C3 JAN 2010

1. Express

$$\frac{x+1}{3x^2-3} - \frac{1}{3x+1}$$

as a single fraction in its simplest form.

(4)

$$\frac{x+1}{3(x^2-1)} - \frac{1}{3x+1} = \frac{1\cancel{(x+1)}}{3\cancel{(x+1)}(x-1)} - \frac{1}{3x+1}$$

$$= \frac{1}{3(x-1)} - \frac{1}{3x+1} = \frac{1}{3(x-1)} \times \frac{3x+1}{3x+1} - \frac{1}{3x+1} \times \frac{3(x-1)}{3(x-1)}$$

$$= \frac{3x+1-3x+3}{3(x-1)(3x+1)} = \frac{4}{3(x-1)(3x+1)}$$

2.

$$f(x) = x^3 + 2x^2 - 3x - 11$$

(a) Show that $f(x) = 0$ can be rearranged as

$$x = \sqrt{\left(\frac{3x+11}{x+2}\right)}, \quad x \neq -2.$$

(2)

The equation $f(x) = 0$ has one positive root α.

The iterative formula $x_{n+1} = \sqrt{\left(\frac{3x_n+11}{x_n+2}\right)}$ is used to find an approximation to α.

(b) Taking $x_1 = 0$, find, to 3 decimal places, the values of x_2, x_3 and x_4.

(3)

(c) Show that $\alpha = 2.057$ correct to 3 decimal places.

(3)

(a) $x^3 + 2x^2 = 3x + 11$

$x^2(x+2) = 3x + 11 \qquad x^2 = \frac{3x+11}{x+2}$

$x = \sqrt{\frac{3x+11}{x+2}}$

(b) $x_1 = 0$
$x_2 = 2.345$
$x_3 = 2.037$
$x_4 = 2.059$

(c) $f(2.0575) = 0.00414$
$f(2.0565) = -0.0138$

by change of sign law $\alpha = 2.057$ (3dp)

3. (a) Express $5\cos x - 3\sin x$ in the form $R\cos(x+\alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$.

(4)

(b) Hence, or otherwise, solve the equation

$$5\cos x - 3\sin x = 4$$

for $0 \leqslant x < 2\pi$, giving your answers to 2 decimal places.

(5)

(a) $R\sin\alpha = 3$
$R\cos\alpha = 5$ $\Rightarrow \tan\alpha = \frac{3}{5} \Rightarrow \alpha = 0.5404$

$R = \sqrt{3^2+5^2} = \sqrt{34}$ $\Rightarrow \sqrt{34}\cos(x+0.5404)$

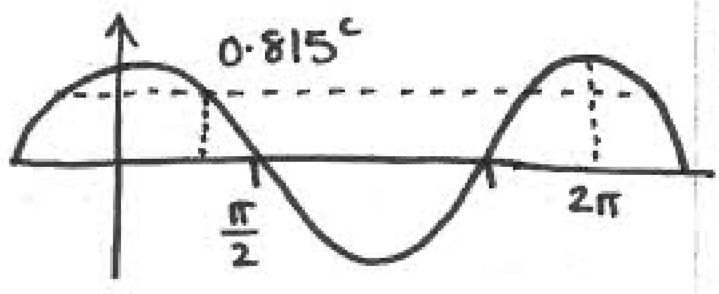
(b) $\sqrt{34}\cos(x+0.5404) = 4$

$x + 0.5404 = \cos^{-1}\left(\frac{4}{\sqrt{34}}\right)$

$x + 0.5405 = 0.815^c$

$= 2\pi - 0.815$

$x = \underline{0.27^c}, \underline{4.93^c}$



4. (i) Given that $y = \frac{\ln(x^2+1)}{x}$, find $\frac{dy}{dx}$.

(4)

(ii) Given that $x = \tan y$, show that $\frac{dy}{dx} = \frac{1}{1+x^2}$.

(5)

(i) $u = \ln(x^2+1)$ $\quad v = x$

$u' = \frac{2x}{x^2+1}$ $\quad v' = 1$

$$\frac{d}{dx}(uv) = \frac{\frac{2x^2}{x^2+1} - \ln(x^2+1)}{x^2} = \frac{2}{x^2+1} - \frac{\ln(x^2+1)}{x^2}$$

