June 2006
6676 Further Pure Mathematics FP3
Mark Scheme

Question Number	Scheme	Marks
1.	$\mathbf{A}^1 = \begin{pmatrix} 1 & 1 & \frac{1}{2}(1+3) \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{A}$	
	(Hence true for n =1)	
	$\mathbf{A}^{k+1} = \mathbf{A}^k.\mathbf{A} = \begin{pmatrix} 1 & k & \frac{1}{2}(k^2+3k) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	
	$= \begin{pmatrix} 1 & k+1 & 2+k+\frac{1}{2}(k^2+3k) \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix}$	
	$+k+\frac{1}{2}(k^2+3k) = \frac{1}{2}(k^2+5k+4) = \frac{1}{2}(k^2+2k+1+3k+3)$	M1 Dep
	$= \frac{1}{2}((k+1)^2+3(k+1))$	
	(Hence, if result is true for $n = k$, then it is true for $n = k + 1$).	
	By Mathematical Induction, above implies true for all positive integers. cso	**(5)**
		[5]

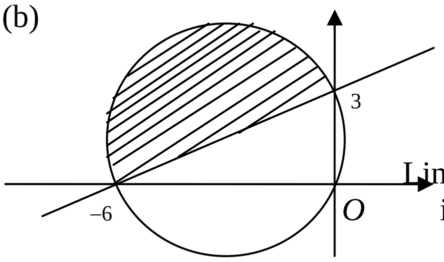
Question Number	Scheme	Marks
2.	(a) $\mathrm{f}(x)=\cos 2x$, $\mathrm{f}\left(\frac{\pi}{4}\right)=0$	
	$\mathrm{f}'(x)=-2\sin 2x$, $\mathrm{f}'\left(\frac{\pi}{4}\right)=-2$	
	$\mathrm{f}''(x)=-4\cos 2x$, $\mathrm{f}''\left(\frac{\pi}{4}\right)=0$	
	$\mathrm{f}'''(x)=8\sin 2x$, $\mathrm{f}'''\left(\frac{\pi}{4}\right)=8$	
	$\mathrm{f}^{(\text{iv})}(x)=16\cos 2x$, $\mathrm{f}^{(\text{iv})}\left(\frac{\pi}{4}\right)=0$	
	$\mathrm{f}^{(\text{v})}(x)=-32\sin 2x$, $\mathrm{f}^{(\text{v})}\left(\frac{\pi}{4}\right)=-32$	
	$\cos 2x=\mathrm{f}\left(\frac{\pi}{4}\right)+\mathrm{f}'\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)+\frac{\mathrm{f}''\left(\frac{\pi}{4}\right)}{2}\left(x-\frac{\pi}{4}\right)^2+\frac{\mathrm{f}'\left(\frac{\pi}{4}\right)}{3!}\left(x-\frac{\pi}{4}\right)^3+\dots$	
	Three terms are sufficient to establish method	
	$\cos 2x=-2\left(x-\frac{\pi}{4}\right)+\frac{4}{3}\left(x-\frac{\pi}{4}\right)^3-\frac{4}{15}\left(x-\frac{\pi}{4}\right)^5+\dots$	A1 **(5)**
	(b) Substitute $x = 1$ $\left(1-\frac{\pi}{4}\approx 0.21460\right)$	
	$\cos 2=-2\left(1-\frac{\pi}{4}\right)+\frac{4}{3}\left(1-\frac{\pi}{4}\right)^3-\frac{4}{15}\left(1-\frac{\pi}{4}\right)^5+\dots$	
	≈ -0.416147 cao	M1 A1 **(3)**
		[8]

Question Number	Scheme	Marks
3.	(a) In this solution $\cos\theta = c$ and $\sin\theta = s$	
	$\cos 5\theta + \mathrm{i}\sin 5\theta = (c + \mathrm{i}s)^5$	M1
	$\left(= c^5 + 5c^4\,\mathrm{i}s + 10c^3(\mathrm{i}s)^2 + 10c^2(\mathrm{i}s)^3 + 5c(\mathrm{i}s)^4 + (\mathrm{i}s)^5\right)$	
	$\Im$ $\sin 5\theta = 5c^4 s - 10c^2 s^3 + s^5$ equate	M1 A1
	$= 5c^4 s - 10c^2(1-c^2)s + (1-c^2)^2 s$ $s^2 = 1 - c^2$	M1
	$= s(16c^4 - 12c^2 + 1)$ *	A1 **(5)**
	(b) $\sin\theta(16\cos^4\theta - 12\cos^2\theta + 1) + 2\cos^2\theta\sin\theta = 0$	M1
	$\sin\theta = 0 \Rightarrow \theta = 0$	B1
	$16c^4 - 10c^2 + 1 = (8c^2 - 1)(2c^2 - 1) = 0$	M1
	$c = \pm\frac{1}{2\sqrt{2}}, \quad c = \pm\frac{1}{\sqrt{2}}$ any two	A1
	$\theta \approx 1.21, 1.93; \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}$ any two	A1
	all four	A1 **(6)**
	accept awrt 0.79, 1.21,1.93,2.36	**[11]**
	Ignore any solutions out of range.	

