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Edexcel AS and A Level Modular Mathematics

The binomial expansion

Exercise A, Question 1

Question:

Find the binomial expansion of the following up to and including the terms in x^3 . State the range values of x for which these expansions are valid.

(a) $(1 + 2x)^3$

(b) $\frac{1}{1-x}$

(c) $\sqrt{1+x}$

(d) $\frac{1}{(1+2x)^3}$

(e) $\sqrt[3]{1-3x}$

(f) $(1-10x)^{\frac{3}{2}}$

(g) $\left(1 + \frac{x}{4}\right)^{-4}$

(h) $\frac{1}{(1+2x^2)}$

Solution:

(a) $(1 + 2x)^3$ Use expansion with $n = 3$ and x replaced with $2x$

$$= 1 + 3 \binom{3}{1} 2x + \frac{3 \times 2 \times (2x)^2}{2!} + \frac{3 \times 2 \times 1 \times (2x)^3}{3!} +$$

$$\frac{3 \times 2 \times 1 \times 0 \times (2x)^4}{4!} + \dots$$

$$= 1 + 6x + 12x^2 + 8x^3 + 0x^4 \quad \text{All terms after } 0x^4 \text{ will also be zero}$$

$$= 1 + 6x + 12x^2 + 8x^3$$

Expansion is finite and exact. Valid for all values of x .

(b) $\frac{1}{1-x}$ Write in index form

$$= (1-x)^{-1} \quad \text{Use expansion with } n = -1 \text{ and } x \text{ replaced with } -x$$

$$= 1 + \binom{-1}{1} \binom{-x}{1} + \frac{(-1)(-2)(-x)^2}{2!} +$$

$$\frac{(-1)(-2)(-3)(-x)^3}{3!} + \dots$$

$$= 1 + 1x + 1x^2 + 1x^3 + \dots$$

$$= 1 + x + x^2 + x^3 + \dots$$

Expansion is infinite. Valid when $|-x| < 1 \Rightarrow |x| < 1$.

(c) $\sqrt{1+x}$ Write in index form

$$= (1+x)^{\frac{1}{2}} \quad \text{Use expansion with } n = \frac{1}{2} \text{ and } x \text{ replaced with } x$$

$$= 1 + \binom{\frac{1}{2}}{1} \binom{x}{1} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(x)^2}{2!} +$$

$$\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(x)^3}{3!} + \dots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots$$

Expansion is infinite. Valid when $|x| < 1$.

(d) $\frac{1}{(1+2x)^3}$ Write in index form

$$= (1+2x)^{-3} \quad \text{Use expansion with } n = -3 \text{ and } x \text{ replaced with } 2x$$

$$= 1 + \binom{-3}{1} \binom{2x}{1} + \frac{(-3)(-4)(2x)^2}{2!} +$$

$$\frac{(-3)(-4)(-5)(2x)^3}{3!} + \dots$$

$$= 1 - 6x + 24x^2 - 80x^3 + \dots$$

Expansion is infinite. Valid when $|2x| < 1 \Rightarrow |x| < \frac{1}{2}$.

(e) $\sqrt[3]{(1-3x)}$ Write in index form

$$= (1-3x)^{\frac{1}{3}} \quad \text{Use expansion with } n = \frac{1}{3} \text{ and } x \text{ replaced with } -3x$$

$$= 1 + \binom{\frac{1}{3}}{1} (-3x) + \frac{\binom{\frac{1}{3}}{2} (-3x)^2}{2!} +$$

$$\frac{\binom{\frac{1}{3}}{3} (-3x)^3}{3!} + \dots$$

$$= 1 - x - x^2 - \frac{5}{3}x^3 + \dots$$

Expansion is infinite. Valid when $|-3x| < 1 \Rightarrow |x| < \frac{1}{3}$.

(f) $(1-10x)^{\frac{3}{2}}$ Use expansion with $n = \frac{3}{2}$ and x replaced with $-10x$

$$= 1 + \binom{\frac{3}{2}}{1} (-10x) + \frac{\binom{\frac{3}{2}}{2} (-10x)^2}{2!} +$$

$$\frac{\binom{\frac{3}{2}}{3} (-10x)^3}{3!} + \dots$$

$$= 1 - 15x + \frac{3}{8} \times 100x^2 - \frac{1}{16} \times (-1000x^3) + \dots$$

$$= 1 - 15x + \frac{75}{2}x^2 + \frac{125}{2}x^3 + \dots$$

Expansion is infinite. Valid when $|-10x| < 1 \Rightarrow |x| < \frac{1}{10}$.

(g) $\left(1 + \frac{x}{4}\right)^{-4}$ Use expansion with $n = -4$ and x replaced with $\frac{x}{4}$

$$= 1 + \binom{-4}{1} \left(\frac{x}{4}\right) + \frac{(-4)(-5)}{2!} \left(\frac{x}{4}\right)^2 +$$

$$\frac{(-4)(-5)(-6)}{3!} \left(\frac{x}{4}\right)^3 + \dots$$

$$= 1 - x + 10 \times \frac{x^2}{16} - 20 \times \frac{x^3}{64} + \dots$$

$$= 1 - x + \frac{5}{8}x^2 - \frac{5}{16}x^3 + \dots$$

Expansion is infinite. Valid when $\left|\frac{x}{4}\right| < 1 \Rightarrow |x| < 4$.

(h) $\frac{1}{1+2x^2}$ Write in index form

$$= (1 + 2x^2)^{-1} \quad \text{Use expansion with } n = -1 \text{ and } x \text{ replaced with } 2x^2$$

$$= 1 + \binom{-1}{1} (2x^2) + \frac{(-1)(-2)(2x^2)^2}{2!} + \dots$$

$$= 1 - 2x^2 + \dots$$

Expansion is infinite. Valid when $|2x^2| < 1 \Rightarrow |x| < \frac{1}{\sqrt{2}}$.

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The binomial expansion
Exercise A, Question 2

Question:

By first writing $\frac{(1+x)}{(1-2x)}$ as $(1+x)(1-2x)^{-1}$ show that the cubic approximation to $\frac{(1+x)}{(1-2x)}$ is $1 + 3x + 6x^2 + 12x^3$. State the range of values of x for which this expansion is valid.

Solution:

$\frac{1+x}{1-2x} = (1+x)(1-2x)^{-1}$ Expand $(1-2x)^{-1}$ using binomial expansion

$$= \left(1+x \right) \left[1 + \left(-1 \right) \left(-2x \right) + \frac{(-1)(-2)(-2x)^2}{2!} + \frac{(-1)(-2)(-3)(-2x)^3}{3!} + \dots \right]$$

$$= (1+x)(1+2x+4x^2+8x^3+\dots) \quad \text{Multiply out}$$

$$= 1+2x+4x^2+8x^3+\dots + x+2x^2+4x^3+8x^4+\dots \quad \text{Add like terms}$$

$$= 1+3x+6x^2+12x^3+\dots$$

$$(1-2x)^{-1} \text{ is only valid when } |-2x| < 1 \Rightarrow |x| < \frac{1}{2}$$

So expansion of $\frac{1+x}{1-2x}$ is only valid when $|x| < \frac{1}{2}$.

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Edexcel AS and A Level Modular Mathematics

The binomial expansion

Exercise A, Question 3

Question:

Find the binomial expansion of $\sqrt{1 + 3x}$ in ascending powers of x up to and including the term in x^3 . By substituting $x = 0.01$ in the expansion, find an approximation to $\sqrt{103}$. By comparing it with the exact value, comment on the accuracy of your approximation.

Solution:

$$\begin{aligned}\sqrt{1 + 3x} &= (1 + 3x)^{\frac{1}{2}} \\ &= 1 + \binom{\frac{1}{2}}{1} (3x) + \frac{\binom{\frac{1}{2}}{2} (-\frac{1}{2}) (3x)^2}{2!} + \\ &\quad \frac{\binom{\frac{1}{2}}{3} (-\frac{1}{2}) (-\frac{3}{2}) (3x)^3}{3!} + \dots \\ &= 1 + \frac{3}{2}x - \frac{9}{8}x^2 + \frac{27}{16}x^3 + \dots\end{aligned}$$

This expansion is valid if $|3x| < 1 \Rightarrow |x| < \frac{1}{3}$

Substitute $x = 0.01$ (OK, as $|x| < \frac{1}{3}$) into both sides to give

$$\sqrt{1 + 3 \times 0.01} \approx 1 + \frac{3}{2} \times 0.01 - \frac{9}{8} \times 0.01^2 + \frac{27}{16} \times 0.01^3$$

$$\begin{aligned}\sqrt{1.03} &\approx 1 + 0.015 - 0.0001125 + 0.0000016875 \\ \sqrt{\frac{103}{100}} &\approx 1.014889188 \quad \left(\sqrt{\frac{103}{100}} = \frac{\sqrt{103}}{\sqrt{100}} = \frac{\sqrt{103}}{10} \right)\end{aligned}$$

$$\frac{\sqrt{103}}{10} \approx 1.014889188 \quad \left(\times 10 \right)$$

$$\sqrt{103} \approx 10.14889188$$

Using a calculator

$$\sqrt{103} = 10.14889157$$

Hence approximation correct to 6 d.p.

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The binomial expansion
Exercise A, Question 5

Question:

Show that if x is small, the expression $\sqrt{\left(\frac{1+x}{1-x}\right)}$ is approximated by $1 + x + \frac{1}{2}x^2$.