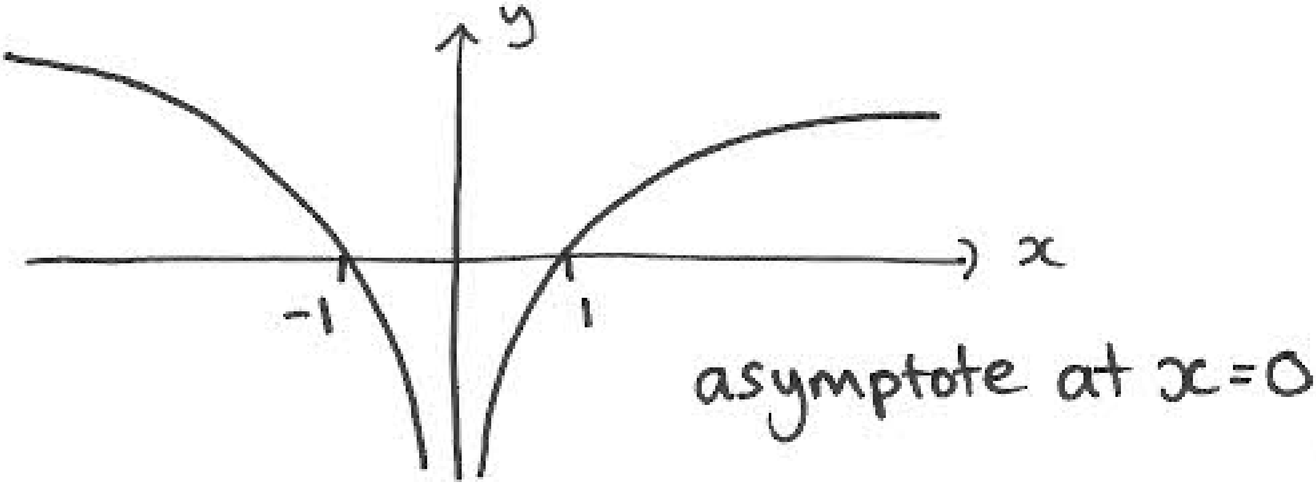
(ii) $x = \tan y$

$$\frac{dx}{dy} = \sec^2 y \Rightarrow \frac{dx}{dy} = 1 + \tan^2 y \Rightarrow \frac{dx}{dy} = 1 + x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$$

5. Sketch the graph of $y = \ln|x|$, stating the coordinates of any points of intersection with the axes.

(3)



6.

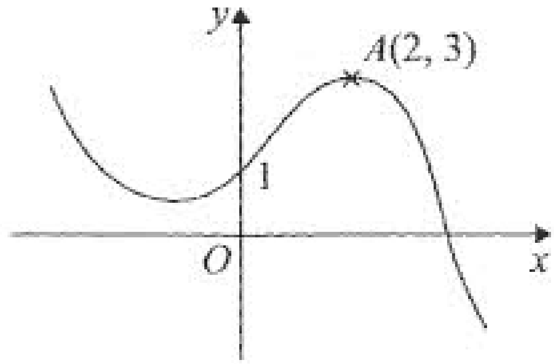


Figure 1

Figure 1 shows a sketch of the graph of $y = \mathrm{f}(x)$.

The graph intersects the y-axis at the point (0, 1) and the point $A(2, 3)$ is the maximum turning point.

Sketch, on separate axes, the graphs of

(i) $y = \mathrm{f}(-x) + 1$, flip ↑+1

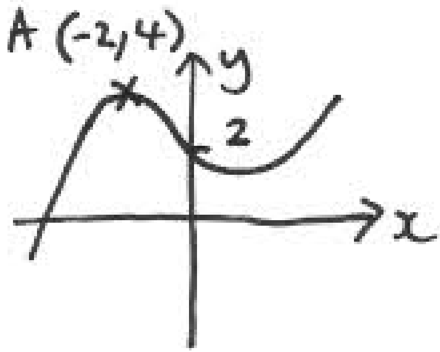
(ii) $y = \mathrm{f}(x + 2) + 3$, 2 ← ↑+3

(iii) $y = 2\mathrm{f}(2x)$. ×2 ↕ → ÷2 ←

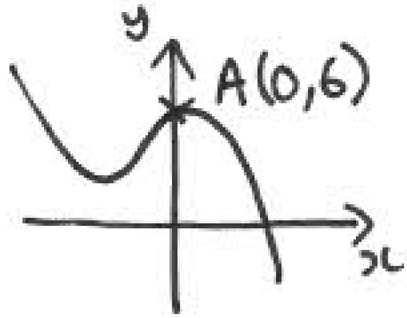
On each sketch, show the coordinates of the point at which your graph intersects the y-axis and the coordinates of the point to which A is transformed.

