} = -k\sqrt{h}$, where k is a positive constant.

Solution:

Let the rate of flow of fluid be $\frac{-dV}{dt}$, as fluid is flowing *out* of the tank, and the volume left in the tank is decreasing.

$$\therefore \frac{-dV}{dt} \propto \sqrt{V}$$

$$\therefore \frac{dV}{dt} = -k'\sqrt{V}, \text{ where } k' \text{ is a positive constant.}$$

But $V = Ah$, where A is the constant cross section.

$$\therefore \frac{dV}{dh} = A$$

Use the chain rule to find $\frac{dh}{dt}$:

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh} = \frac{-k'\sqrt{V}}{A}$$

But $V = Ah$,

$$\therefore \frac{dh}{dt} = \frac{-k\sqrt{Ah}}{A} = \left(\frac{-k'}{\sqrt{A}}\right)\sqrt{h} = -k\sqrt{h}\text{, where } \frac{k'}{\sqrt{A}} \text{ is a positive constant.}$$

Solutionbank
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise E, Question 10

Question:

At time t seconds the surface area of a cube is A cm^2 and the volume is V cm^3.
The surface area of the cube is expanding at a constant rate 2 cm^{2}s^{-1}.
Show that $\frac{dV}{dt} = \frac{1}{2}V^{\frac{1}{3}}$.

Solution:

Rate of expansion of surface area is $\frac{dA}{dt}$.

Need $\frac{dV}{dt}$ so use the chain rule.

$$\frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$$

As $\frac{dA}{dt} = 2$, $\frac{dV}{dt} = 2\frac{dV}{dA}$ or $2 \div \left(\frac{dA}{dV}\right)$ ①

Let the cube have edge of length x cm.
Then $V = x^3$ and $A = 6x^2$.

Eliminate x to give $A = 6V^{\frac{2}{3}}$

$\therefore \frac{dA}{dV} = 4V^{\frac{-1}{3}}$

From ① $\frac{dV}{dt} = \frac{2}{4V^{-\frac{1}{3}}} = \frac{2V^{\frac{1}{3}}}{4} = \frac{1}{2}V^{\frac{1}{3}}$

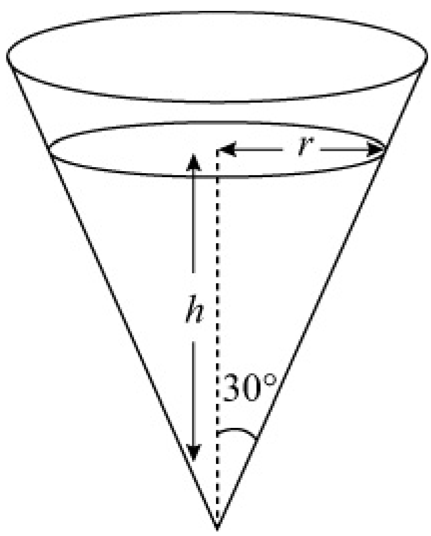
Solutionbank
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise E, Question 11

Question:

An inverted conical funnel is full of salt. The salt is allowed to leave by a small hole in the vertex. It leaves at a constant rate of 6 $\text{cm}^3\ \text{s}^{-1}$.
Given that the angle of the cone between the slanting edge and the vertical is 30 degrees, show that the volume of the salt is $\frac{1}{9}\pi h^3$, where h is the height of salt at time t seconds.
Show that the rate of change of the height of the salt in the funnel is inversely proportional to h^2. Write down the differential equation relating h and t.

Solution:



Use $V = \frac{1}{3}\pi r^2 h$

As $\tan 30° = \frac{r}{h}$

$\therefore r = h \tan 30° = \frac{h}{\sqrt{3}}$

$\therefore V = \frac{1}{3}\pi \left(\frac{h^2}{3} \right) \times h = \frac{1}{9}\pi h^3$ ①

It is given that $\frac{dV}{dt} = -6$.

To find $\frac{dh}{dt}$ use the chain rule:

$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV} = \frac{dV}{dt} \div \frac{dV}{dh}$$

From ① $\frac{dV}{dh} = \frac{1}{3}\pi h^2$

$$\therefore \frac{dh}{dt} = -6 \div \frac{1}{3}\pi h^2$$

$$\therefore \frac{dh}{dt} = \frac{-18}{\pi h^2}$$

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Differentiation
Exercise F, Question 1

Question:

The curve C is given by the equations

$x = 4t - 3, y = \frac{8}{t^2}, t > 0$

where t is a parameter.
At A, $t = 2$. The line l is the normal to C at A.

(a) Find $\frac{dy}{dx}$ in terms of t.