Solution:

$$\begin{aligned} \sqrt{\frac{1+x}{1-x}} &= \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} \\ &= (1+x)^{\frac{1}{2}} (1-x)^{-\frac{1}{2}} \quad \text{Expand using the binomial expansion} \\ &= \left[1 + \left(\frac{1}{2}\right)(x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(x)^2}{2!} + \dots\right] \left[1 + \left(-\frac{1}{2}\right)(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-x)^2}{2!} + \dots\right] \\ &= \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \dots\right) \left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \dots\right) \\ &= 1 \left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \dots\right) + \frac{1}{2}x \left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \dots\right) - \frac{1}{8}x^2 \left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \dots\right) \\ &= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{8}x^2 + \dots \quad \text{Add like terms} \\ &= 1 + x + \frac{1}{2}x^2 + \dots \end{aligned}$$

Hence $\sqrt{\frac{1+x}{1-x}} \simeq 1 + x + \frac{1}{2}x^2$

If terms larger than or equal to x^3 are ignored.

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Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise A, Question 6

Question:

Find the first four terms in the expansion of $(1 - 3x)^{\frac{3}{2}}$. By substituting in a suitable value of x , find an approximation to $97^{\frac{3}{2}}$.

Solution:

$$\begin{aligned} (1 - 3x)^{\frac{3}{2}} &= 1 + \binom{\frac{3}{2}}{1} (-3x) + \frac{\binom{\frac{3}{2}}{2} \left(\frac{1}{2}\right) (-3x)^2}{2!} + \frac{\binom{\frac{3}{2}}{3} \left(\frac{1}{2}\right) \left(-\frac{1}{2}\right) (-3x)^3}{3!} + \dots \\ &= 1 - \frac{9x}{2} + \frac{27x^2}{8} + \frac{27x^3}{16} + \dots \end{aligned}$$

Expansion is valid if $|-3x| < 1 \Rightarrow |x| < \frac{1}{3}$.

Substitute $x = 0.01$ into both sides of expansion to give

$$(1 - 3 \times 0.01)^{\frac{3}{2}} = 1 - \frac{9 \times 0.01}{2} + \frac{27 \times (0.01)^2}{8} + \frac{27 \times (0.01)^3}{16} + \dots$$

$$(0.97)^{\frac{3}{2}} \simeq 1 - 0.045 + 0.0003375 + 0.000001687$$

$$(0.97)^{\frac{3}{2}} \simeq 0.955339187$$

$$\left(\frac{97}{100} \right)^{\frac{3}{2}} \simeq 0.955339187, \quad \left[\left(\frac{97}{100} \right)^{\frac{3}{2}} = \frac{97^{\frac{3}{2}}}{100^{\frac{3}{2}}} = \frac{97^{\frac{3}{2}}}{(\sqrt{100})^3} = \frac{97^{\frac{3}{2}}}{1000} \right]$$

$$\frac{97^{\frac{3}{2}}}{1000} \simeq 0.955339187 \quad \left(\times 1000 \right)$$

$$97^{\frac{3}{2}} \simeq 955.339187$$

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Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise B, Question 1

Question:

Find the binomial expansions of the following in ascending powers of x as far as the term in x^3 . State the range of values of x for which the expansions are valid.

(a) $\sqrt{(4 + 2x)}$

(b) $\frac{1}{2 + x}$

(c) $\frac{1}{(4 - x)^2}$

(d) $\sqrt{(9 + x)}$

(e) $\frac{1}{\sqrt{(2 + x)}}$

(f) $\frac{5}{3 + 2x}$

(g) $\frac{1 + x}{2 + x}$

(h) $\sqrt{\left(\frac{2 + x}{1 - x}\right)}$

Solution:

(a) $\sqrt{(4 + 2x)}$ Write in index form.

$$= (4 + 2x)^{\frac{1}{2}} \quad \text{Take out a factor of 4}$$

$$= \left[4 \left(1 + \frac{2x}{4} \right) \right]^{\frac{1}{2}} \quad \text{Remember to put the 4 to the power } \frac{1}{2}$$

$$= 4^{\frac{1}{2}} \left(1 + \frac{x}{2} \right)^{\frac{1}{2}} \quad 4^{\frac{1}{2}} = 2$$

$$= 2 \left(1 + \frac{x}{2} \right)^{\frac{1}{2}} \quad \text{Use the binomial expansion with } n = \frac{1}{2} \text{ and } x = \frac{x}{2}$$

$$\begin{aligned}
&= 2 \left[1 + \binom{\frac{1}{2}}{\frac{x}{2}} + \frac{\left(\frac{1}{2}\right) \left(-\frac{1}{2}\right) \left(\frac{x}{2}\right)^2}{2!} + \right. \\
&\quad \left. \frac{\left(\frac{1}{2}\right) \left(-\frac{1}{2}\right) \left(-\frac{3}{2}\right) \left(\frac{x}{2}\right)^3}{3!} + \dots \right] \\
&= 2 \left(1 + \frac{x}{4} - \frac{x^2}{32} + \frac{x^3}{128} + \dots \right) \quad \text{Multiply by the 2} \\
&= 2 + \frac{x}{2} - \frac{x^2}{16} + \frac{x^3}{64}
\end{aligned}$$

Valid if $\left| \frac{x}{2} \right| < 1 \Rightarrow |x| < 2$

(b) $\frac{1}{2+x}$ Write in index form

$$= (2+x)^{-1} \quad \text{Take out a factor of 2}$$

$$= \left[2 \left(1 + \frac{x}{2} \right) \right]^{-1} \quad \text{Remember to put 2 to the power } -1$$

$$= 2^{-1} \left(1 + \frac{x}{2} \right)^{-1}, \quad 2^{-1} = \frac{1}{2}. \text{ Use the binomial expansion with}$$

$$n = -1 \text{ and } x = \frac{x}{2}$$

$$\begin{aligned}
&= \frac{1}{2} \left[1 + \binom{-1}{-1} \left(\frac{x}{2}\right) + \frac{(-1)(-2)}{2!} \left(\frac{x}{2}\right)^2 + \right. \\
&\quad \left. \frac{(-1)(-2)(-3)}{3!} \left(\frac{x}{2}\right)^3 + \dots \right]
\end{aligned}$$

$$= \frac{1}{2} \left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \dots \right) \quad \text{Multiply by the } \frac{1}{2}$$

$$= \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16}$$

$$\text{Valid if } \left| \frac{x}{2} \right| < 1 \Rightarrow |x| < 2$$

(c) $\frac{1}{(4-x)^2}$ Write in index form

$$= (4-x)^{-2} \quad \text{Take 4 out as a factor}$$

$$= \left[4 \left(1 - \frac{x}{4} \right) \right]^{-2}$$

$$= 4^{-2} \left(1 - \frac{x}{4} \right)^{-2}, \quad 4^{-2} = \frac{1}{16}. \text{ Use the binomial expansion with}$$

$$n = -2 \text{ and } x = \frac{x}{4}$$

$$= \frac{1}{16} \left[1 + \binom{-2}{1} \left(-\frac{x}{4} \right) + \frac{(-2)(-3)}{2!} \left(-\frac{x}{4} \right)^2 + \right.$$

$$\left. \frac{(-2)(-3)(-4)}{3!} \left(-\frac{x}{4} \right)^3 + \dots \right]$$

$$= \frac{1}{16} \left(1 + \frac{x}{2} + \frac{3x^2}{16} + \frac{x^3}{16} + \dots \right) \quad \text{Multiply by } \frac{1}{16}$$

$$= \frac{1}{16} + \frac{x}{32} + \frac{3x^2}{256} + \frac{x^3}{256}$$

$$\text{Valid for } \left| \frac{x}{4} \right| < 1 \Rightarrow |x| < 4$$

(d) $\sqrt{9+x}$ Write in index form

$$= (9+x)^{\frac{1}{2}} \quad \text{Take 9 out as a factor}$$

$$= \left[9 \left(1 + \frac{x}{9} \right) \right]^{\frac{1}{2}}$$

$$= 9^{\frac{1}{2}} \left(1 + \frac{x}{9} \right)^{\frac{1}{2}}, \quad 9^{\frac{1}{2}} = 3. \text{ Use binomial expansion with } n = \frac{1}{2} \text{ and}$$

$$x = \frac{x}{9}$$

$$\begin{aligned}
&= 3 \left[1 + \binom{\frac{1}{2}}{\frac{x}{9}} + \frac{\binom{\frac{1}{2}}{2} \binom{-\frac{1}{2}}{2}}{2!} \left(\frac{x}{9}\right)^2 + \right. \\
&\quad \left. \frac{\binom{\frac{1}{2}}{3} \binom{-\frac{1}{2}}{3} \binom{-\frac{3}{2}}{3}}{3!} \left(\frac{x}{9}\right)^3 + \dots \right] \\
&= 3 \left(1 + \frac{x}{18} - \frac{x^2}{648} + \frac{x^3}{11664} + \dots \right) \quad \text{Multiply by 3} \\
&= 3 + \frac{x}{6} - \frac{x^2}{216} + \frac{x^3}{3888}
\end{aligned}$$

