June 2006
6675 Further Pure Mathematics FP2
Mark Scheme

Question number	Scheme	Marks
1.	$5\left(\frac{e^x+e^{-x}}{2}\right)-2\left(\frac{e^x-e^{-x}}{2}\right)=11$	B1
	$3e^{2x}-22e^x+7=0$ M: Simplify to form <u>quadratic</u> in e^x.	M1 A1
	$(3e^x-1)(e^x-7)=0$ $e^x=\frac{1}{3}$, $e^x=7$ M: Solve 3 term quadratic.	M1 A1
	$x=\ln\frac{1}{3}\ (\text{or} -\ln 3)$ $x=\ln 7$	A1 (6)
		6
2.	(a) Using $b^2=a^2(1-e^2)$ or equiv. to find *e* or *ae*: (a = 2 and b = 1) $e=\frac{\sqrt{3}}{2}$	M1 A1
	Using $y^2=4(ae)x$ $y^2=4\sqrt{3}x$ (M requires <u>values</u> for *a* and *e*)	M1 A1 (4)
	(b) $x=-\sqrt{3}$	B1ft (1)
		5
3.	$\rho=\frac{ds}{d\psi}$ $s=\int e^{\sin\psi}\cos\psi\,d\psi=e^{\sin\psi}$ $(+k)$	M1 A1
	(M mark may be scored by a full substitution method)	
	$s=0$ at $\psi=0$: $k=-1$ $s=e^{\sin\psi}-1$	M1 A1cso (4)
	(M mark requires a *k*, or use of limits)	
		4

Question number	Scheme	Marks
4.	$\dfrac{dy}{dx} = \dfrac{2x}{1+(x^2)^2}$ M: $\dfrac{dy}{dx} = \dfrac{*}{1+(x^2)^2}$ or $\dfrac{dy}{dx} = \dfrac{2x}{1+x^2}$	M1 A1
	$\dfrac{d^2y}{dx^2} = \dfrac{2(1+x^4) - 2x.4x^3}{(1+x^4)^2}$ $\left(= \dfrac{2-6x^4}{(1+x^4)^2} \right)$	M1
	$\rho = \dfrac{\left(1+\left(\dfrac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}} = \dfrac{(1+1)^{\frac{3}{2}}}{-1}$	M1 A1cso
	$= -2\sqrt{2}$ (or $2\sqrt{2}$) (or exact equivalent, e.g. $\sqrt{8}$, $2^{\frac{3}{2}}$)	A1cso (6) **6**
	2nd M: Quotient or product rule attempt. (Chain rule, if used, must be 'good'.) 3rd M: Attempt ρ with derivative values… A: Correct derivative values (1 and –1) seen or implied by working. <u>Alternative:</u> (involving implicit differentiation). $\sec^2 y \dfrac{dy}{dx} = 2x$ [M1 A1] (allow one 'slip' for M1) $\sec^2 y \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} \cdot 2\sec y(\sec y \tan y)\dfrac{dy}{dx} = 2$ [M1] (or alternative e.g. $\dfrac{dy}{dx} = 2x\cos^2 y$, so $\dfrac{d^2y}{dx^2} = 2\cos^2 y + 2x \cdot 2\cos y(-\sin y)\dfrac{dy}{dx}$) Then marks as in main scheme (n.b. $\tan y = 1$, $\sec^2 y = 2$).	

