June 2006
6676 Further Pure Mathematics FP3
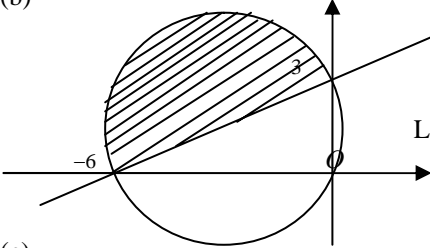## Mark Scheme

Question Number	Scheme	Marks
1.	$$\mathbf{A}^1 = \begin{pmatrix} 1 & 1 & \frac{1}{2}(1+3) \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{A}$$ (Hence true for $n = 1$) $$\mathbf{A}^{k+1} = \mathbf{A}^k . \mathbf{A} = \begin{pmatrix} 1 & k & \frac{1}{2}(k^2 + 3k) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$ $$= \begin{pmatrix} 1 & k+1 & 2+k+\frac{1}{2}(k^2+3k) \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix}$$ $$2+k+\tfrac{1}{2}(k^2+3k) = \tfrac{1}{2}(k^2+5k+4) = \tfrac{1}{2}(k^2+2k+1+3k+3)$$ $$= \tfrac{1}{2}\left((k+1)^2 + 3(k+1)\right)$$ (Hence, if result is true for $n = k$, then it is true for $n = k + 1$). By Mathematical Induction, above implies true for all positive integers. cso	B1 M1 M1 Dep A1 A1 **(5)** **[5]**

Question Number	Scheme	Marks
2.	(a) $\quad \text{f}(x) = \cos 2x, \qquad \text{f}\left(\frac{\pi}{4}\right) = 0$	
	$\quad \text{f}'(x) = -2\sin 2x, \qquad \text{f}'\left(\frac{\pi}{4}\right) = -2$	M1
	$\quad \text{f}''(x) = -4\cos 2x, \qquad \text{f}''\left(\frac{\pi}{4}\right) = 0$	
	$\quad \text{f}'''(x) = 8\sin 2x, \qquad \text{f}'''\left(\frac{\pi}{4}\right) = 8$	A1
	$\quad \text{f}^{(iv)}(x) = 16\cos 2x, \qquad \text{f}^{(iv)}\left(\frac{\pi}{4}\right) = 0$	
	$\quad \text{f}^{(v)}(x) = -32\sin 2x, \qquad \text{f}^{(v)}\left(\frac{\pi}{4}\right) = -32$	A1
	$\cos 2x = \text{f}\left(\frac{\pi}{4}\right) + \text{f}'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \dfrac{\text{f}''\left(\frac{\pi}{4}\right)}{2}\left(x - \frac{\pi}{4}\right)^2 + \dfrac{\text{f}'\left(\frac{\pi}{4}\right)}{3!}\left(x - \frac{\pi}{4}\right)^3 + \ldots$	
	Three terms are sufficient to establish method	M1
	$\cos 2x = -2\left(x - \frac{\pi}{4}\right) + \frac{4}{3}\left(x - \frac{\pi}{4}\right)^3 - \frac{4}{15}\left(x - \frac{\pi}{4}\right)^5 + \ldots$	A1 **(5)**
	(b) $\qquad$ Substitute $x = 1 \qquad \left(1 - \frac{\pi}{4} \approx 0.21460\right)$	B1
	$\cos 2 = -2\left(1 - \frac{\pi}{4}\right) + \frac{4}{3}\left(1 - \frac{\pi}{4}\right)^3 - \frac{4}{15}\left(1 - \frac{\pi}{4}\right)^5 + \ldots$	
	$\approx -0.416147 \qquad\qquad\qquad\qquad$ cao	M1 A1 **(3)**
		[8]