Valid for $\left| \frac{x}{9} \right| < 1 \Rightarrow |x| < 9$

(e) $\frac{1}{\sqrt{2+x}}$ Write in index form

$$= (2+x)^{-\frac{1}{2}} \quad \text{Take out a factor of 2}$$

$$= \left[2 \left(1 + \frac{x}{2} \right) \right]^{-\frac{1}{2}}$$

$$= 2^{-\frac{1}{2}} \left(1 + \frac{x}{2} \right)^{-\frac{1}{2}}, \quad 2^{-\frac{1}{2}} = \frac{1}{2^{\frac{1}{2}}} = \frac{1}{\sqrt{2}}. \quad \text{Use binomial}$$

expansion with $n = -\frac{1}{2}$ and $x = \frac{x}{2}$

$$= \frac{1}{\sqrt{2}} \left[1 + \binom{-\frac{1}{2}}{\frac{x}{2}} + \frac{\binom{-\frac{1}{2}}{2} \binom{-\frac{3}{2}}{2}}{2!} \left(\frac{x}{2}\right)^2 + \dots \right]$$

$$\begin{aligned}
& \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!} \left(\frac{x}{2}\right)^3 + \dots \\
& = \frac{1}{\sqrt{2}} \left(1 - \frac{x}{4} + \frac{3x^2}{32} - \frac{5x^3}{128} + \dots\right) \quad \text{Multiply by } \frac{1}{\sqrt{2}} \\
& = \frac{1}{\sqrt{2}} - \frac{x}{4\sqrt{2}} + \frac{3x^2}{32\sqrt{2}} - \frac{5x^3}{128\sqrt{2}} + \dots \quad \text{Rationalise surds} \\
& = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}x}{8} + \frac{3\sqrt{2}x^2}{64} - \frac{52x^3}{256} \\
\text{Valid if } & \left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2
\end{aligned}$$

(f) $\frac{5}{3+2x}$ Write in index form

$$= 5(3+2x)^{-1} \quad \text{Take out a factor of 3}$$

$$= 5 \left[3 \left(1 + \frac{2x}{3} \right) \right]^{-1}$$

$$= 5 \times 3^{-1} \left(1 + \frac{2x}{3} \right)^{-1}, \quad 3^{-1} = \frac{1}{3}. \text{ Use binomial expansion with}$$

$$n = -1 \text{ and } x = \frac{2x}{3}$$

$$= \frac{5}{3} \left[1 + \binom{-1}{1} \left(\frac{2x}{3}\right) + \frac{(-1)(-2)}{2!} \left(\frac{2x}{3}\right)^2 + \dots \right]$$

$$\frac{(-1)(-2)(-3)}{3!} \left(\frac{2x}{3}\right)^3 + \dots$$

$$= \frac{5}{3} \left(1 - \frac{2x}{3} + \frac{4x^2}{9} - \frac{8x^3}{27} + \dots \right) \quad \text{Multiply by } \frac{5}{3}$$

$$= \frac{5}{3} - \frac{10x}{9} + \frac{20x^2}{27} - \frac{40x^3}{81}$$

$$\text{Valid if } \left|\frac{2x}{3}\right| < 1 \Rightarrow |x| < \frac{3}{2}$$

(g) $\frac{1+x}{2+x}$ Write $\frac{1}{2+x}$ in index form

$$= (1+x)(2+x)^{-1} \quad \text{Take out a factor of 2}$$

$$= (1+x) \left[2 \left(1 + \frac{x}{2} \right) \right]^{-1}$$

$$= (1+x) 2^{-1} \left(1 + \frac{x}{2} \right)^{-1} \quad \text{Expand } \left(1 + \frac{x}{2} \right)^{-1} \text{ using the}$$

binomial expansion

$$= (1+x) \frac{1}{2} \left[1 + \binom{-1}{1} \left(\frac{x}{2} \right) + \frac{(-1)(-2)}{2!} \left(\frac{x}{2} \right)^2 + \right.$$

$$\left. \frac{(-1)(-2)(-3)}{3!} \left(\frac{x}{2} \right)^3 + \dots \right]$$

$$= (1+x) \frac{1}{2} \left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \dots \right) \quad \text{Multiply } \left(1 - \right.$$

$$\left. \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \dots \right) \text{ by } \frac{1}{2}$$

$$= (1+x) \left(\frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \dots \right) \quad \text{Multiply your answer}$$

by $(1+x)$

$$= 1 \left(\frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \dots \right) + x \left(\frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \dots \right)$$

)

$$= \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \dots + \frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{8} + \dots \quad \text{Collect like}$$

terms

$$= \frac{1}{2} + \frac{1}{4}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$$

$$\text{Valid if } \left| \frac{x}{2} \right| < 1 \Rightarrow |x| < 2$$

(h) $\sqrt{\frac{2+x}{1-x}}$

$$= (2+x)^{\frac{1}{2}} (1-x)^{-\frac{1}{2}} \quad \text{Put both in index form}$$

$$= 2^{\frac{1}{2}} \left(1 + \frac{x}{2} \right)^{\frac{1}{2}} (1 - x)^{-\frac{1}{2}} \quad \text{Expand both using the binomial}$$

expansion

$$= \sqrt{2} \left[1 + \binom{\frac{1}{2}}{1} \left(\frac{x}{2} \right) + \frac{\binom{\frac{1}{2}}{2} \left(-\frac{1}{2} \right)}{2!} \left(\frac{x}{2} \right)^2 + \right.$$

$$\left. \frac{\binom{\frac{1}{2}}{3} \left(-\frac{1}{2} \right) \left(-\frac{3}{2} \right)}{3!} \left(\frac{x}{2} \right)^3 + \dots \right] \left[1 + \binom{-\frac{1}{2}}{1} (-x) + \right.$$

$$\left. \frac{\binom{-\frac{1}{2}}{2} \left(-\frac{3}{2} \right) (-x)^2}{2!} + \frac{\binom{-\frac{1}{2}}{3} \left(-\frac{3}{2} \right) \left(-\frac{5}{2} \right) (-x)^3}{3!} + \dots \right]$$

$$= \sqrt{2} \left(1 + \frac{1}{4}x - \frac{1}{32}x^2 + \frac{1}{128}x^3 + \dots \right) \left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \dots \right) \quad \text{Multiply out}$$

$$= \sqrt{2} \left[1 \left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \dots \right) + \frac{1}{4}x \left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \dots \right) - \frac{1}{32}x^2 \left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \dots \right) + \frac{1}{128}x^3 \left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \dots \right) \dots \right]$$

$$= \sqrt{2} \left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{1}{4}x + \frac{1}{8}x^2 + \frac{3}{32}x^3 - \frac{1}{32}x^2 - \frac{1}{64}x^3 + \frac{1}{128}x^3 + \dots \right) \quad \text{Collect like terms}$$

$$= \sqrt{2} \left(1 + \frac{3}{4}x + \frac{15}{32}x^2 + \frac{51}{128}x^3 + \dots \right) \quad \text{Multiply by } \sqrt{2}$$

$$= \sqrt{2} + \frac{3\sqrt{2}}{4}x + \frac{15\sqrt{2}}{32}x^2 + \frac{51\sqrt{2}}{128}x^3$$

Valid if $\left| \frac{x}{2} \right| < 1$ and $|-x| < 1 \Rightarrow |x| < 1$ for both to be valid

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Edexcel AS and A Level Modular Mathematics

The binomial expansion

Exercise B, Question 2

Question:

Prove that if x is sufficiently small, $\frac{3 + 2x - x^2}{4 - x}$ may be approximated by $\frac{3}{4} +$

$\frac{11}{16}x - \frac{5}{64}x^2$. What does 'sufficiently small' mean in this question?

Solution:

$$\frac{3 + 2x - x^2}{4 - x} \equiv \left(3 + 2x - x^2 \right) (4 - x)^{-1} \quad \text{Write } \frac{1}{4 - x} \text{ as } (4 - x)^{-1}$$

$$= \left(3 + 2x - x^2 \right) \left[4 \left(1 - \frac{x}{4} \right) \right]^{-1} \quad \text{Take out a factor of 4}$$

$$= \left(3 + 2x - x^2 \right) \frac{1}{4} \left(1 - \frac{x}{4} \right)^{-1} \quad \text{Expand } \left(1 - \frac{x}{4} \right)^{-1} \text{ using the}$$

binomial expansion

$$= \left(3 + 2x - x^2 \right) \frac{1}{4} \left[1 + \left(-1 \right) \left(-\frac{x}{4} \right) + \frac{(-1)(-2)}{2!} \left(-\frac{x}{4} \right)^2 + \dots \right]$$

$$\left. \right] \quad \text{Ignore terms higher than } x^2$$

$$= \left(3 + 2x - x^2 \right) \frac{1}{4} \left(1 + \frac{x}{4} + \frac{x^2}{16} + \dots \right) \quad \text{Multiply expansion by}$$

$$\frac{1}{4}$$

$$= \left(3 + 2x - x^2 \right) \left(\frac{1}{4} + \frac{x}{16} + \frac{x^2}{64} + \dots \right) \quad \text{Multiply result by}$$

$$(3 + 2x - x^2)$$

$$= 3 \left(\frac{1}{4} + \frac{x}{16} + \frac{x^2}{64} + \dots \right) + 2x \left(\frac{1}{4} + \frac{x}{16} + \frac{x^2}{64} + \dots \right) - x^2 \left(\frac{1}{4} + \frac{x}{16} + \frac{x^2}{64} + \dots \right)$$

$$\left. \right)$$

$$= \frac{3}{4} + \frac{3}{16}x + \frac{3}{64}x^2 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{4}x^2 + \dots \quad \text{Ignore any terms}$$

bigger than x^2

$$= \frac{3}{4} + \frac{11}{16}x - \frac{5}{64}x^2$$

$$\text{Expansion is valid if } \left| \frac{-x}{4} \right| < 1 \Rightarrow |x| < 4$$

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Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise B, Question 3

Question:

Find the first four terms in the expansion of $\sqrt{4-x}$. By substituting $x = \frac{1}{9}$, find a fraction that is an approximation to $\sqrt{35}$. By comparing this to the exact value, state the degree of accuracy of your approximation.