(b) Hence find an equation of l. Ⓔ

Solution:

(a) $x = 4t - 3, y = \frac{8}{t^2} = 8t^{-2}$

$\therefore \frac{dx}{dt} = 4$ and $\frac{dy}{dt} = -16t^{-3}$

$\therefore \frac{dy}{dx} = \frac{-16t^{-3}}{4} = \frac{-4}{t^3}$

(b) When $t = 2$ the curve has gradient $\frac{-4}{8} = -\frac{1}{2}$.

$\therefore$ the normal has gradient 2.
Also the point A has coordinates (5, 2)
$\therefore$ the equation of the normal is
$y - 2 = 2(x - 5)$
i.e. $y = 2x - 8$

Solutionbank
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Differentiation
Exercise F, Question 2

Question:

The curve C is given by the equations $x = 2t$, $y = t^2$, where t is a parameter. Find an equation of the normal to C at the point P on C where $t = 3$. (E)

Solution:

$x = 2t, y = t^2$

$\frac{dx}{dt} = 2, \frac{dy}{dt} = 2t$

$\therefore \frac{dy}{dx} = \frac{2t}{2} = t$

When $t = 3$ the gradient of the curve is 3.

$\therefore$ the gradient of the normal is $-\frac{1}{3}$.

Also at the point P where $t = 3$, the coordinates are (6, 9).

$\therefore$ the equation of the normal is

$y - 9 = -\frac{1}{3}\left(x - 6\right)$

i.e. $3y - 27 = -x + 6$

$\therefore 3y + x = 33$

Solutionbank
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Differentiation
Exercise F, Question 3

Question:

The curve C has parametric equations
$x = t^3, y = t^2, t > 0$
Find an equation of the tangent to C at A (1, 1). Ⓔ

Solution:

$x = t^3, y = t^2$

$\frac{dx}{dt} = 3t^2$ and $\frac{dy}{dt} = 2t$

$\therefore \frac{dy}{dx} = \frac{2t}{3t^2} = \frac{2}{3t}$

At the point (1, 1) the value of t is 1.

$\therefore$ the gradient of the curve is $\frac{2}{3}$, which is also the gradient of the tangent.

$\therefore$ the equation of the tangent is

$y - 1 = \frac{2}{3}\left(x - 1\right)$

i.e. $y = \frac{2}{3}x + \frac{1}{3}$

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Differentiation
Exercise F, Question 4

Question:

A curve C is given by the equations
$x = 2\cos t + \sin 2t$, $y = \cos t - 2\sin 2t$, $0 < t < \pi$
where t is a parameter.

(a) Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in terms of t.

(b) Find the value of $\frac{dy}{dx}$ at the point P on C where $t = \frac{\pi}{4}$.

(c) Find an equation of the normal to the curve at P. **E**

Solution:

(a) $x = 2\cos t + \sin 2t$, $y = \cos t - 2\sin 2t$

$$\frac{dx}{dt} = -2\sin t + 2\cos 2t, \quad \frac{dy}{dt} = -\sin t - 4\cos 2t$$

(b) $\therefore \frac{dy}{dx} = \frac{-\sin t - 4\cos 2t}{-2\sin t + 2\cos 2t}$

When $t = \frac{\pi}{4}$, $\frac{dy}{dx} = \frac{\frac{-1}{\sqrt{2}} - 0}{\frac{-2}{\sqrt{2}} + 0} = \frac{1}{2}$

(c) $\therefore$ the gradient of the normal at the point P, where $t = \frac{\pi}{4}$, is -2.

The coordinates of P are found by substituting $t = \frac{\pi}{4}$ into the parametric equations, to give

$x = \frac{2}{\sqrt{2}} + 1$, $y = \frac{1}{\sqrt{2}} - 2$

$\therefore$ the equation of the normal is

$$y - \left(\frac{1}{\sqrt{2}} - 2\right) = -2\left[x - \left(\frac{2}{\sqrt{2}} + 1\right)\right]$$

i.e. $y - \frac{1}{\sqrt{2}} + 2 = -2x + \frac{4}{\sqrt{2}} + 2$

$\therefore y + 2x = \frac{5\sqrt{2}}{2}$

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Differentiation
Exercise F, Question 5

Question:

A curve is given by $x = 2t + 3$, $y = t^3 - 4t$, where t is a parameter. The point A has parameter $t = -1$ and the line l is the tangent to C at A. The line l also cuts the curve at B.

(a) Show that an equation for l is $2y + x = 7$.