Question Number	Scheme	Marks
3.	(a) In this solution $\cos\theta = c$ and $\sin\theta = s$	
	$\cos 5\theta + \mathrm{i}\sin 5\theta = (c+\mathrm{i}s)^5$	M1
	$\left(= c^5 + 5c^4\mathrm{i}s + 10c^3(\mathrm{i}s)^2 + 10c^2(\mathrm{i}s)^3 + 5c(\mathrm{i}s)^4 + (\mathrm{i}s)^5 \right)$	
	$\Im$ $\quad\quad \sin 5\theta = 5c^4 s - 10c^2 s^3 + s^5$	M1 A1
	equate	
	$= 5c^4 s - 10c^2\left(1-c^2\right)s + \left(1-c^2\right)^2 s \quad\quad s^2 = 1 - c^2$	M1
	$= s\left(16c^4 - 12c^2 + 1\right)$ ✳	A1 (5)
	(b) $\quad \sin\theta\left(16\cos^4\theta - 12\cos^2\theta + 1\right) + 2\cos^2\theta\sin\theta = 0$	M1
	$\sin\theta = 0 \quad\Rightarrow\quad \theta = 0$	B1
	$16c^4 - 10c^2 + 1 = \left(8c^2 - 1\right)\left(2c^2 - 1\right) = 0$	M1
	$c = \pm\dfrac{1}{2\sqrt{2}}, \quad c = \pm\dfrac{1}{\sqrt{2}} \quad\quad$ any two	A1
	$\theta \approx 1.21, 1.93; \quad \theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4} \quad\quad$ any two	A1
	$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ all four	A1 (6)
	accept awrt 0.79, 1.21,1.93,2.36	[11]
	Ignore any solutions out of range.	

Question Number	Scheme	Marks
4. (a)	$\left(\dfrac{dx}{dt}\right)_0 = 0.4 \approx \dfrac{x_{0.1}-0}{0.1} \;\Rightarrow\; x_{0.1} \approx 0.04$	B1
	$\left(\dfrac{d^2x}{dt^2}\right)_{0.1} = -3\sin x_{0.1} \approx \dfrac{x_{0.2}-2x_{0.1}+0}{0.01}$	M1
	Must have their $x_{0.1}$	
	$x_{0.2} \approx 0.0788$ awrt	A1
	$\left(\dfrac{d^2x}{dt^2}\right)_{0.2} = -3\sin x_{0.2} \approx \dfrac{x_{0.3}-2x_{0.2}+x_{0.1}}{0.01}$	M1
	Must have their $x_{0.1}$, $x_{0.2}$	
	$x_{0.3} \approx 0.115$ awrt	A1 **(5)**
(b)	$f''(t) = -3\sin x, \qquad f''(0) = 0$	
	$f'''(t) = -3\cos x \dfrac{dx}{dt}, \quad f'''(0) = -3\times 0.4 = -1.2$	M1 A1
	$f(t) = f(0)+t\,f'(0)+\dfrac{t^2}{2}f''(0)+\dfrac{t^3}{3!}f'''(0)+\ldots$	
	$= 0.4t - 0.2t^3$	M1 A1 **(4)**
(c) Substituting $t=0.3$ into their answer to (b) and evaluating		M1
	$f(0.3) \approx 0.1146$ cao	A1 **(2)**
		[11]