Solution:

$$\begin{aligned}\sqrt{4-x} &= (4-x)^{\frac{1}{2}} \\ &= \left[4 \left(1 - \frac{x}{4} \right) \right]^{\frac{1}{2}} \\ &= 4^{\frac{1}{2}} \left(1 - \frac{x}{4} \right)^{\frac{1}{2}} \\ &= 2 \left[1 + \binom{\frac{1}{2}}{1} \left(-\frac{x}{4} \right) + \frac{\binom{\frac{1}{2}}{2} \left(-\frac{x}{4} \right)^2}{2!} + \frac{\binom{\frac{1}{2}}{3} \left(-\frac{x}{4} \right)^3}{3!} \right. \\ &\quad \left. + \dots \right] \\ &= 2 \left(1 - \frac{x}{8} - \frac{x^2}{128} - \frac{x^3}{1024} + \dots \right) \\ &= 2 - \frac{x}{4} - \frac{x^2}{64} - \frac{x^3}{512} + \dots\end{aligned}$$

$$\text{Valid for } \left| -\frac{x}{4} \right| < 1 \Rightarrow |x| < 4$$

Substitute $x = \frac{1}{9}$ into both sides of the expansion:

$$\sqrt{\left(4 - \frac{1}{9} \right)} \approx 2 - \frac{\frac{1}{9}}{4} - \frac{\left(\frac{1}{9} \right)^2}{64} - \frac{\left(\frac{1}{9} \right)^3}{512}$$

$$\sqrt{\frac{35}{9}} \approx 2 - \frac{1}{36} - \frac{1}{5184} - \frac{1}{373248}$$

$$\frac{\sqrt{35}}{3} \approx \frac{736055}{373248}$$

$$\sqrt{35} \approx 3 \times \frac{736055}{373248} = \frac{736055}{124416} = 5.916079 \quad \left| \quad 925 \right.$$

By calculator $\sqrt{35} = 5.916079 \dots$
Fraction accurate to 6 decimal places

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The binomial expansion
Exercise B, Question 4

Question:

The expansion of $(a + bx)^{-2}$ may be approximated by $\frac{1}{4} + \frac{1}{4}x + cx^2$. Find the values of the constants a , b and c .

Solution:

$$\begin{aligned}(a + bx)^{-2} &= \left[a \left(1 + \frac{bx}{a} \right) \right]^{-2} \quad \text{Take out a factor of } a \\ &= a^{-2} \left(1 + \frac{bx}{a} \right)^{-2} \\ &= \frac{1}{a^2} \left(1 + \frac{bx}{a} \right)^{-2} \\ &= \frac{1}{a^2} \left[1 + (-2) \left(\frac{bx}{a} \right) + \frac{(-2)(-3)}{2!} \left(\frac{bx}{a} \right)^2 + \dots \right] \\ &= \frac{1}{a^2} - \frac{2bx}{a^3} + \frac{3b^2x^2}{a^4} + \dots\end{aligned}$$

Compare this to $\frac{1}{4} + \frac{1}{4}x + cx^2$

Comparing constant terms: $\frac{1}{a^2} = \frac{1}{4}$

$$\Rightarrow a^2 = 4 \quad (\sqrt{\quad})$$

$$\Rightarrow a = \pm 2$$

Comparing terms in x : $\frac{-2b}{a^3} = \frac{1}{4}$

$$\Rightarrow b = \frac{a^3}{-8} \quad \text{Substitute } a = \pm 2$$

$$\Rightarrow b = \frac{(\pm 2)^3}{-8}$$

$$\Rightarrow b = \pm 1$$

Compare terms in x^2 : $c = \frac{3b^2}{a^4}$ Substitute $a^4 = 16$, $b^2 = 1$

$$\Rightarrow c = \frac{3 \times 1}{16}$$

$$\Rightarrow c = \frac{3}{16}$$

$$\text{Hence } a = \pm 2, b \pm 1, c = \frac{3}{16}$$

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The binomial expansion
Exercise C, Question 1

Question:

(a) Express $\frac{8x+4}{(1-x)(2+x)}$ as partial fractions.

(b) Hence or otherwise expand $\frac{8x+4}{(1-x)(2+x)}$ in ascending powers of x as far as the term in x^2 .

(c) State the set of values of x for which the expansion is valid.

Solution:

$$(a) \text{ Let } \frac{8x+4}{(1-x)(2+x)} \equiv \frac{A}{(1-x)} + \frac{B}{(2+x)} \equiv \frac{A(2+x) + B(1-x)}{(1-x)(2+x)}$$

$$\text{Set the numerators equal: } 8x+4 \equiv A(2+x) + B(1-x)$$

$$\text{Substitute } x=1: 8 \times 1 + 4 = A \times 3 + B \times 0$$

$$\Rightarrow 12 = 3A$$

$$\Rightarrow A = 4$$

$$\text{Substitute } x=-2: 8 \times (-2) + 4 = A \times 0 + B \times 3$$

$$\Rightarrow -12 = 3B$$

$$\Rightarrow B = -4$$

$$\text{Hence } \frac{8x+4}{(1-x)(2+x)} \equiv \frac{4}{(1-x)} - \frac{4}{(2+x)}$$

$$(b) \frac{4}{(1-x)} = 4(1-x)^{-1}$$

$$= 4 \left[1 + \binom{-1}{1} (-x) + \frac{(-1)(-2)(-x)^2}{2!} + \dots \right]$$

$$= 4(1+x+x^2+\dots)$$

$$= 4 + 4x + 4x^2 + \dots$$

$$\frac{4}{(2+x)} = 4(2+x)^{-1}$$

$$\begin{aligned}
&= 4 \left[2 \left(1 + \frac{x}{2} \right) \right]^{-1} \\
&= 4 \times 2^{-1} \left(1 + \frac{x}{2} \right)^{-1} \\
&= 4 \times \frac{1}{2} \times \left[1 + \binom{-1}{1} \left(\frac{x}{2} \right) + \frac{(-1)(-2)}{2!} \left(\frac{x}{2} \right)^2 + \dots \right]^{-1} \\
&= 2 \left(1 - \frac{x}{2} + \frac{x^2}{4} + \dots \right) \\
&= 2 - x + \frac{1}{2}x^2 + \dots
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{8x+4}{(1-x)(2+x)} &\equiv \frac{4}{(1-x)} - \frac{4}{(2+x)} \\
&= \left(4 + 4x + 4x^2 + \dots \right) - \left(2 - x + \frac{1}{2}x^2 + \dots \right) \\
&= 2 + 5x + \frac{7x^2}{2}
\end{aligned}$$

(c) $\frac{4}{(1-x)}$ is valid for $|x| < 1$

$\frac{4}{(2+x)}$ is valid for $|x| < 2$

Both are valid when $|x| < 1$.

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Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise C, Question 2

Question:

(a) Express $\frac{-2x}{(2+x)^2}$ as a partial fraction.

(b) Hence prove that $\frac{-2x}{(2+x)^2}$ can be expressed in the form $0 - \frac{1}{2}x + Bx^2 + Cx^3$ where constants B and C are to be determined.

(c) State the set of values of x for which the expansion is valid.

Solution:

$$(a) \text{ Let } \frac{-2x}{(2+x)^2} \equiv \frac{A}{(2+x)} + \frac{B}{(2+x)^2} \equiv \frac{A(2+x) + B}{(2+x)^2}$$

$$\text{Set the numerators equal: } -2x \equiv A(2+x) + B$$

$$\text{Substitute } x = -2: \quad 4 = A \times 0 + B \quad \Rightarrow \quad B = 4$$

$$\text{Equate terms in } x: \quad -2 = A \quad \Rightarrow \quad A = -2$$

$$\text{Hence } \frac{-2x}{(2+x)^2} \equiv \frac{-2}{(2+x)} + \frac{4}{(2+x)^2}$$

$$(b) \frac{-2}{2+x} = -2(2+x)^{-1}$$

$$= -2 \left[2 \left(1 + \frac{x}{2} \right) \right]^{-1}$$

$$= -2 \times 2^{-1} \times \left(1 + \frac{x}{2} \right)^{-1}$$

$$= -1 \times \left[1 + \left(-1 \right) \left(\frac{x}{2} \right) + \frac{(-1)(-2)}{2!} \left(\frac{x}{2} \right)^2 + \right.$$

$$\left. \frac{(-1)(-2)(-3)}{3!} \left(\frac{x}{2} \right)^3 + \dots \right]$$

$$= -1 \times \left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \dots \right)$$

$$= -1 + \frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{8} + \dots$$

$$\begin{aligned} \frac{4}{(2+x)^2} &= 4(2+x)^{-2} \\ &= 4 \left[2 \left(1 + \frac{x}{2} \right) \right]^{-2} \\ &= 4 \times 2^{-2} \times \left(1 + \frac{x}{2} \right)^{-2} \\ &= 1 \times \left[1 + \binom{-2}{-2} \left(\frac{x}{2} \right) + \frac{(-2)(-3)}{2!} \left(\frac{x}{2} \right)^2 + \right. \\ &\quad \left. \frac{(-2)(-3)(-4)}{3!} \left(\frac{x}{2} \right)^3 + \dots \right] \\ &= 1 \times \left(1 - x + \frac{3x^2}{4} - \frac{x^3}{2} + \dots \right) \\ &= 1 - x + \frac{3x^2}{4} - \frac{x^3}{2} + \dots \end{aligned}$$

Hence

$$\begin{aligned} \frac{-2x}{(2+x)^2} &\equiv \frac{-2}{(2+x)} + \frac{4}{(2+x)^2} \\ &= -1 + \frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{8} + 1 - x + \frac{3x^2}{4} - \frac{x^3}{2} + \dots \\ &= 0 - \frac{1}{2}x + \frac{1}{2}x^2 - \frac{3}{8}x^3 \end{aligned}$$

Hence $B = \frac{1}{2}$, (coefficient of x^2) and $C = -\frac{3}{8}$, (coefficients of x^3)

(c) $\frac{-2}{(2+x)}$ is valid for $|x| < 2$

$\frac{4}{(2+x)^2}$ is valid for $|x| < 2$

Hence whole expression is valid $|x| < 2$.