(b) Find the value of t at B. E

Solution:

(a) $x = 2t + 3$, $y = t^3 - 4t$
At point A, $t = -1$.
$\therefore$ the coordinates of the point A are (1, 3)

$\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 3t^2 - 4$

$\therefore \frac{dy}{dx} = \frac{3t^2 - 4}{2}$

At the point A, $\frac{dy}{dx} = -\frac{1}{2}$

$\therefore$ the gradient of the tangent at A is $-\frac{1}{2}$.

$\therefore$ the equation of the tangent at A is

$y - 3 = -\frac{1}{2}\left(x - 1\right)$

i.e. $2y - 6 = -x + 1$

$\therefore 2y + x = 7$

(b) This line cuts the curve at the point B.
$\therefore 2\left(t^3 - 4t\right) + \left(2t + 3\right) = 7$ gives the values of t at A and B.
i.e. $2t^3 - 6t - 4 = 0$
At A, $t = -1$
$\therefore \left(t + 1\right)$ is a root of this equation

$$2t^3 - 6t - 4 = \left(t + 1 \right) \left(2t^2 - 2t - 4 \right) = \left(t + 1 \right) \left(t + 1 \right) \left(2t - 4 \right) = 2\,(t + 1)^2 \left(t - 2 \right)$$

So when the line meets the curve, $t = -1$ (repeated root because the line touches the curve) or $t = 2$.

$\therefore$ at the point B, $t = 2$.

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Differentiation
Exercise F, Question 6

Question:

A Pancho car has value £V at time t years. A model for V assumes that the rate of decrease of V at time t is proportional to V. Form an appropriate differential equation for V. (E)

Solution:

$\frac{dV}{dt}$ is the rate of change of V.

$\frac{dV}{dt} \propto -V$, as a decrease indicates a negative quantity.

$\therefore \frac{dV}{dt} = -kV$, where k is a positive constant of proportionality.

Solutionbank
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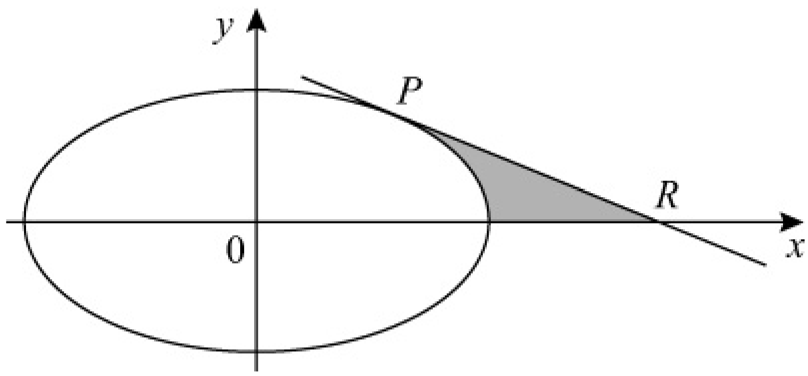
Differentiation
Exercise F, Question 7

Question:

The curve shown has parametric equations
$x = 5\cos\theta$, $y = 4\sin\theta$, $0 \leq \theta < 2\pi$



(a) Find the gradient of the curve at the point P at which $\theta = \frac{\pi}{4}$.

(b) Find an equation of the tangent to the curve at the point P.

(c) Find the coordinates of the point R where this tangent meets the x-axis. E

Solution:

(a) $x = 5\cos\theta$, $y = 4\sin\theta$

$\frac{dx}{d\theta} = -5\sin\theta$ and $\frac{dy}{d\theta} = 4\cos\theta$

$\therefore \frac{dy}{dx} = \frac{-4\cos\theta}{5\sin\theta}$

At the point P, where $\theta = \frac{\pi}{4}$, $\frac{dy}{dx} = \frac{-4}{5}$.

(b) At the point P, $x = \frac{5}{\sqrt{2}}$ and $y = \frac{4}{\sqrt{2}}$.

$\therefore$ the equation of the tangent at P is

$$y - \frac{4}{\sqrt{2}} = \frac{-4}{5}\left(x - \frac{5}{\sqrt{2}}\right)$$

i.e. $y - \frac{4}{\sqrt{2}} = \frac{-4}{5}x + \frac{4}{\sqrt{2}}$

$\therefore y = \frac{-4}{5}x + \frac{8}{\sqrt{2}}$

Multiply equation by 5 and rationalise the denominator of the surd:
$5y + 4x = 20\sqrt{2}$

(c) The tangent meets the x-axis when $y = 0$.
$\therefore x = 5\sqrt{2}$ and R has coordinates $(5\sqrt{2}, 0)$.

Solutionbank
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise F, Question 8

Question:

The curve C has parametric equations

$x = 4\cos 2t,\ y = 3\sin t,\ -\frac{\pi}{2} < t < \frac{\pi}{2}$

A is the point $\left(2,\ 1\frac{1}{2}\right)$, and lies on C.