Question Number	Scheme	Marks
4.	(a) $\left(\frac{dx}{dt}\right)_0 = 0.4 \approx \frac{x_{0.1} - 0}{0.1} \quad \Rightarrow \quad x_{0.1} \approx 0.04$	B1
	$\left(\frac{d^2x}{dt^2}\right)_{0.1} = -3\sin x_{0.1} \approx \frac{x_{0.2} - 2x_{0.1} + 0}{0.01}$	M1
	Must have their $x_{0.1}$	
	$x_{0.2} \approx 0.0788$ awrt	A1
	$\left(\frac{d^2x}{dt^2}\right)_{0.2} = -3\sin x_{0.2} \approx \frac{x_{0.3} - 2x_{0.2} + x_{0.1}}{0.01}$	M1
	Must have their $x_{0.1}, x_{0.2}$	
	$x_{0.3} \approx 0.115$ awrt	A1 **(5)**
	(b) $f''(t) = -3\sin x, \quad f''(0) = 0$	
	$f'''(t) = -3\cos x \frac{dx}{dt}, \quad f'''(0) = -3 \times 0.4 = -1.2$	M1 A1
	$f(t) = f(0) + t f'(0) + \frac{t^2}{2} f''(0) + \frac{t^3}{3!} f'''(0) + \ldots$	
	$= 0.4t - 0.2t^3$	M1 A1 **(4)**
	(c) Substituting $t = 0.3$ into their answer to (b) and evaluating	M1
	$f(0.3) \approx 0.1146$ cao	A1 **(2)**
		[11]

Question Number	Scheme	Marks
5.	(a) $(4-\lambda)(1-\lambda)+2=0$	M1
	$\lambda^2-5\lambda+6=(\lambda-3)(\lambda-2)=0$	M1
	$\lambda_1=2,\ \lambda_2=3$ both	A1 **(3)**
	(b) $M^{-1}=\frac{1}{6}\begin{pmatrix}1 & 2\\ -1 & 4\end{pmatrix}$	B1 B1 **(2)**
	(c) $\begin{vmatrix}\frac{1}{6}-\frac{1}{2} & \frac{1}{3}\\ -\frac{1}{6} & \frac{2}{3}-\frac{1}{2}\end{vmatrix}=-\frac{1}{3}\times\frac{1}{6}+\frac{1}{3}\times\frac{1}{6}=0$ M1 for either value	M1 A1
	(hence $\frac{1}{2}$ is an eigenvalue of $\mathbf{M}^{-1}$)	
	$\begin{vmatrix}\frac{1}{6}-\frac{1}{3} & \frac{1}{3}\\ -\frac{1}{6} & \frac{2}{3}-\frac{1}{3}\end{vmatrix}=-\frac{1}{6}\times\frac{1}{3}+\frac{1}{3}\times\frac{1}{6}=0$	A1
	(hence $\frac{1}{3}$ is an eigenvalue of $\mathbf{M}^{-1}$)	**(3)**
	(d) Using eigenvalues	
	$\begin{pmatrix}4 & -2\\ 1 & 1\end{pmatrix}.\begin{pmatrix}x\\ y\end{pmatrix}=2\begin{pmatrix}x\\ y\end{pmatrix}$	
	$4x-2y=2x \Rightarrow y=x$	M1 A1
	$\begin{pmatrix}4 & -2\\ 1 & 1\end{pmatrix}.\begin{pmatrix}x\\ y\end{pmatrix}=3\begin{pmatrix}x\\ y\end{pmatrix}$	
	$4x-2y=3x \Rightarrow y=\frac{1}{2}x$	M1 A1 **(4)**
		[12]
	Alternative to (c), using characteristic polynomial of $\mathbf{M}^{-1}$	
	$\left(\frac{1}{6}-\lambda\right)\left(\frac{2}{3}-\lambda\right)+\frac{1}{3}\times\frac{1}{6}=0$	M1
	Leading to $6\lambda^2-5\lambda+1=(3\lambda-1)(2\lambda-1)=0 \Rightarrow \lambda=\frac{1}{2},\frac{1}{3}$	A1, A1 **(3)**
	Alternative to (d) $\begin{pmatrix}4 & -2\\ 1 & 1\end{pmatrix}.\begin{pmatrix}x\\ mx\end{pmatrix}=\begin{pmatrix}x'\\ mx'\end{pmatrix}$	
	$4x-2mx=x',\quad x+mx=mx'$ both	M1
	$\frac{1+m}{4-2m}=m$	A1
	Leading to $2m^2-3m+1=(2m-1)(m-1)=0 \Rightarrow m=\frac{1}{2},1$	M1
	$y=\frac{1}{2}x,\ y=x$ both	A1 **(4)**