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Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise C, Question 3

Question:

(a) Express $\frac{6 + 7x + 5x^2}{(1+x)(1-x)(2+x)}$ as a partial fraction.

(b) Hence or otherwise expand $\frac{6 + 7x + 5x^2}{(1+x)(1-x)(2+x)}$ in ascending powers of x as far as the term in x^3 .

(c) State the set of values of x for which the expansion is valid.

Solution:

$$\begin{aligned} \text{(a) Let } \frac{6 + 7x + 5x^2}{(1+x)(1-x)(2+x)} &\equiv \frac{A}{(1+x)} + \frac{B}{(1-x)} + \frac{C}{(2+x)} \\ &\equiv \\ &\frac{A(1-x)(2+x) + B(1+x)(2+x) + C(1+x)(1-x)}{(1+x)(1-x)(2+x)} \end{aligned}$$

Set the numerators equal:

$$\begin{aligned} 6 + 7x + 5x^2 &\equiv A \begin{pmatrix} 1-x \\ 1+x \end{pmatrix} \begin{pmatrix} 2+x \\ 1-x \end{pmatrix} + B \begin{pmatrix} 1+x \\ 1-x \end{pmatrix} \begin{pmatrix} 2+x \\ 1-x \end{pmatrix} + C \begin{pmatrix} 1+x \\ 1-x \end{pmatrix} \begin{pmatrix} 1-x \\ 1-x \end{pmatrix} \end{aligned}$$

$$\text{Substitute } x = 1: \quad 6 + 7 + 5 = A \times 0 + B \times 2 \times 3 + C \times 0$$

$$\Rightarrow 18 = 6B$$

$$\Rightarrow B = 3$$

$$\text{Substitute } x = -1: \quad 6 - 7 + 5 = A \times 2 \times 1 + B \times 0 + C \times 0$$

$$\Rightarrow 4 = 2A$$

$$\Rightarrow A = 2$$

$$\text{Substitute } x = -2: \quad 6 - 14 + 20 = A \times 0 + B \times 0 + C \times (-1) \times 3$$

$$\Rightarrow 12 = -3C$$

$$\Rightarrow C = -4$$

$$\text{Hence } \frac{6+7x+5x^2}{(1+x)(1-x)(2+x)} \equiv \frac{2}{(1+x)} + \frac{3}{(1-x)} - \frac{4}{(2+x)}$$

$$(b) \frac{2}{1+x} = 2(1+x)^{-1}$$

$$= 2 \left[1 + \binom{-1}{1} \binom{x}{1} + \frac{(-1)(-2)(x)^2}{2!} + \frac{(-1)(-2)(-3)(x)^3}{3!} + \dots \right]$$

$$= 2(1 - x + x^2 - x^3 + \dots)$$

$$\approx 2 - 2x + 2x^2 - 2x^3 \quad \text{Valid for } |x| < 1$$

$$\frac{3}{1-x} = 3(1-x)^{-1}$$

$$= 3 \left[1 + \binom{-1}{1} \binom{-x}{1} + \frac{(-1)(-2)(-x)^2}{2!} + \frac{(-1)(-2)(-3)(-x)^3}{3!} + \dots \right]$$

$$= 3(1 + x + x^2 + x^3 + \dots)$$

$$\approx 3 + 3x + 3x^2 + 3x^3 \quad \text{Valid for } |x| < 1$$

$$\frac{4}{2+x} = 4(2+x)^{-1}$$

$$= 4 \left[2 \left(1 + \frac{x}{2} \right) \right]^{-1}$$

$$= 4 \times 2^{-1} \times \left(1 + \frac{x}{2} \right)^{-1}$$

$$= 2 \left[1 + \binom{-1}{1} \binom{\frac{x}{2}}{1} + \frac{(-1)(-2)}{2!} \left(\frac{x}{2} \right)^2 + \frac{(-1)(-2)(-3)}{3!} \left(\frac{x}{2} \right)^3 + \dots \right]$$

$$= 2 \left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \dots \right)$$

$$\approx 2 - x + \frac{x^2}{2} - \frac{x^3}{4} \quad \text{Valid for } |x| < 2$$

Hence

$$\begin{aligned}
 \frac{6 + 7x + 5x^2}{(1+x)(1-x)(2+x)} &\equiv \frac{2}{(1+x)} + \frac{3}{(1-x)} - \frac{4}{(2+x)} \\
 &= \left(2 - 2x + 2x^2 - 2x^3 \right) + \left(3 + 3x + 3x^2 + 3x^3 \right) \\
 - \left(2 - x + \frac{x^2}{2} - \frac{x^3}{4} \right) \\
 &= 2 + 3 - 2 - 2x + 3x + x + 2x^2 + 3x^2 - \\
 \frac{x^2}{2} - 2x^3 + 3x^3 + \frac{x^3}{4} \\
 &= 3 + 2x + \frac{9}{2}x^2 + \frac{5}{4}x^3
 \end{aligned}$$

(c) All expansions are valid when $|x| < 1$.

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Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise D, Question 1

Question:

Find binomial expansions of the following in ascending powers of x as far as the term in x^3 . State the set of values of x for which the expansion is valid.

(a) $(1 - 4x)^3$

(b) $\sqrt{(16 + x)}$

(c) $\frac{1}{(1 - 2x)}$

(d) $\frac{4}{2 + 3x}$

(e) $\frac{4}{\sqrt{(4 - x)}}$

(f) $\frac{1 + x}{1 + 3x}$

(g) $\left(\frac{1 + x}{1 - x}\right)^2$

(h) $\frac{x - 3}{(1 - x)(1 - 2x)}$

Solution:

(a) $(1 - 4x)^3$ Use binomial expansion with $n = 3$ and $x = -4x$

$$= 1 + \binom{3}{1} \binom{-4x}{1} + \frac{\binom{3}{2} \binom{-4x}{2}}{2!} +$$

$$\frac{\binom{3}{3} \binom{-4x}{3}}{3!} \quad \text{As } n = 3 \text{ expansion is finite}$$

and exact

$$= 1 - 12x + 48x^2 - 64x^3 \quad \text{Valid for all } x$$

(b) $\sqrt{16 + x}$ Write in index form

$$= (16 + x)^{\frac{1}{2}} \quad \text{Take out a factor of 16}$$

$$\begin{aligned}
&= \left[16 \left(1 + \frac{x}{16} \right) \right]^{\frac{1}{2}} \\
&= 16^{\frac{1}{2}} \left(1 + \frac{x}{16} \right)^{\frac{1}{2}} \quad \text{Use binomial expansion with } n = \frac{1}{2} \text{ and } x = \frac{x}{16} \\
&= 4 \left[1 + \frac{1}{2} \left(\frac{x}{16} \right) + \frac{\left(\frac{1}{2} \right) \left(-\frac{1}{2} \right)}{2!} \left(\frac{x}{16} \right)^2 + \frac{\left(\frac{1}{2} \right) \left(-\frac{1}{2} \right) \left(-\frac{3}{2} \right)}{3!} \left(\frac{x}{16} \right)^3 + \dots \right] \\
&= 4 \left(1 + \frac{x}{32} - \frac{x^2}{2048} + \frac{x^3}{65536} + \dots \right) \quad \text{Multiply by 4} \\
&= 4 + \frac{x}{8} - \frac{x^2}{512} + \frac{x^3}{16384} + \dots \\
\text{Valid for } &\left| \frac{x}{16} \right| < 1 \Rightarrow |x| < 16
\end{aligned}$$

(c) $\frac{1}{1-2x}$ Write in index form

$$\begin{aligned}
&= (1-2x)^{-1} \quad \text{Use binomial expansion with } n = -1 \text{ and } x = -2x \\
&= 1 + \left(\begin{matrix} -1 \\ -1 \end{matrix} \right) \left(-2x \right) + \frac{(-1)(-2)(-2x)^2}{2!} + \\
&\quad \frac{(-1)(-2)(-3)(-2x)^3}{3!} + \dots \\
&= 1 + 2x + 4x^2 + 8x^3 + \dots \\
\text{Valid for } &|2x| < 1 \Rightarrow |x| < \frac{1}{2}
\end{aligned}$$

