C3 June 2009

1 (a) $x_{n+1} = \dfrac{2}{(x_n)^2} + 2$

$x_0 = 2.5$, $x_1 = 2.32$, $x_2 = 2.372$,
$x_3 = 2.356$, $x_4 = 2.360$

(b) $f(x) = -x^3 + 2x^2 + 2$

$f(2.3585) = 0.0058$, $f(2.3595) = -0.0014$

There is a sign change and $f(x)$ is continuous, so the root α lies in the interval $(2.3585, 2.3595)$.

Hence $\alpha = 2.359$ (3 d.p.)

2 (a) $\cos^2\theta + \sin^2\theta = 1$

$\Rightarrow \dfrac{\cos^2\theta}{\cos^2\theta} + \dfrac{\sin^2\theta}{\cos^2\theta} = \dfrac{1}{\cos^2\theta}$

$\Rightarrow 1 + \tan^2\theta = \sec^2\theta \Rightarrow \tan^2\theta = \sec^2\theta - 1$.

(b) $2\tan^2\theta + 4\sec\theta + \sec^2\theta = 2$, $0 \leq \theta < 360^\circ$

$\Rightarrow 2(\sec^2\theta - 1) + 4\sec\theta + \sec^2\theta = 2$

$\Rightarrow 3\sec^2\theta + 4\sec\theta - 4 = 0$

$\Rightarrow (3\sec\theta - 2)(\sec\theta + 2) = 0 \Rightarrow \sec\theta = \frac{2}{3}, -2$

$\Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = 120^\circ, 240^\circ$.

③ $P = 80e^{t/5}, \quad t \in \mathbb{R},\ t \geq 0.$

(a) $t=0 \Rightarrow P = 80$ rabbits.

(b) $80e^{t/5} > 1000 \Rightarrow e^{t/5} > 12.5$

$\Rightarrow \frac{t}{5} > \ln 12.5 \Rightarrow t > 12.6$

So population will exceed 1000 after 13 years.

(c) $\frac{dP}{dt} = 16e^{t/5}$

(d) $\frac{dP}{dt} = 50 \Rightarrow 16e^{t/5} = 50 \Rightarrow e^{t/5} = 3.125$

$\Rightarrow P = 80 \times 3.125 = \underline{250 \text{ rabbits}}.$

④ (i) (a) $y = x^2 \cos 3x$

$\Rightarrow \frac{dy}{dx} = x^2(-3\sin 3x) + 2x\cos 3x$

$= \underline{-3x^2 \sin 3x + 2x\cos 3x}.$

(b) $y = \frac{\ln(x^2+1)}{x^2+1} \Rightarrow \frac{dy}{dx} = \frac{(x^2+1)\left(\frac{2x}{x^2+1}\right) - \left(\ln(x^2+1)\right)2x}{(x^2+1)^2}$

$= \underline{\frac{2x - 2x\ln(x^2+1)}{(x^2+1)^2}}$

(ii) $y = \sqrt{4x+1}$, $x > -\frac{1}{4}$, $y > 0$.

$x = 2 \Rightarrow y = 3 \Rightarrow P(2, 3)$.

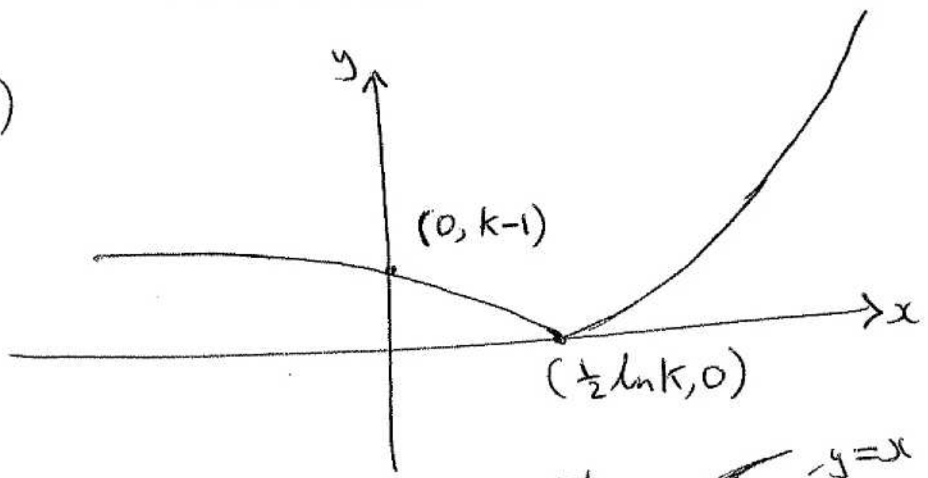
$$\frac{dy}{dx} = \frac{1}{2}(4)(4x+1)^{-1/2} = \frac{2}{\sqrt{4x+1}}$$