Question Number	Scheme	Marks
5.	(a) $\qquad (4-\lambda)(1-\lambda)+2=0$	M1
	$\qquad \lambda^2 - 5\lambda + 6 = (\lambda - 3)(\lambda - 2) = 0$	M1
	$\qquad \lambda_1 = 2, \; \lambda_2 = 3 \qquad\qquad$ both	A1 (3)
	(b) $\qquad M^{-1} = \dfrac{1}{\underline{6}}\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$	<u>B1</u> B1 (2)
	(c) $\quad \begin{vmatrix} \frac{1}{6} - \frac{1}{2} & \frac{1}{3} \\ -\frac{1}{6} & \frac{2}{3} - \frac{1}{2} \end{vmatrix} = -\frac{1}{3} \times \frac{1}{6} + \frac{1}{3} \times \frac{1}{6} = 0 \quad$ M1 for either value	M1 A1
	$\qquad \left(\text{hence } \tfrac{1}{2} \text{ is an eigenvalue of } \mathbf{M}^{-1}\right)$	
	$\quad \begin{vmatrix} \frac{1}{6} - \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{2}{3} - \frac{1}{3} \end{vmatrix} = -\frac{1}{6} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{6} = 0$	A1
	$\qquad \left(\text{hence } \tfrac{1}{3} \text{ is an eigenvalue of } \mathbf{M}^{-1}\right)$	(3)
	(d) Using eigenvalues	
	$\qquad \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$	
	$\qquad 4x - 2y = 2x \;\Rightarrow\; y = x$	M1 A1
	$\qquad \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$	
	$\qquad 4x - 2y = 3x \;\Rightarrow\; y = \tfrac{1}{2}x$	M1 A1 (4)
		[12]
	Alternative to (c), using characteristic polynomial of $\mathbf{M}^{-1}$	
	$\qquad \left(\tfrac{1}{6} - \lambda\right)\left(\tfrac{2}{3} - \lambda\right) + \tfrac{1}{3} \times \tfrac{1}{6} = 0$	M1
	Leading to $\; 6\lambda^2 - 5\lambda + 1 = (3\lambda - 1)(2\lambda - 1) = 0 \;\Rightarrow\; \lambda = \tfrac{1}{2}, \tfrac{1}{3}$	A1, A1 (3)
	Alternative to (d) $\quad \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\cdot\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} x' \\ mx' \end{pmatrix}$	
	$\qquad 4x - 2mx = x', \quad x + mx = mx' \qquad$ both	M1
	$\qquad \dfrac{1+m}{4-2m} = m$	A1
	Leading to $\; 2m^2 - 3m + 1 = (2m - 1)(m - 1) = 0 \;\Rightarrow\; m = \tfrac{1}{2}, 1$	M1
	$\qquad y = \tfrac{1}{2}x, \;\; y = x \qquad\qquad$ both	A1 (4)

Question Number	Scheme	Marks
6.	(a) Let $z = x + iy$	
	$$(x-6)^2 + (y+3)^2 = 9\left[(x+2)^2 + (y-1)^2\right]$$	M1
	Leading to $\qquad 8x^2 + 8y^2 + 48x - 24y = 0$	M1 A1
	This is a circle; the coefficients of x^2 and y^2 are the same and there is no xy term.	
	Allow equivalent arguments and ft their $f(x, y)$ if appropriate.	A1ft
	$$\left(x^2 + 6x + y^2 - 3y = 0\right)$$	
	Leading to $\qquad (x+3)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{45}{4}$	M1
	Centre: $\left(-3, \frac{3}{2}\right)$	A1
	Radius: $\frac{3}{2}\sqrt{5}$ $\qquad$ or equivalent	A1 **(7)**
	(b) Circle	B1
	centre in correct quadrant	B1 ft
	through origin	B1
	Line cuts $-$ve x and $+$ve y axes	B1
	intersects with circle on axes and all correct	B1 **(5)**
	(c) $\qquad$ Shading inside circle	B1
	and above line with all correct	B1 **(2)**
		[14]
	Having 3 instead of 9 in first equation gains maximum of	
	$\quad$ M1M1A0A1ftM1A0A0 $\;$ B1B1B0B1B0 $\;$ B1B0 $\;$ 8/14	
	Alternative to (a)	
	$\quad$ Accept the following argument:-	
	The locus of P is a Circle of Apollonius, which is a circle with diameter XY, where the points X and Y cut $(6, -3)$ and $(-2, 1)$ internally and externally in the ratio 3 : 1.	M1 A1
	$X: (0, 0) \quad Y: (-6, 3)$	M1 A1
	Centre: $\left(-3, \frac{3}{2}\right)$	M1 A1
	Radius: $\frac{3}{2}\sqrt{5}$ $\qquad$ or equivalent	A1 **(7**