(d) $\frac{4}{2+3x}$ Write in index form

$$\begin{aligned}
&= 4(2+3x)^{-1} \quad \text{Take out a factor of 2} \\
&= 4 \left[2 \left(1 + \frac{3x}{2} \right) \right]^{-1} \\
&= 4 \times 2^{-1} \times \left(1 + \frac{3x}{2} \right)^{-1} \quad \text{Use binomial expansion with } n = -1 \text{ and}
\end{aligned}$$

$$\begin{aligned}
 x &= \frac{3x}{2} \\
 &= 2 \left[1 + \binom{-1}{1} \left(\frac{3x}{2} \right) + \frac{(-1)(-2)}{2!} \left(\frac{3x}{2} \right)^2 + \right. \\
 &\quad \left. \frac{(-1)(-2)(-3)}{3!} \left(\frac{3x}{2} \right)^3 + \dots \right] \\
 &= 2 \left(1 - \frac{3x}{2} + \frac{9x^2}{4} - \frac{27x^3}{8} + \dots \right) \quad \text{Multiply by 2} \\
 &= 2 - 3x + \frac{9x^2}{2} - \frac{27x^3}{4} + \dots
 \end{aligned}$$

$$\text{Valid for } \left| \frac{3x}{2} \right| < 1 \Rightarrow |x| < \frac{2}{3}$$

$$(e) \frac{4}{\sqrt{4-x}} = 4 (\sqrt{4-x})^{-1} \quad \text{Write in index form}$$

$$= 4 (4-x)^{-\frac{1}{2}} \quad \text{Take out a factor of 4}$$

$$= 4 \left[4 \left(1 - \frac{x}{4} \right) \right]^{-\frac{1}{2}}$$

$$= 4 \times 4^{-\frac{1}{2}} \left(1 - \frac{x}{4} \right)^{-\frac{1}{2}} \quad \text{Use binomial expansion with } n = -\frac{1}{2} \text{ and}$$

$$x = -\frac{x}{4}$$

$$\begin{aligned}
 &= 4^{\frac{1}{2}} \left[1 + \binom{-\frac{1}{2}}{1} \left(-\frac{x}{4} \right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!} \left(-\frac{x}{4} \right)^2 + \right. \\
 &\quad \left. \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!} \left(-\frac{x}{4} \right)^3 + \dots \right] \\
 &= 2 \left(1 + \frac{x}{8} + \frac{3}{128}x^2 + \frac{5}{1024}x^3 + \dots \right) \quad \text{Multiply by 2}
 \end{aligned}$$

$$= 2 + \frac{x}{4} + \frac{3}{64}x^2 + \frac{5}{512}x^3 + \dots$$

$$\text{Valid } \left| -\frac{x}{4} \right| < 1 \Rightarrow |x| < 4$$

(f) $\frac{1+x}{1+3x} = \left(1+x\right) (1+3x)^{-1}$ Write $\frac{1}{1+3x}$ in index form then expand

$$= \left(1+x\right) \left[1 + \binom{-1}{1} (3x) + \frac{(-1)(-2)(3x)^2}{2!} + \frac{(-1)(-2)(-3)(3x)^3}{3!} + \dots \right]$$

$$= (1+x) (1 - 3x + 9x^2 - 27x^3 + \dots)$$
 Multiply out

$$= 1 - 3x + 9x^2 - 27x^3 + x - 3x^2 + 9x^3 + \dots$$
 Collect like terms

$$= 1 - 2x + 6x^2 - 18x^3 + \dots$$

$$\text{Valid for } |3x| < 1 \Rightarrow |x| < \frac{1}{3}$$

(g) $\left(\frac{1+x}{1-x}\right)^2 = \frac{(1+x)^2}{(1-x)^2}$ Write in index form

$$= (1+x)^2 (1-x)^{-2}$$
 Expand $(1-x)^{-2}$ using binomial expansion

$$= \left(1+2x+x^2\right) \left[1 + \binom{-2}{1} (-x) + \frac{(-2)(-3)(-x)^2}{2!} + \frac{(-2)(-3)(-4)(-x)^3}{3!} + \dots \right]$$

$$= (1+2x+x^2) (1+2x+3x^2+4x^3+\dots)$$
 Multiply out brackets

$$= 1+2x+3x^2+4x^3+2x+4x^2+6x^3+x^2+2x^3+\dots$$
 Collect like terms

terms

$$= 1+4x+8x^2+12x^3+\dots$$

$$\text{Valid for } |x| < 1$$

(h) Let $\frac{x-3}{(1-x)(1-2x)} \equiv \frac{A}{(1-x)} + \frac{B}{(1-2x)}$ Put in partial fraction form

$$\equiv \frac{A(1-2x) + B(1-x)}{(1-x)(1-2x)}$$
 Add fractions.

Set the numerators equal: $x-3 \equiv A(1-2x) + B(1-x)$

Substitute $x=1$: $1-3 = A \times -1 + B \times 0$

$$\Rightarrow -2 = -1A$$

$$\Rightarrow A = 2$$

$$\text{Substitute } x = \frac{1}{2}: \quad \frac{1}{2} - 3 = A \times 0 + B \times \frac{1}{2}$$

$$\Rightarrow -2 \frac{1}{2} = \frac{1}{2}B$$

$$\Rightarrow B = -5$$

$$\text{Hence } \frac{x-3}{(1-x)(1-2x)} \equiv \frac{2}{(1-x)} - \frac{5}{(1-2x)}$$

$$\begin{aligned} \frac{2}{(1-x)} &= 2(1-x)^{-1} \\ &= 2 \left[1 + (-1)(-x) + \frac{(-1)(-2)(-x)^2}{2!} + \right. \\ &\quad \left. \frac{(-1)(-2)(-3)(-x)^3}{3!} + \dots \right] \\ &= 2(1+x+x^2+x^3+\dots) \\ &\approx 2+2x+2x^2+2x^3 \end{aligned}$$

$$\begin{aligned} \frac{5}{(1-2x)} &= 5(1-2x)^{-1} \\ &= 5 \left[1 + (-1)(-2x) + \frac{(-1)(-2)(-2x)^2}{2!} + \right. \\ &\quad \left. \frac{(-1)(-2)(-3)(-2x)^3}{3!} + \dots \right] \\ &= 5(1+2x+4x^2+8x^3+\dots) \\ &\approx 5+10x+20x^2+40x^3 \end{aligned}$$

$$\begin{aligned} \text{Hence } \frac{x-3}{(1-x)(1-2x)} &\equiv \frac{2}{(1-x)} - \frac{5}{(1-2x)} \\ &\approx (2+2x+2x^2+2x^3) - (5+10x+20x^2+40x^3) \\ &\approx -3-8x-18x^2-38x^3 \end{aligned}$$

$$\frac{2}{1-x} \text{ is valid for } |x| < 1$$

$$\frac{5}{1-2x} \text{ is valid for } |2x| < 1 \Rightarrow |x| < \frac{1}{2}$$

$$\text{Both are valid when } |x| < \frac{1}{2}.$$

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The binomial expansion
Exercise D, Question 2

Question:

Find the first four terms of the expansion in ascending powers of x of:

$$\left(1 - \frac{1}{2}x\right)^{\frac{1}{2}}, \quad |x| < 2$$

and simplify each coefficient. **E** (adapted)

Solution:

$$\begin{aligned} \left(1 - \frac{1}{2}x\right)^{\frac{1}{2}} &= 1 + \binom{\frac{1}{2}}{1} \left(-\frac{1}{2}x\right) + \frac{\binom{\frac{1}{2}}{2} \left(-\frac{1}{2}\right) \left(-\frac{1}{2}x\right)^2}{2!} + \\ &\frac{\binom{\frac{1}{2}}{3} \left(-\frac{1}{2}\right) \left(-\frac{3}{2}\right) \left(-\frac{1}{2}x\right)^3}{3!} + \dots \\ &= 1 - \frac{1}{4}x + \left(-\frac{1}{8}\right) \times \left(\frac{1}{4}x^2\right) + \left(\frac{1}{16}\right) \times \left(-\frac{1}{8}x^3\right) + \dots \\ &= 1 - \frac{1}{4}x - \frac{1}{32}x^2 - \frac{1}{128}x^3 \end{aligned}$$

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The binomial expansion
Exercise D, Question 3

Question:

Show that if x is sufficiently small then $\frac{3}{\sqrt{(4+x)}}$ can be approximated by $\frac{3}{2} - \frac{3}{16}x + \frac{9}{256}x^2$.