Question number	Scheme	Marks	
5.	(a) $\dfrac{dy}{dx} = 4\text{sech}^2 4x - 1$	B1	
	Put $\dfrac{dy}{dx} = 0$ $\quad \left(\cosh^2 4x = 4 \quad \cosh 4x = 2\right)$	M1	
	$4x = \ln(2 \pm \sqrt{3})$ or $8x = \ln(7 \pm 4\sqrt{3})$ or $e^{4x} = 2 \pm \sqrt{3}$ or $e^{4x} = 7 \pm 4\sqrt{3}$ (± or +)	A1	
	$x = \dfrac{1}{4}\ln(2+\sqrt{3})$ or $x = \dfrac{1}{8}\ln(7+4\sqrt{3})$ (or equiv.)	A1	(4)
	(b) $y = -\dfrac{1}{4}\ln(2+\sqrt{3}) + \tanh(....)$ (Substitute for x)	M1	
	$\text{sech}\,4x = \dfrac{1}{2} = \sqrt{1-\tanh^2 4x}$, $\quad \tanh 4x = \dfrac{\sqrt{3}}{2}$	M1	
	$y = \dfrac{\sqrt{3}}{2} - \dfrac{1}{4}\ln(2+\sqrt{3}) = \dfrac{1}{4}\left\{2\sqrt{3} - \ln\left(2+\sqrt{3}\right)\right\}$ (*)	A1	(3)
			7
	(a) 'Second solution', if seen, must be rejected to score the final mark. (b) 2nd M requires an expression in terms of √3 without hyperbolics, exponentials and logarithms.		

Question number	Scheme	Marks
6.	(a) $\frac{dx}{dt} = 1 - \frac{1}{t}$ $\quad \frac{dy}{dt} = 2t^{-\frac{1}{2}}$	B1 B1
	$\sqrt{\left(1-\frac{1}{t}\right)^2 + \left(2t^{-\frac{1}{2}}\right)^2}$, $\quad = \sqrt{1+\frac{2}{t}+\frac{1}{t^2}} = 1+\frac{1}{t}$ or $\frac{t+1}{t}$	M1, A1
	Length $= \int_1^4 \left(1+\frac{1}{t}\right) dt = [t + \ln t]_1^4 = (4+\ln 4) - 1 = 3 + \ln 4$ $\quad$ (*)	M1 M1 A1 (7)
	(b) Surface area $= 2\pi \int_1^4 4\sqrt{t}\sqrt{\left(1-\frac{1}{t}\right)^2 + \left(2t^{-\frac{1}{2}}\right)^2}\, dt$ $\quad \left(= 8\pi \int_1^4 \left(t^{\frac{1}{2}} + t^{-\frac{1}{2}}\right) dt\right)$	M1
	$= (8\pi)\left[\frac{2t^{\frac{3}{2}}}{3} + 2t^{\frac{1}{2}}\right]_1^4 = (8\pi)\left\{\left(\frac{16}{3}+4\right) - \left(\frac{2}{3}+2\right)\right\} = \frac{160\pi}{3}$ $\quad \left(53\frac{1}{3}\pi\right)$	M1 M1 A1 (4)
		11
	Note dependence of M's. (a) 2nd M: Complete integration attempt. 3rd M: Subs. correct limits and subtract. (b) 2nd M: Complete integration attempt. 3rd M: Subs. correct limits and subtract. In (b), missing 2 or π or 2π throughout could score M0 M1 M1 A0. If 2π is there initially, then lost, could score M1 M1 M1 A0.	