Question Number	Scheme	Marks
7.	**(a)** $\quad (\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -3 \\ 4 & -5 & -1 \end{vmatrix}$	M1
	$= -15\mathbf{i} - 10\mathbf{j} - 10\mathbf{k}$	A1+A1+A1 **(4)**
	Allow M1 A1 for negative of above	
	(b) $\quad \mathbf{r}.(3\mathbf{i}+2\mathbf{j}+2\mathbf{k}) = (3\mathbf{i}+2\mathbf{j}+2\mathbf{k}).(\mathbf{i}+3\mathbf{j}-\mathbf{k})$ $\quad$ or equivalent	M1
	$\quad \mathbf{r}.(3\mathbf{i}+2\mathbf{j}+2\mathbf{k}) = 7$ $\qquad\qquad\qquad$ or multiple	A1 $\qquad$ **(2)**
	(c) Let $x = \lambda$, $\quad z = 3-\lambda$,	
	$\qquad$ then $2y = 7 - 3\lambda - 2(3-\lambda) \implies y = \tfrac{1}{2} - \tfrac{1}{2}\lambda$	
	$\qquad\qquad$ x, y and z in terms of a single parameter	M1
	$\quad$ The direction of l is any multiple of $\left(2\mathbf{i} - \mathbf{j} - 2\mathbf{k}\right)$	M1
	$\quad \left(\mathbf{r} - \left(\tfrac{1}{2}\mathbf{j}+3\mathbf{k}\right)\right)\times\left(2\mathbf{i}-\mathbf{j}-2\mathbf{k}\right) = \mathbf{0}$ $\quad$ or equivalent	M1 A1 $\qquad$ **(4)**
	Possible equivalents are $\left(\mathbf{r} - (\mathbf{i}+2\mathbf{k})\right)\times\left(-2\mathbf{i}+\mathbf{j}+2\mathbf{k}\right) = \mathbf{0}$	
	$\qquad$ and $\left(\mathbf{r} - (3\mathbf{i}-\mathbf{j})\right)\times\left(-\mathbf{i}+\tfrac{1}{2}\mathbf{j}+\mathbf{k}\right) = \mathbf{0}$	
	The general form is	
	$\quad \left\{\mathbf{r} - \left[\mathbf{i}+2\mathbf{k} + c_1\left(2\mathbf{i}-\mathbf{j}-2\mathbf{k}\right)\right]\right\}\times c_2\left(2\mathbf{i}-\mathbf{j}-2\mathbf{k}\right) = \mathbf{0}$	
	(d) $\quad \left(\lambda\mathbf{i} + \left(\tfrac{1}{2}-\tfrac{1}{2}\lambda\right)\mathbf{j} + (3-\lambda)\mathbf{k}\right).\left(2\mathbf{i}-\mathbf{j}-2\mathbf{k}\right) = 0$	M1
	$\qquad 2\lambda - \tfrac{1}{2} + \tfrac{1}{2}\lambda - 6 + 2\lambda = 0$	
	$\qquad$ Leading to $\lambda = \tfrac{13}{9}$	M1 A1
	$\qquad P : \left(\tfrac{13}{9}, -\tfrac{2}{9}, \tfrac{14}{9}\right)$	A1 $\qquad$ **(4)**
		[14]
	Alternative to (d)	
	$\quad OP^2 = \lambda^2 + \left(\tfrac{1}{2}-\tfrac{1}{2}\lambda\right)^2 + (3-\lambda)^2 \quad \left(= \tfrac{1}{4}\left(9\lambda^2 - 26\lambda + 37\right)\right)$	M1
	$\qquad \dfrac{\mathrm{d}}{\mathrm{d}\lambda}\left(OP^2\right) = 0 \implies \lambda = \tfrac{13}{9}$	M1 A1
	$\qquad P : \left(\tfrac{13}{9}, -\tfrac{2}{9}, \tfrac{14}{9}\right)$	A1 $\qquad$ **(4)**