At P, $\frac{dy}{dx} = \frac{2}{3} \Rightarrow$ tangent at P is

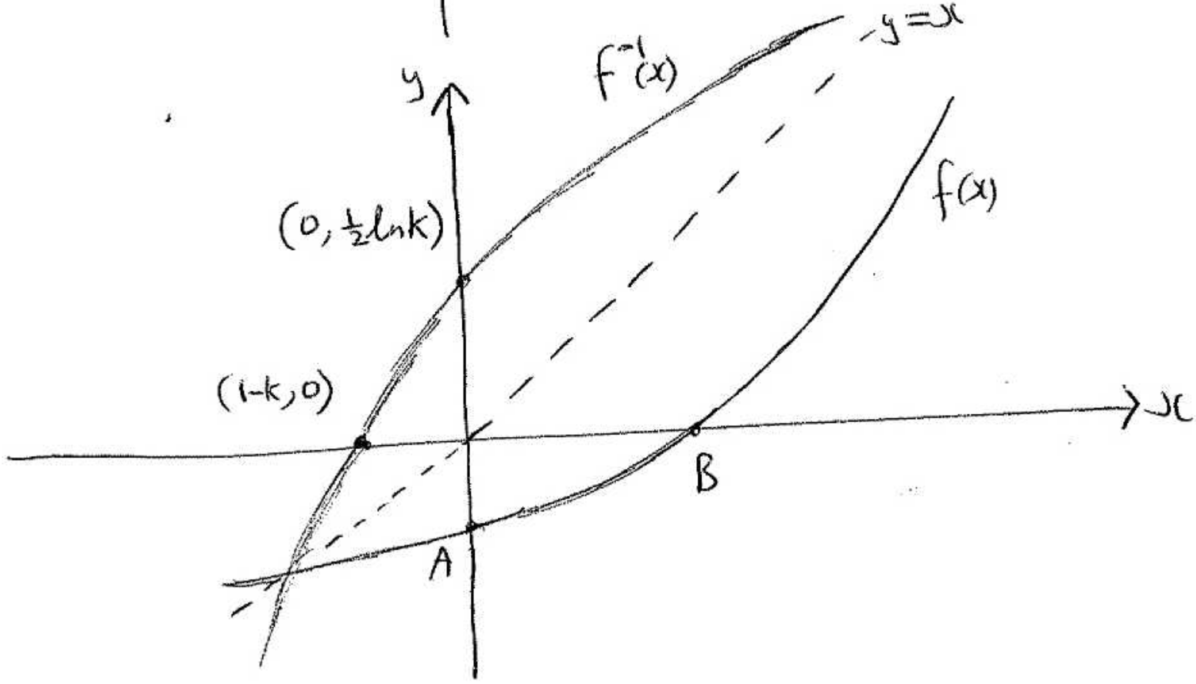
$$y - 3 = \frac{2}{3}(x-2)$$

$$\Rightarrow \underline{2x - 3y + 5 = 0}.$$

5 (a)



(b)



(c) $f(x) = e^{2x} - k \Rightarrow$ range is $f(x) > -k$.

(d) $y = e^{2x} - k \Rightarrow e^{2x} = y + k$

$\Rightarrow 2x = \ln(y+k) \Rightarrow x = \frac{1}{2}\ln(y+k)$

so $\underline{f^{-1}(x) = \frac{1}{2}\ln(x+k)}$

(e) Domain of f^{-1} = Range of f,

so domain is $x > -k$ (i.e. $x = -k$ is an asymptote to $f^{-1}(x)$).