(a) Find the value of t at the point A.

(b) Find $\frac{dy}{dx}$ in terms of t.

(c) Show that an equation of the normal to C at A is $6y - 16x + 23 = 0$.
The normal at A cuts C again at the point B.

(d) Find the y-coordinate of the point B.

E

Solution:

(a) $x = 4\cos 2t$ and $y = 3\sin t$

A is the point $\left(2,\ 1\frac{1}{2}\right)$ and so

$4\cos 2t = 2$ and $3\sin t = 1\frac{1}{2}$

$\therefore \cos 2t = \frac{1}{2}$ and $\sin t = \frac{1}{2}$

As $-\frac{\pi}{2} < t < \frac{\pi}{2}$, $t = \frac{\pi}{6}$ at the point A.

(b) $\frac{dx}{dt} = -8\sin 2t$ and $\frac{dy}{dt} = 3\cos t$

$\therefore \frac{dy}{dx} = \frac{3\cos t}{-8\sin 2t}$

$= \frac{-3\cos t}{16\sin t\cos t}$ (using the double angle formula)

$$= \frac{-3}{16 \sin t}$$

$$= \frac{-3}{16} \operatorname{cosec} t$$

(c) When $t = \frac{\pi}{6}$, $\frac{dy}{dx} = \frac{-3}{8}$

$\therefore$ the gradient of the normal at the point A is $\frac{8}{3}$.

$\therefore$ the equation of the normal is

$$y - 1\frac{1}{2} = \frac{8}{3}\left(x - 2 \right)$$

Multiply equation by 6:

$6y - 9 = 16x - 32$

$\therefore 6y - 16x + 23 = 0$

(d) The normal cuts the curve when

$6(3 \sin t) - 16(4 \cos 2t) + 23 = 0$

$\therefore 18 \sin t - 64 \cos 2t + 23 = 0.$

$\therefore 18 \sin t - 64(1 - 2 \sin^2 t) + 23 = 0$ (using the double angle formula)

$\therefore 128 \sin^2 t + 18 \sin t - 41 = 0$

But $\sin t = \frac{1}{2}$ is one solution of this equation, as point A lies on the line and on the curve.

$\therefore 128 \sin^2 t + 18 \sin t - 41 = (2 \sin t - 1)(64 \sin t + 41)$

$\therefore (2 \sin t - 1)(64 \sin t + 41) = 0$

$\therefore$ at point B, $\sin t = \frac{-41}{64}$

$\therefore$ the y coordinate of point B is $\frac{-123}{64}$.

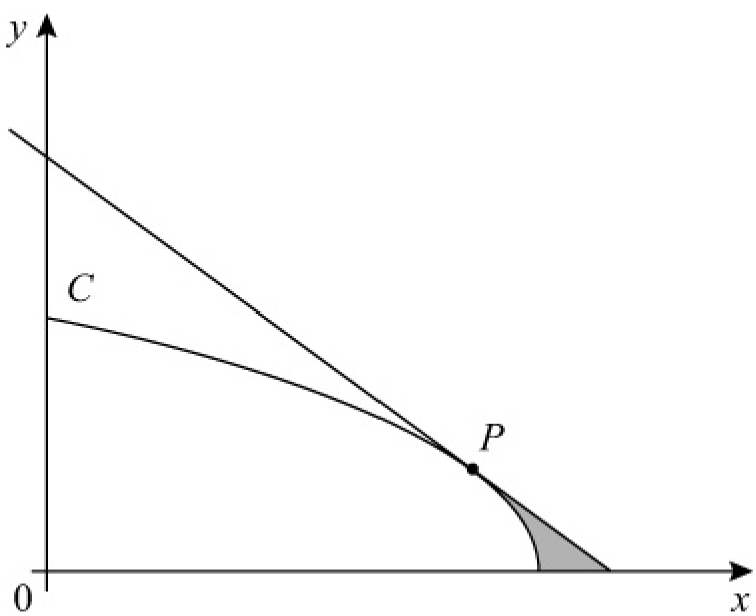
Solutionbank
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise F, Question 9

Question:

The diagram shows the curve C with parametric equations
$x = a\ \sin^2 t,\ y = a\cos t,\ 0 \le t \le \frac{1}{2}\pi$

where a is a positive constant. The point P lies on C and has coordinates $\left(\frac{3}{4}a\ ,\ \frac{1}{2}a\right)$.



(a) Find $\frac{dy}{dx}$, giving your answer in terms of t.

(b) Find an equation of the tangent at P to C.