Question Number	Scheme	Marks
6.	(a) Let $z = x + iy$	
	$(x-6)^2 + (y+3)^2 = 9\left[(x+2)^2 + (y-1)^2\right]$	M1
	Leading to $8x^2 + 8y^2 + 48x - 24y = 0$	M1 A1
	This is a circle; the coefficients of x^2 and y^2 are the same and there is no *xy* term. Allow equivalent arguments and ft their $f(x, y)$ if appropriate.	A1ft
	$(x^2 + 6x + y^2 - 3y = 0)$	
	Leading to $(x+3)^2 + (y - \frac{3}{2})^2 = \frac{45}{4}$	M1
	Centre: $(-3, \frac{3}{2})$	A1
	Radius: $\frac{3}{2}\sqrt{5}$ or equivalent	A1 **(7)**
	(b)	
	Circle	B1
	centre in correct quadrant	B1 ft
	through origin	B1
	Line cuts –ve *x* and +ve *y* axes	B1
	intersects with circle on axes and all correct	B1 **(5)**
	(c) Shading inside circle	B1
	and above line with all correct	B1 **(2)**
		[14]



Having 3 instead of 9 in first equation gains maximum of
M1M1A0A1ftM1A0A0 B1B1B0B1B0 B1B0 8/14

Alternative to (a)

Accept the following argument:-
The locus of *P* is a Circle of Apollonius, which is a circle with diameter *XY*, where the points *X* and *Y* cut (6, –3) and (–2, 1) internally and externally in the ratio 3 : 1. — M1 A1

X: (0, 0) *Y*: (–6, 3)	M1 A1
Centre: $(-3, \frac{3}{2})$	M1 A1
Radius: $\frac{3}{2}\sqrt{5}$ or equivalent	A1 **(7)**

Question Number	Scheme	Marks
7.	(a) $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -3 \\ 4 & -5 & -1 \end{vmatrix}$	M1
	$= -15\mathbf{i} - 10\mathbf{j} - 10\mathbf{k}$	A1+A1+A1 **(4)**
	Allow M1 A1 for negative of above	
	(b) $\mathbf{r}.(3\mathbf{i}+2\mathbf{j}+2\mathbf{k}) = (3\mathbf{i}+2\mathbf{j}+2\mathbf{k}).(\mathbf{i}+3\mathbf{j}-\mathbf{k})$ or equivalent	M1
	$\mathbf{r}.(3\mathbf{i}+2\mathbf{j}+2\mathbf{k}) = 7$ or multiple	A1 **(2)**
	(c) Let $x = \lambda$, $z = 3-\lambda$,	
	then $2y = 7 - 3\lambda - 2(3-\lambda) \Rightarrow y = \frac{1}{2} - \frac{1}{2}\lambda$	
	x, y and z in terms of a single parameter	M1
	The direction of l is any multiple of $(2\mathbf{i}-\mathbf{j}-2\mathbf{k})$	M1
	$\left(\mathbf{r} - \left(\frac{1}{2}\mathbf{j}+3\mathbf{k}\right)\right)\times(2\mathbf{i}-\mathbf{j}-2\mathbf{k}) = \mathbf{0}$ or equivalent	M1 A1 **(4)**
	Possible equivalents are $(\mathbf{r}-(\mathbf{i}+2\mathbf{k}))\times(-2\mathbf{i}+\mathbf{j}+2\mathbf{k}) = \mathbf{0}$	
	and $(\mathbf{r}-(3\mathbf{i}-\mathbf{j}))\times\left(-\mathbf{i}+\frac{1}{2}\mathbf{j}+\mathbf{k}\right) = \mathbf{0}$	
	The general form is	
	$\{\mathbf{r} - [\mathbf{i}+2\mathbf{k}+c_1(2\mathbf{i}-\mathbf{j}-2\mathbf{k})]\}\times c_2(2\mathbf{i}-\mathbf{j}-2\mathbf{k}) = \mathbf{0}$	
	(d) $\left(\lambda\mathbf{i} + \left(\frac{1}{2}-\frac{1}{2}\lambda\right)\mathbf{j} + (3-\lambda)\mathbf{k}\right).(2\mathbf{i}-\mathbf{j}-2\mathbf{k}) = 0$	M1
	$2\lambda - \frac{1}{2} + \frac{1}{2}\lambda - 6 + 2\lambda = 0$	
	Leading to $\lambda = \frac{13}{9}$	M1 A1
	$P: \left(\frac{13}{9}, -\frac{2}{9}, \frac{14}{9}\right)$	A1 **(4)**
		[14]
	Alternative to (d)	
	$OP^2 = \lambda^2 + \left(\frac{1}{2}-\frac{1}{2}\lambda\right)^2 + (3-\lambda)^2 \quad \left(= \frac{1}{4}\left(9\lambda^2 - 26\lambda + 37\right)\right)$	M1
	$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(OP^2\right) = 0 \Rightarrow \lambda = \frac{13}{9}$	M1 A1
	$P: \left(\frac{13}{9}, -\frac{2}{9}, \frac{14}{9}\right)$	A1 **(4)**