6 (a) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

Put $B = A \Rightarrow \cos 2A = \cos^2 A - \sin^2 A$

$= (1 - \sin^2 A) - \sin^2 A$

so $\underline{\cos 2A = 1 - 2\sin^2 A}$.

(b) $C_1: y = 3\sin 2x$, $C_2: y = 4\sin^2 x - 2\cos 2x$.

Intersecting $\Rightarrow 3\sin 2x = 4\sin^2 x - 2\cos 2x$

$\boxed{\text{Use } 2\sin^2 A = 1 - \cos 2A}$

$3\sin 2x = 2(1 - \cos 2x) - 2\cos 2x$

$3\sin 2x = 2 - 4\cos 2x$

$\Rightarrow \underline{4\cos 2x + 3\sin 2x = 2}$

(c) $4\cos 2x + 3\sin 2x = R\cos(2x-\alpha)$

$= R(\cos 2x\cos\alpha + \sin 2x\sin\alpha)$

$\Rightarrow \left.\begin{array}{l} 4 = R\cos\alpha \\ 3 = R\sin\alpha \end{array}\right\} \Rightarrow \tan\alpha = \frac{3}{4} \Rightarrow \underline{\alpha = 36.87^\circ}$

$R = \sqrt{3^2+4^2} = \underline{\underline{5}}$

So $4\cos 2x + 3\sin 2x = 5\cos(2x - 36.87^\circ)$.

(d) $4\cos 2x + 3\sin 2x = 2, \quad 0 \le x < 180^\circ$

$\Rightarrow 5\cos(2x - \alpha) = 2$

$\Rightarrow \cos(2x - \alpha) = 0.4$

$\Rightarrow 2x - \alpha = \left\{\begin{array}{l} 66.4 \\ -66.4 \end{array}\right. + 360n$

$\Rightarrow 2x = \left\{\begin{array}{l} 103.3 \\ -29.6 \end{array}\right. + 360n$

$\Rightarrow x = \left\{\begin{array}{l} 51.6 \\ -14.8 \end{array}\right. + 180n$

$\Rightarrow \underline{x = 51.6^\circ, 165.2^\circ}$

7 $f(x) = 1 - \dfrac{2}{x+4} + \dfrac{x-8}{(x-2)(x+4)}$, $x \in \mathbb{R}$, $x \neq -4$, $x \neq 2$.

(a) $f(x) = \dfrac{(x-2)(x+4) - 2(x-2) + x-8}{(x-2)(x+4)}$

$$= \frac{x^2+2x-8-2x+4+x-8}{(x-2)(x+4)} = \frac{x^2+x-12}{(x-2)(x+4)}$$

$$= \frac{(x-3)(x+4)}{(x-2)(x+4)} = \frac{x-3}{x-2}$$

(b) $g(x) = \dfrac{e^x-3}{e^x-2}$, $x \in \mathbb{R}$, $x \neq \ln 2$.

$$g'(x) = \frac{(e^x-2)(e^x) - (e^x-3)(e^x)}{(e^x-2)^2}$$

$$= \frac{e^{2x}-2e^x-e^{2x}+3e^x}{(e^x-2)^2} = \frac{e^x}{(e^x-2)^2}.$$

(c) $g'(x) = 1 \Rightarrow (e^x-2)^2 = e^x$

$\Rightarrow (e^x)^2 - 5e^x + 4 = 0$

$\Rightarrow (e^x-4)(e^x-1) = 0 \Rightarrow e^x = 4, 1$

$\Rightarrow \underline{x = \ln 4 \text{ or } 0}.$

(8) (a) $\sin 2x = 2\sin x \cos x$.

(b) $\operatorname{cosec} x - 8\cos x = 0, \quad 0 < x < \pi$.

$$\Rightarrow \frac{1}{\sin x} - 8\cos x = 0$$

$$\Rightarrow 1 - 8\sin x\cos x = 0$$

$$\Rightarrow 1 - 4\sin 2x = 0 \Rightarrow \sin 2x = \frac{1}{4}$$

$$\Rightarrow 2x = \begin{cases} 0.25268\ldots \\ 2.8889\ldots \end{cases} + 2\pi n$$

$$x = \begin{cases} 0.12634\ldots \\ 1.44445\ldots \end{cases} + \pi n$$

$$\Rightarrow \underline{x = 0.13,\ 1.44}$$