E

Solution:

(a) $x = a\ \sin^2 t,\ y = a\cos t$

$$\frac{dx}{dt} = 2a\sin t\cos t \text{ and } \frac{dy}{dt} = -a\sin t$$

$$\therefore\ \frac{dy}{dx} = \frac{-a\sin t}{2a\sin t\cos t} = \frac{-1}{2\cos t} = \frac{-1}{2}\sec\ t$$

(b) P is the point $\left(\frac{3}{4}a , \frac{1}{2}a \right)$ and lies on the curve.

$\therefore a \sin^2 t = \frac{3}{4}a$ and $a \cos t = \frac{1}{2}a$

$\therefore \sin t = \pm \frac{\sqrt{3}}{2}$ and $\cos t = \frac{1}{2}$ and $0 \le t \le \frac{1}{2}\pi$

$\therefore t = \frac{\pi}{3}$

$\therefore$ the gradient of the curve at point P is $-\frac{1}{2}\sec\frac{\pi}{3} = -1.$

The equation of the tangent at P is

$y - \frac{1}{2}a = -1\left(x - \frac{3}{4}a \right)$

$\therefore y + x = \frac{1}{2}a + \frac{3}{4}a$

Multiply equation by 4 to give $4y + 4x = 5a$

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Edexcel AS and A Level Modular Mathematics

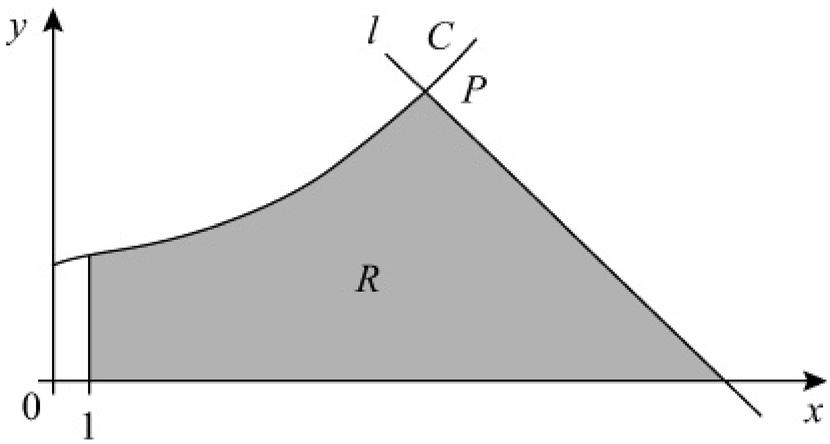
Differentiation
Exercise F, Question 10

Question:

This graph shows part of the curve C with parametric equations

$x = (t+1)^2, y = \frac{1}{2}t^3 + 3, t \geq -1$

P is the point on the curve where $t = 2$. The line l is the normal to C at P.
Find the equation of l.



E

Solution:

$x = (t+1)^2, y = \frac{1}{2}t^3 + 3$

$\frac{dx}{dt} = 2\left(t+1\right)$ and $\frac{dy}{dt} = \frac{3}{2}t^2$

$\therefore \frac{dy}{dx} = \frac{\left(\frac{3}{2}t^2\right)}{2(t+1)} = \frac{3t^2}{4(t+1)}$

When $t = 2$, $\frac{dy}{dx} = \frac{3 \times 4}{4 \times 3} = 1$

The gradient of the normal at the point P where $t = 2$, is -1.
The coordinates of P are (9, 7).
$\therefore$ the equation of the normal is
$y - 7 = -1(x - 9)$
i.e. $y - 7 = -x + 9$

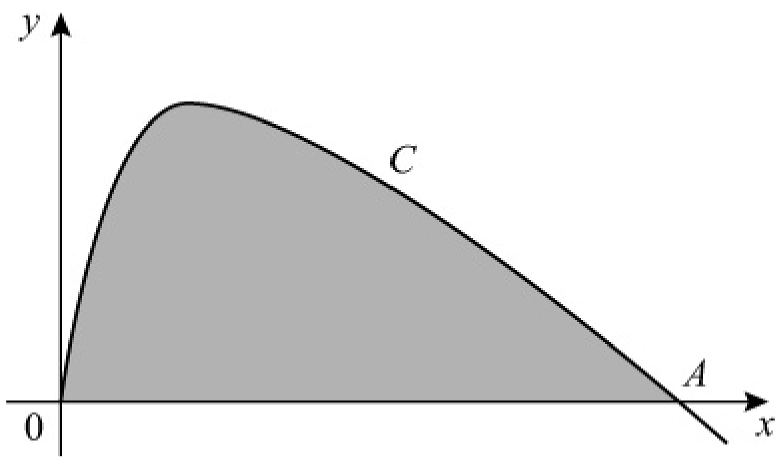
$\therefore y + x = 16$

Solutionbank
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise F, Question 11

Question:

The diagram shows part of the curve C with parametric equations
$x = t^2$, $y = \sin 2t$, $t \geq 0$
The point A is an intersection of C with the x-axis.