Solution:

$$\begin{aligned}
 \frac{3}{\sqrt{4+x}} &= 3(\sqrt{4+x})^{-1} && \text{Write in index form} \\
 &= 3(4+x)^{-\frac{1}{2}} && \text{Take out a factor of 4} \\
 &= 3\left[4\left(1+\frac{x}{4}\right)\right]^{-\frac{1}{2}} \\
 &= 3 \times 4^{-\frac{1}{2}} \times \left(1+\frac{x}{4}\right)^{-\frac{1}{2}} && 4^{-\frac{1}{2}} = \frac{1}{4^{\frac{1}{2}}} = \frac{1}{2} \\
 &= \frac{3}{2} \times \left[1 + \left(-\frac{1}{2}\right)\left(\frac{x}{4}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x}{4}\right)^2}{2!} + \right. \\
 &\quad \left. \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{x}{4}\right)^3}{3!} + \dots \right] \\
 &= \frac{3}{2} \left(1 - \frac{x}{8} + \frac{3}{128}x^2 + \dots\right) && \text{Multiply by } \frac{3}{2} \\
 &= \frac{3}{2} - \frac{3}{16}x + \frac{9}{256}x^2 + \dots \\
 &= \frac{3}{2} - \frac{3}{16}x + \frac{9}{256}x^2 && \text{If terms higher than } x^2 \text{ are ignored}
 \end{aligned}$$

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The binomial expansion

Exercise D, Question 4

Question:

Given that $|x| < 4$, find, in ascending powers of x up to and including the term in x^3 , the series expansion of:

(a) $(4 - x)^{\frac{1}{2}}$

(b) $(4 - x)^{\frac{1}{2}} (1 + 2x)^E$ (adapted)

Solution:

(a) $(4 - x)^{\frac{1}{2}}$ Take out a factor of 4

$$= \left[4 \left(1 - \frac{x}{4} \right) \right]^{\frac{1}{2}}$$

$$= 4^{\frac{1}{2}} \left(1 - \frac{x}{4} \right)^{\frac{1}{2}} \quad \text{Use binomial expansion with } n = \frac{1}{2} \text{ and } x = -\frac{x}{4}$$

$$= 2 \left[1 + \binom{\frac{1}{2}}{1} \left(-\frac{x}{4} \right) + \frac{\binom{\frac{1}{2}}{2} \left(-\frac{x}{4} \right)^2}{2!} + \right.$$

$$\left. \frac{\binom{\frac{1}{2}}{3} \left(-\frac{x}{4} \right)^3}{3!} + \dots \right]$$

$$= 2 \left(1 - \frac{x}{8} - \frac{x^2}{128} - \frac{x^3}{1024} + \dots \right) \quad \text{Multiply by 2}$$

$$= 2 - \frac{x}{4} - \frac{x^2}{64} - \frac{x^3}{512} + \dots$$

(b) $(4 - x) \frac{1}{2} (1 + 2x)$ Use answer from part (a)

$$= \left(2 - \frac{x}{4} - \frac{x^2}{64} - \frac{x^3}{512} + \dots \right) (1 + 2x)$$

Multiply out

brackets

$$= 2 - \frac{x}{4} - \frac{x^2}{64} - \frac{x^3}{512} + \dots + 4x - \frac{x^2}{2} - \frac{x^3}{32} + \dots$$

Collect

like terms

$$= 2 + \frac{15}{4}x - \frac{33}{64}x^2 - \frac{17}{512}x^3 + \dots$$

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The binomial expansion

Exercise D, Question 5

Question:

(a) Find the first four terms of the expansion, in ascending powers of x , of

$$(2 + 3x)^{-1}, \quad |x| < \frac{2}{3}$$

(b) Hence or otherwise, find the first four non-zero terms of the expansion, in ascending powers of x , of:

$$\frac{1+x}{2+3x}, \quad |x| < \frac{2}{3} \text{ (E)}$$

Solution:

(a) $(2 + 3x)^{-1}$ Take out factor of 2

$$= \left[2 \left(1 + \frac{3x}{2} \right) \right]^{-1}$$

$$= 2^{-1} \left(1 + \frac{3x}{2} \right)^{-1} \quad \text{Use binomial expansion with } n = -1 \text{ and } x =$$

$$\frac{3x}{2}$$

$$= \frac{1}{2} \left[1 + \binom{-1}{1} \left(\frac{3x}{2} \right) + \frac{(-1)(-2)}{2!} \left(\frac{3x}{2} \right)^2 + \right.$$

$$\left. \frac{(-1)(-2)(-3)}{3!} \left(\frac{3x}{2} \right)^3 + \dots \right]$$

$$= \frac{1}{2} \left(1 - \frac{3}{2}x + \frac{9}{4}x^2 - \frac{27}{8}x^3 + \dots \right) \quad \text{Multiply by } \frac{1}{2}$$

$$= \frac{1}{2} - \frac{3}{4}x + \frac{9}{8}x^2 - \frac{27}{16}x^3 + \dots$$

$$\text{Valid for } \left| \frac{3x}{2} \right| < 1 \Rightarrow |x| < \frac{2}{3}$$

(b) $\frac{1+x}{2+3x}$ Put in index form

$$= (1+x)(2+3x)^{-1} \quad \text{Use expansion from part (a)}$$

$$= \left(1 + x \right) \left(\frac{1}{2} - \frac{3}{4}x + \frac{9}{8}x^2 - \frac{27}{16}x^3 + \dots \right) \quad \text{Multiply out}$$

$$= \frac{1}{2} - \frac{3}{4}x + \frac{9}{8}x^2 - \frac{27}{16}x^3 + \frac{1}{2}x - \frac{3}{4}x^2 + \frac{9}{8}x^3 + \dots \quad \text{Collect like}$$

terms

$$= \frac{1}{2} - \frac{1}{4}x + \frac{3}{8}x^2 - \frac{9}{16}x^3 + \dots$$

$$\text{Valid for } \left| \frac{3x}{2} \right| < 1 \quad \Rightarrow \quad |x| < \frac{2}{3}$$

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The binomial expansion
Exercise D, Question 6

Question:

Find, in ascending powers of x up to and including the term in x^3 , the series expansion of $(4 + x)^{-\frac{1}{2}}$, giving your coefficients in their simplest form. **E**

Solution:

$$\begin{aligned}
 (4 + x)^{-\frac{1}{2}} &= \left[4 \left(1 + \frac{x}{4} \right) \right]^{-\frac{1}{2}} \quad \text{Take out factor of 4} \\
 &= 4^{-\frac{1}{2}} \left(1 + \frac{x}{4} \right)^{-\frac{1}{2}} \quad 4^{-\frac{1}{2}} = \frac{1}{4^{\frac{1}{2}}} = \frac{1}{2} \\
 &= \frac{1}{2} \left(1 + \frac{x}{4} \right)^{-\frac{1}{2}} \quad \text{Use binomial expansion with } n = -\frac{1}{2} \text{ and } x = \frac{x}{4} \\
 &= \frac{1}{2} \left[1 + \left(-\frac{1}{2} \right) \left(\frac{x}{4} \right) + \frac{\left(-\frac{1}{2} \right) \left(-\frac{3}{2} \right) \left(\frac{x}{4} \right)^2}{2!} + \right. \\
 &\quad \left. \frac{\left(-\frac{1}{2} \right) \left(-\frac{3}{2} \right) \left(-\frac{5}{2} \right) \left(\frac{x}{4} \right)^3}{3!} + \dots \right] \\
 &= \frac{1}{2} \left(1 - \frac{1}{8}x + \frac{3}{128}x^2 - \frac{5}{1024}x^3 + \dots \right) \\
 &\approx \frac{1}{2} - \frac{1}{16}x + \frac{3}{256}x^2 - \frac{5}{2048}x^3
 \end{aligned}$$

Valid for $\left| \frac{x}{4} \right| < 1 \Rightarrow |x| < 4$

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The binomial expansion
Exercise D, Question 7

Question:

$$f(x) = (1 + 3x)^{-1}, \quad |x| < \frac{1}{3}.$$

(a) Expand $f(x)$ in ascending powers of x up to and including the term in x^3 .

(b) Hence show that, for small x :

$$\frac{1+x}{1+3x} \approx 1 - 2x + 6x^2 - 18x^3.$$

(c) Taking a suitable value for x , which should be stated, use the series expansion

in part (b) to find an approximate value for $\frac{101}{103}$, giving your answer to 5 decimal places. **E**

Solution:

(a) $(1 + 3x)^{-1}$ Use binomial expansion with $n = -1$ and $x = 3x$

$$= 1 + \binom{-1}{1} \binom{3x}{1} + \frac{(-1)(-2)(3x)^2}{2!} +$$

$$\frac{(-1)(-2)(-3)(3x)^3}{3!} + \dots$$

$$= 1 - 3x + 9x^2 - 27x^3 + \dots$$

(b) $\frac{1+x}{1+3x} = \left(1+x\right) (1+3x)^{-1}$ Use expansion from part (a)

$$= (1+x) (1 - 3x + 9x^2 - 27x^3 + \dots) \quad \text{Multiply out}$$

$$= 1 - 3x + 9x^2 - 27x^3 + x - 3x^2 + 9x^3 + \dots \quad \text{Collect like terms}$$

$$= 1 - 2x + 6x^2 - 18x^3 + \dots \quad \text{Ignore terms greater than } x^3$$

$$\text{Hence } \frac{1+x}{1+3x} \approx 1 - 2x + 6x^2 - 18x^3$$

(c) Substitute $x = 0.01$ into both sides of the above

$$\frac{1 + 0.01}{1 + 3 \times 0.01} = 1 - 2 \times 0.01 + 6 \times 0.01^2 - 18 \times 0.01^3$$

$$\frac{1.01}{1.03} \approx 1 - 0.02 + 0.0006 - 0.000018, \quad \left[\frac{1.01}{1.03} = \frac{101}{103} \right]$$

$$\frac{101}{103} \approx 0.980582 \quad \text{Round to 5 d.p.}$$

$$\frac{101}{103} \approx 0.98058 \quad (5 \text{ d.p.})$$

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The binomial expansion
Exercise D, Question 8

Question:

Obtain the first four non-zero terms in the expansion, in ascending powers of x , of the function $f(x)$ where $f(x) = \frac{1}{\sqrt{(1+3x^2)}}$, $3x^2 < 1$. **E**

Solution:

$$\begin{aligned}
 f(x) &= \frac{1}{\sqrt{1+3x^2}} = (1+3x^2)^{-1/2} \\
 &= (1+3x^2)^{-1/2} \quad \text{Use binomial expansion with } n = -\frac{1}{2} \text{ and } x = 3x^2 \\
 &= 1 + \left(-\frac{1}{2}\right)(3x^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(3x^2)^2}{2!} + \\
 &\quad \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(3x^2)^3}{3!} + \dots \\
 &\simeq 1 - \frac{3x^2}{2} + \frac{27x^4}{8} - \frac{135x^6}{16}
 \end{aligned}$$

Valid for $|3x^2| < 1$

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The binomial expansion
Exercise D, Question 9

Question:

Give the binomial expansion of $(1+x)^{\frac{1}{2}}$ up to and including the term in x^3 .
By substituting $x = \frac{1}{4}$, find the fraction that is an approximation to $\sqrt{5}$.