(9)

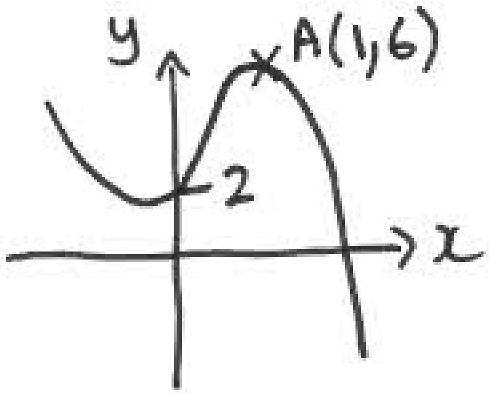
i)



(ii)



iii)



7. (a) By writing $\sec x$ as $\frac{1}{\cos x}$, show that $\frac{\mathrm{d}(\sec x)}{\mathrm{d}x} = \sec x \tan x$.

(3)

Given that $y = e^{2x} \sec 3x$,

(b) find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

(4)

The curve with equation $y = e^{2x} \sec 3x$, $-\frac{\pi}{6} < x < \frac{\pi}{6}$, has a minimum turning point at (a, b).

(c) Find the values of the constants a and b, giving your answers to 3 significant figures.

(4)

(a) $\frac{d}{dx} \sec x = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$ $u=1$ $v=\cos x$

$u'=0$ $v'=-\sin x$

$\frac{d}{dx}(uv) = \frac{0+\sin x}{\cos^2 x} = \frac{1}{\cos x} \times \frac{\sin x}{\cos x} = \underline{\sec x \tan x}$

(b) $\frac{d}{dx} e^{2x} \sec 3x$ $u=e^{2x}$ $v=\sec 3x$

$u'=2e^{2x}$ $v'=3\sec 3x \tan 3x$

$\frac{d}{dx}(uv) = 2e^{2x}\sec 3x + 3e^{2x}\sec 3x \tan 3x$

$= e^{2x}\sec 3x(2+3\tan 3x)$

(c) Minimum point $\Rightarrow \frac{dy}{dx}=0$ $e^{2x}\sec 3x \neq 0$

$\Rightarrow 2+3\tan 3x = 0 \Rightarrow \tan 3x = -\frac{2}{3}$

$3x = \tan^{-1}\left(-\frac{2}{3}\right) \Rightarrow 3x = -0.588$ $(+k\pi)$

$x = -0.196$ $y = \frac{e^{2x}}{\cos 3x} = 0.812$ $a=-0.196$

$\underline{b=0.812}$

8. Solve

$$\operatorname{cosec}^2 2x - \cot 2x = 1$$

for $0 \leqslant x \leqslant 180°$.

(7)

$(\cancel{1} + \cot^2 2x) - \cot 2x = \cancel{1}$

$\cot^2 2x - \cot 2x = 0$

$\cot 2x(\cot 2x - 1) = 0$

i) $\cot 2x - 1 = 0$ ii) $\cot 2x = 0$

i) $\Rightarrow \cot 2x - 1 = 0 \Rightarrow \cot 2x = 1$

$\Rightarrow \dfrac{1}{\tan 2x} = 1 \Rightarrow \tan 2x = 1$

$\Rightarrow 2x = \tan^{-1}(1) = 45°$

$2x = 45°, 225°$

$x = \underline{22.5°}, \underline{112.5°}$

ii) $\Rightarrow \cot 2x = 0 \Rightarrow 2x = 90, 270$

$x = \underline{45°}, \underline{135°}$

9. (i) Find the exact solutions to the equations

(a) $\ln(3x-7)=5$ **(3)**

(b) $3^x e^{7x+2}=15$ **(5)**

(ii) The functions f and g are defined by

$$f(x)=e^{2x}+3, \qquad x\in\mathbb{R}$$
$$g(x)=\ln(x-1), \qquad x\in\mathbb{R},\ x>1$$

(a) Find f^{-1} and state its domain. **(4)**

(b) Find fg and state its range. **(3)**

(a) $3x-7=e^5 \Rightarrow 3x=7+e^5 \Rightarrow x=\dfrac{7+e^5}{3}$

(b) $\ln 3^x e^{7x+2}=\ln 15$

$\ln 3^x+\ln e^{7x+2}=x\ln 3+7x+2=\ln 15$

$x(7+\ln 3)=\ln(15)-2 \Rightarrow x=\dfrac{\ln(15)-2}{7+\ln 3}$

(ii) $y=e^{2x}+3 \Rightarrow x=e^{2y}+3 \Rightarrow e^{2y}=x-3$

$\Rightarrow 2y=\ln(x-3) \Rightarrow y=f^{-1}(x)=\dfrac{\ln(x-3)}{2}$ $\quad x>3$

(b) $fg(x)=e^{2\ln(x-1)}+3=(e^{\ln(x-1)})^2+3$

$fg(x)=(x-1)^2+3 \qquad fg(x)>3$