(a) Find, in terms of π, the x-coordinate of A.

(b) Find $\frac{dy}{dx}$ in terms of t, $t > 0$.

(c) Show that an equation of the tangent to C at A is $4x + 2\pi y = \pi^2$.

E

Solution:

(a) $x = t^2$ and $y = \sin 2t$
At the point A, $y = 0$.
$\therefore \sin 2t = 0$
$\therefore 2t = \pi$
$\therefore t = \frac{\pi}{2}$

The point A is $\left(\frac{\pi^2}{4} , 0 \right)$

(b) $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 2\cos 2t$

$\therefore \frac{dy}{dx} = \frac{\cos 2t}{t}$

(c) At point A, $\frac{dy}{dx} = \frac{-1}{\left(\frac{\pi}{2}\right)} = \frac{-2}{\pi}$

$\therefore$ the gradient of the tangent at A is $\frac{-2}{\pi}$.

$\therefore$ the equation of the tangent at A is

$y - 0 = \frac{-2}{\pi}\left(x - \frac{\pi^2}{4}\right)$

i.e. $y = \frac{-2x}{\pi} + \frac{\pi}{2}$

Multiply equation by 2π to give

$2\pi y + 4x = \pi^2$

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Differentiation
Exercise F, Question 12

Question:

Find the gradient of the curve with equation
$5x^2 + 5y^2 - 6xy = 13$
at the point (1, 2).

E

Solution:

$5x^2 + 5y^2 - 6xy = 13$
Differentiate implicitly with respect to x:

$$10x + 10y\frac{dy}{dx} - \left(6x\frac{dy}{dx} + 6y\right) = 0$$

$$\therefore \frac{dy}{dx}\left(10y - 6x\right) + 10x - 6y = 0$$

At the point (1, 2)

$$\frac{dy}{dx}\left(14\right) + 10 - 12 = 0$$

$$\therefore \frac{dy}{dx} = \frac{2}{14} = \frac{1}{7}$$

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Differentiation
Exercise F, Question 13

Question:

Given that $e^{2x} + e^{2y} = xy$, find $\frac{dy}{dx}$ in terms of x and y.

E

Solution:

$e^{2x} + e^{2y} = xy$

Differentiate with respect to x:

$$2e^{2x} + 2e^{2y} \frac{dy}{dx} = x \frac{dy}{dx} + y \times 1$$

$$\therefore 2e^{2y} \frac{dy}{dx} - x \frac{dy}{dx} = y - 2e^{2x}$$

$$\therefore \frac{dy}{dx} \left(2e^{2y} - x \right) = y - 2e^{2x}$$

$$\therefore \frac{dy}{dx} = \frac{y - 2e^{2x}}{2e^{2y} - x}$$

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Differentiation
Exercise F, Question 14

Question:

Find the coordinates of the turning points on the curve $y^3 + 3xy^2 - x^3 = 3$.

E

Solution:

$y^3 + 3xy^2 - x^3 = 3$

Differentiate with respect to x:

$$3y^2 \frac{dy}{dx} + \left(3x \times 2y \frac{dy}{dx} + y^2 \times 3 \right) - 3x^2 = 0$$

$$\therefore \frac{dy}{dx} \left(3y^2 + 6xy \right) = 3x^2 - 3y^2$$

$$\therefore \frac{dy}{dx} = \frac{3 (x^2 - y^2)}{3y (y + 2x)} = \frac{x^2 - y^2}{y (y + 2x)}$$

When $\frac{dy}{dx} = 0$, $x^2 = y^2$, i.e. $x = \pm y$

When $x = + y$, $y^3 + 3y^3 - y^3 = 3 \Rightarrow 3y^3 = 3 \Rightarrow y = 1$ and $x = 1$

When $x = - y$, $y^3 - 3y^3 + y^3 = 3 \Rightarrow - y^3 = 3 \Rightarrow y = \sqrt[3]{(- 3)}$ and $x = - \sqrt[3]{(- 3)}$

$\therefore$ the coordinates are (1, 1) and $\left(- \sqrt[3]{(- 3)} , \sqrt[3]{(- 3)} \right)$.

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Differentiation
Exercise F, Question 15

Question:

Given that $y\left(x+y\right)=3$, evaluate $\frac{dy}{dx}$ when $y=1$.