Solution:

Using binomial expansion

$$\begin{aligned} (1+x)^{\frac{1}{2}} &= 1 + \binom{\frac{1}{2}}{1} (x) + \frac{\binom{\frac{1}{2}}{2} \left(-\frac{1}{2}\right) (x)^2}{2!} + \\ &\frac{\binom{\frac{1}{2}}{3} \left(-\frac{1}{2}\right) \left(-\frac{3}{2}\right) (x)^3}{3!} + \dots \\ &\approx 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 \end{aligned}$$

Expansion is valid if $|x| < 1$.

Substituting $x = \frac{1}{4}$ in both sides of expansion gives

$$\left(1 + \frac{1}{4}\right)^{\frac{1}{2}} \approx 1 + \frac{1}{2} \times \frac{1}{4} - \frac{1}{8} \times \left(\frac{1}{4}\right)^2 + \frac{1}{16} \times \left(\frac{1}{4}\right)^3$$

$$\left(\frac{5}{4}\right)^{\frac{1}{2}} \approx 1 + \frac{1}{8} - \frac{1}{128} + \frac{1}{1024} \quad \left[\left(\frac{5}{4}\right)^{\frac{1}{2}} = \sqrt{\frac{5}{4}} \right]$$

$$\sqrt{\frac{5}{4}} \approx \frac{1145}{1024} \quad \left[\sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{\sqrt{4}} = \frac{\sqrt{5}}{2} \right]$$

$$\frac{\sqrt{5}}{2} \approx \frac{1145}{1024} \quad \text{Multiply both sides by 2}$$

$$\sqrt{5} \approx \frac{1145}{512}$$

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The binomial expansion
Exercise D, Question 10

Question:

When $(1 + ax)^n$ is expanded as a series in ascending powers of x , the coefficients of x and x^2 are -6 and 27 respectively.

- (a) Find the values of a and n .
- (b) Find the coefficient of x^3 .
- (c) State the values of x for which the expansion is valid. **E**

Solution:

(a) Using binomial expansion

$$(1 + ax)^n = 1 + n \binom{n}{1} ax + \frac{n(n-1)}{2!} (ax)^2 + \frac{n(n-1)(n-2)}{3!} (ax)^3 + \dots$$

If coefficient of x is -6 then $na = -6$ ①

If coefficient of x^2 is 27 then $\frac{n(n-1)a^2}{2} = 27$ ②

From ① $a = \frac{-6}{n}$. Substitute in ②:

$$\frac{n(n-1)}{2} \left(\frac{-6}{n} \right)^2 = 27$$

$$\frac{n(n-1)}{2} \times \frac{36}{n^2} = 27$$

$$\frac{(n-1)18}{n} = 27$$

$$(n-1)18 = 27n$$

$$18n - 18 = 27n$$

$$-18 = 9n$$

$$n = -2$$

Substitute $n = -2$ back in ①: $-2a = -6 \Rightarrow a = 3$

(b) Coefficient of x^3 is

$$\frac{n(n-1)(n-2)a^3}{3!} = \frac{(-2) \times (-3) \times (-4) \times 3^3}{3 \times 2 \times 1} = -108$$

(c) $(1 + 3x)^{-2}$ is valid if $|3x| < 1 \Rightarrow |x| < \frac{1}{3}$

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The binomial expansion
Exercise D, Question 11

Question:

(a) Express $\frac{9x^2 + 26x + 20}{(1+x)(2+x)^2}$ as a partial fraction.

(b) Hence or otherwise show that the expansion of $\frac{9x^2 + 26x + 20}{(1+x)(2+x)^2}$ in ascending powers of x can be approximated to $5 - \frac{7x}{2} + Bx^2 + Cx^3$ where B and C are constants to be found.

(c) State the set of values of x for which this expansion is valid.

Solution:

$$(a) \text{ Let } \frac{9x^2 + 26x + 20}{(1+x)(2+x)^2} \equiv \frac{A}{(1+x)} + \frac{B}{(2+x)} + \frac{C}{(2+x)^2}$$

$$\Rightarrow \frac{9x^2 + 26x + 20}{(1+x)(2+x)^2} \equiv \frac{A(2+x)^2 + B(1+x)(2+x) + C(1+x)}{(1+x)(2+x)^2}$$

Set the numerators equal:

$$9x^2 + 26x + 20 \equiv A(2+x)^2 + B(1+x)(2+x) + C(1+x)$$

Substitute $x = -2$: $36 - 52 + 20 = A \times 0 + B \times 0 + C \times (-1)$

$$\Rightarrow 4 = -1C$$

$$\Rightarrow C = -4$$

Substitute $x = -1$: $9 - 26 + 20 = A \times 1 + B \times 0 + C \times 0$

$$\Rightarrow 3 = 1A$$

$$\Rightarrow A = 3$$

Equate terms in x^2 : $9 = A + B$

$$\Rightarrow 9 = 3 + B$$

$$\Rightarrow B = 6$$

Hence $\frac{9x^2 + 26x + 20}{(1+x)(2+x)^2} \equiv \frac{3}{(1+x)} + \frac{6}{(2+x)} - \frac{4}{(2+x)^2}$

(b) Using binomial expansion

$$\begin{aligned}\frac{3}{(1+x)} &= 3(1+x)^{-1} \\ &= 3 \left[1 + (-1)(x) + \frac{(-1)(-2)(x)^2}{2!} + \frac{(-1)(-2)(-3)(x)^3}{3!} + \dots \right] \\ &= 3(1 - x + x^2 - x^3 + \dots) \\ &= 3 - 3x + 3x^2 - 3x^3 + \dots\end{aligned}$$

$$\begin{aligned}\frac{6}{(2+x)} &= 6(2+x)^{-1} \\ &= 6 \left[2 \left(1 + \frac{x}{2} \right) \right]^{-1} \\ &= 6 \times 2^{-1} \left(1 + \frac{x}{2} \right)^{-1} \\ &= 6 \times \frac{1}{2} \left[1 + (-1) \left(\frac{x}{2} \right) + \frac{(-1)(-2)}{2!} \left(\frac{x}{2} \right)^2 + \frac{(-1)(-2)(-3)}{3!} \left(\frac{x}{2} \right)^3 + \dots \right] \\ &= 3 \left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \dots \right) \\ &= 3 - \frac{3x}{2} + \frac{3x^2}{4} - \frac{3x^3}{8} + \dots\end{aligned}$$

$$\begin{aligned}\frac{4}{(2+x)^2} &= 4(2+x)^{-2} \\ &= 4 \left[2 \left(1 + \frac{x}{2} \right) \right]^{-2} \\ &= 4 \times 2^{-2} \times \left(1 + \frac{x}{2} \right)^{-2} \\ &= 4 \times \frac{1}{4} \times \left[1 + (-2) \left(\frac{x}{2} \right) + \frac{(-2)(-3)}{2!} \left(\frac{x}{2} \right)^2 + \frac{(-2)(-3)(-4)}{3!} \left(\frac{x}{2} \right)^3 + \dots \right] \\ &= 1 \times \left(1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3 + \dots \right) \\ &= 1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3 + \dots\end{aligned}$$

Hence

$$\begin{aligned}\frac{9x^2 + 26x + 20}{(1+x)(2+x)^2} &\equiv \frac{3}{(1+x)} + \frac{6}{(2+x)} - \frac{4}{(2+x)^2} \\ &\simeq \left(3 - 3x + 3x^2 - 3x^3 \right) + \left(3 - \frac{3x}{2} + \frac{3}{4}x^2 - \frac{3}{8}x^3 \right) - \left(1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3 \right) \\ &\simeq 3 - 3x + 3x^2 - 3x^3 + 3 - \frac{3x}{2} + \frac{3}{4}x^2 - \frac{3}{8}x^3 - 1 + x - \frac{3}{4}x^2 + \frac{1}{2}x^3\end{aligned}$$

$$- 5 - \frac{7x}{2} + 3x^2 - \frac{23}{8}x^3$$

$$\text{Hence } B = 3 \text{ and } C = \frac{-23}{8}$$

$$(c) \frac{3}{(1+x)} \text{ is valid if } |x| < 1$$

$$\frac{6}{(2+x)} \text{ is valid if } |x| < 2$$

$$\frac{4}{(2+x)^2} \text{ is valid if } |x| < 2$$

Therefore, they *all* become valid if $|x| < 1$.