E

Solution:

$y\left(x+y\right)=3$

$\therefore yx+y^2=3$

Differentiate with respect to x:

$\left(y+x\frac{dy}{dx}\right)+2y\frac{dy}{dx}=0$ ①

When $y=1$, $1\left(x+1\right)=3$ (from original equation)

$\therefore x=2$

Substitute into ①:

$1+2\frac{dy}{dx}+2\frac{dy}{dx}=0$

$\therefore 4\frac{dy}{dx}=-1$

i.e. $\frac{dy}{dx}=\frac{-1}{4}$

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Differentiation
Exercise F, Question 16

Question:

(a) If $(1+x)(2+y)=x^2+y^2$, find $\frac{dy}{dx}$ in terms of x and y.

(b) Find the gradient of the curve $(1+x)(2+y)=x^2+y^2$ at each of the two points where the curve meets the y-axis.

(c) Show also that there are two points at which the tangents to this curve are parallel to the y-axis.

E

Solution:

(a) $(1+x)(2+y)=x^2+y^2$

Differentiate with respect to x:

$$\left(1+x\right)\left(\frac{dy}{dx}\right)+\left(2+y\right)\left(1\right)=2x+2y\frac{dy}{dx}$$

$$\therefore \left(1+x-2y\right)\frac{dy}{dx}=2x-y-2$$

$$\therefore \frac{dy}{dx}=\frac{2x-y-2}{1+x-2y}$$

(b) When the curve meets the y-axis, $x=0$.

Put $x=0$ in original equation $(1+x)(2+y)=x^2+y^2$.

Then $2+y=y^2$

i.e. $y^2-y-2=0$

$\Rightarrow \quad (y-2)(y+1)=0$

$\therefore y=2$ or $y=-1$ when $x=0$

At (0, 2), $\frac{dy}{dx}=\frac{-4}{-3}=\frac{4}{3}$

At $(0,-1)$, $\frac{dy}{dx}=\frac{-1}{3}$

(c) When the tangent is parallel to the y-axis it has infinite gradient and as

$$\frac{dy}{dx} = \frac{2x - y - 2}{1 + x - 2y}$$

So $1 + x - 2y = 0$

Substitute $1 + x = 2y$ into the equation of the curve:

$2y(2 + y) = (2y - 1)^2 + y^2$

$2y^2 + 4y = 4y^2 - 4y + 1 + y^2$

$3y^2 - 8y + 1 = 0$

$$y = \frac{8 \pm \sqrt{64 - 12}}{6} = \frac{4 \pm \sqrt{13}}{3}$$

When $y = \frac{4 + \sqrt{13}}{3}$, $x = \frac{5 + 2\sqrt{13}}{3}$

When $y = \frac{4 - \sqrt{13}}{3}$, $x = \frac{5 - 2\sqrt{13}}{3}$

$\therefore$ there are two points at which the tangents are parallel to the y-axis.

They are $\left(\frac{5 + 2\sqrt{13}}{3}, \frac{4 + \sqrt{13}}{3} \right)$ and $\left(\frac{5 - 2\sqrt{13}}{3}, \frac{4 - \sqrt{13}}{3} \right)$.

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Differentiation

Exercise F, Question 17

Question:

A curve has equation $7x^2 + 48xy - 7y^2 + 75 = 0$. A and B are two distinct points on the curve and at each of these points the gradient of the curve is equal to $\frac{2}{11}$. Use implicit differentiation to show that $x + 2y = 0$ at the points A and B.

E

Solution:

$7x^2 + 48xy - 7y^2 + 75 = 0$

Differentiate with respect to x (implicit differentiation):

$$14x + \left(48x\frac{dy}{dx} + 48y\right) - 14y\frac{dy}{dx} = 0$$

Given that $\frac{dy}{dx} = \frac{2}{11}$

$$\therefore 14x + 48x \times \frac{2}{11} + 48y - 14y \times \frac{2}{11} = 0$$

Multiply equation by 11,

then $154x + 96x + 528y - 28y = 0$

$\therefore 250x + 500y = 0$

i.e. $x + 2y = 0$, after division by 250.

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Differentiation
Exercise F, Question 18

Question:

Given that $y = x^x$, $x > 0$, $y > 0$, by taking logarithms show that

$$\frac{dy}{dx} = x^x \left(1 + \ln x \right)$$

E

Solution:

$y = x^x$

Take natural logs of both sides:

$\ln y = \ln x^x$

$\therefore \ln y = x \ln x$ Property of lns

Differentiate with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{x} + \ln x \times 1$$

$$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x$$

$$\therefore \frac{dy}{dx} = y \left(1 + \ln x \right)$$

But $y = x^x$

$$\therefore \frac{dy}{dx} = x^x \left(1 + \ln x \right)$$

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Differentiation
Exercise F, Question 19

Question:

(a) Given that $x = 2^t$, by using logarithms prove that

$$\frac{dx}{dt} = 2^t \ln 2$$

A curve C has parametric equations $x = 2^t$, $y = 3t^2$. The tangent to C at the point with coordinates (2, 3) cuts the x-axis at the point P.

(b) Find $\frac{dy}{dx}$ in terms of t.

(c) Calculate the x-coordinate of P, giving your answer to 3 decimal places.

E

Solution:

(a) Given $x = 2^t$

Take natural logs of both sides:

$\ln x = \ln 2^t = t \ln 2$

Differentiate with respect to t:

$$\frac{1}{x}\frac{dx}{dt} = \ln 2$$

$$\therefore \frac{dx}{dt} = x \ln 2 = 2^t \ln 2$$

(b) $x = 2^t$, $y = 3t^2$

$$\frac{dx}{dt} = 2^t \ln 2, \frac{dy}{dt} = 6t$$

$$\therefore \frac{dy}{dx} = \frac{6t}{2^t \ln 2}$$

(c) At the point (2, 3), $t = 1$.

The gradient of the curve at (2, 3) is $\frac{6}{2 \ln 2}$.

$\therefore$ the equation of the tangent is

$$y - 3 = \frac{6}{2\ln 2}\left(x - 2\right)$$

i.e. $y = \frac{3}{\ln 2}x - \frac{6}{\ln 2} + 3$

The tangent meets the x-axis when $y = 0$.

$$\therefore \frac{3}{\ln 2}x = \frac{6}{\ln 2} - 3$$

$$\therefore x = 2 - \ln 2 = 1.307 \text{ (3 decimal places)}$$

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Differentiation
Exercise F, Question 20

Question:

(a) Given that $a^x \equiv e^{kx}$, where a and k are constants, $a > 0$ and $x \in \mathbb{R}$, prove that $k = \ln a$.

(b) Hence, using the derivative of e^{kx}, prove that when $y = 2^x$
$\frac{dy}{dx} = 2^x \ln 2$.

(c) Hence deduce that the gradient of the curve with equation $y = 2^x$ at the point (2, 4) is ln 16.

Solution:

(a) $a^x = e^{kx}$
Take lns of both sides:
$\ln a^x = \ln e^{kx}$
i.e. $x \ln a = kx$
As this is true for all values of x, $k = \ln a$.

(b) Therefore, $2^x = e^{\ln 2 \times x}$
When $y = 2^x = e^{\ln 2 \times x}$
$\frac{dy}{dx} = \ln 2 \; e^{\ln 2 \times x} = \ln 2 \times 2^x$

(c) At the point (2, 4), $x = 2$.
$\therefore$ the gradient of the curve is
$2^2 \ln 2$
$= 4 \ln 2$
$= \ln 2^4$ (property of logs)
$= \ln 16$

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Differentiation
Exercise F, Question 21

Question:

A population P is growing at the rate of 9% each year and at time t years may be approximated by the formula

$P = P_0 (1.09)^t, t \geq 0$

where P is regarded as a continuous function of t and P_0 is the starting population at time $t = 0$.

(a) Find an expression for t in terms of P and P_0.

(b) Find the time T years when the population has doubled from its value at $t = 0$,
giving your answer to 3 significant figures.

(c) Find, as a multiple of P_0, the rate of change of population $\frac{dP}{dt}$ at time $t = T$. (E)

Solution:

(a) $P = P_0 (1.09)^t$

Take natural logs of both sides:

$\ln P = \ln [P_0 (1.09)^t] = \ln P_0 + t \ln 1.09$

$\therefore t \ln 1.09 = \ln P - \ln P_0$

$$\Rightarrow \quad t = \frac{\ln P - \ln P_0}{\ln 1.09} \quad \text{or} \quad \frac{\ln \left(\frac{P}{P_0} \right)}{\ln 1.09}$$

(b) When $P = 2P_0$, $t = T$.

$\therefore T = \frac{\ln 2}{\ln 1.09} = 8.04$ (to 3 significant figures)

(c) $\frac{dP}{dt} = P_0 (1.09)^t \ln 1.09$

When $t = T$, $P = 2P_0$ so $(1.09)^T = 2$ and

$$\frac{dP}{dt} = P_0 \times 2 \times \ln\ 1.09$$

$$= \ln\ (1.09^2) \times P_0 = \ln(1.1881) \times P_0$$

$$= 0.172P_0 \text{ (to 3 significant figures)}$$