Question number	Scheme	Marks
7.	$\int x^2 \operatorname{arsinh} x \, dx = \frac{x^3}{3}\operatorname{arsinh} x - \int \frac{x^3}{3\sqrt{x^2+1}} dx$	M1 A1 A1
	$\left[\frac{x^3}{3}\operatorname{arsinh} x\right]_0^3 = 9\operatorname{arsinh}3 \quad \left(\text{or } 9\ln\left(3+\sqrt{10}\right)\right)$	B1
	Let $u = x^2+1 \quad \frac{du}{dx} = 2x \qquad \left[u^2 = x^2+1 \quad 2u\frac{du}{dx} = 2x\right]$	M1
	$\frac{1}{3}\int \frac{x^3}{u^{\frac{1}{2}}.2x} du = \frac{1}{6}\int \frac{u-1}{u^{\frac{1}{2}}} du = \frac{1}{6}\int \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right) du \qquad \left[\frac{1}{3}\int \left(u^2-1\right) du\right]$	M1
	$= \frac{1}{6}\left[\frac{2u^{\frac{3}{2}}}{3} - 2u^{\frac{1}{2}}\right] \qquad \left[= \frac{1}{3}\left[\frac{u^3}{3} - u\right]\right]$	M1
	When $x = 0$, $u = 1$ and when $x = 3$, $u = 10 \qquad \left[\ldots\ldots u = \sqrt{10}\right]$	
	$\frac{1}{6}\left[\frac{2u^{\frac{3}{2}}}{3} - 2u^{\frac{1}{2}}\right]_1^{10} = \frac{1}{6}\left\{\left(\frac{20\sqrt{10}}{3} - 2\sqrt{10}\right) - \left(\frac{2}{3} - 2\right)\right\}$	M1 A1
	Area $= 9\operatorname{arsinh}3 - \frac{1}{6}\left(\frac{14\sqrt{10}}{3} + \frac{4}{3}\right) = 9\ln\left(3+\sqrt{10}\right) - \frac{1}{9}\left(7\sqrt{10}+2\right) \qquad (*)$	A1cso (10)
		(10)
	<u>Dependent M marks:</u> M: Choose an appropriate substitution & find $\frac{du}{dx}$ <u>or</u> 'Set up' integration by parts. M: Get <u>all</u> in terms of 'u' <u>or</u> Use integration by parts. M: Sound integration. M: Substitute both limits (for the correct variable) and subtract.	

Question number	Scheme	Marks
7.	Alternative solution:	
	Let $x = \sinh\theta$ $\quad \frac{\mathrm{d}x}{\mathrm{d}\theta} = \cosh\theta$	M1
	$\int x^2 \operatorname{ar}\sinh x \,\mathrm{d}x = \int \theta \sinh^2\theta \cosh\theta \,\mathrm{d}\theta$	M1
	$= \left[\frac{\theta\sinh^3\theta}{3}\right] - \int \frac{1}{3}\sinh\theta(\cosh^2\theta - 1)\,\mathrm{d}\theta$	M1 A1 A1
	$\left[\frac{\theta\sinh^3\theta}{3}\right]_0^{\operatorname{arsinh}3} = 9\operatorname{arsinh}3$	B1
	$\int \frac{1}{3}\sinh\theta(\cosh^2\theta - 1)\,\mathrm{d}\theta = \frac{1}{3}\left[\frac{\cosh^3\theta}{3} - \cosh\theta\right]$	M1
	$\frac{1}{3}\left[\frac{\cosh^3\theta}{3} - \cosh\theta\right]_0^{\operatorname{arsinh}3} = \frac{1}{3}\left\{\left(\frac{10\sqrt{10}}{3} - \sqrt{10}\right) - \left(\frac{1}{3} - 1\right)\right\}$	M1 A1
	Area $= 9\operatorname{arsinh}3 - \frac{1}{3}\left(\frac{7\sqrt{10}}{3} + \frac{2}{3}\right) = 9\ln(3+\sqrt{10}) - \frac{1}{9}(7\sqrt{10} + 2)$ (*)	A1cso (10)
		(10)

Question number	Scheme	Marks
7.	A few alternatives for: $\int \frac{x^3}{\sqrt{x^2+1}}\mathrm{d}x$.	
	(i) Let $u = x^2$ $\quad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$	
	$\int \frac{u^{\frac{3}{2}}}{\sqrt{1+u}} \cdot \frac{1}{2u^{\frac{1}{2}}}\mathrm{d}u = \frac{1}{2}\int \frac{u}{\sqrt{1+u}}\mathrm{d}u$	
	No marks yet… needs another substitution, or parts, or perhaps…	
	$\frac{u}{\sqrt{1+u}} = \sqrt{1+u} - \frac{1}{\sqrt{1+u}}$	M1
	$\frac{1}{2}\int \sqrt{1+u}\,\mathrm{d}u - \frac{1}{2}\int \frac{1}{\sqrt{1+u}}\mathrm{d}u$	M1
	$\frac{1}{3}(1+u)^{\frac{3}{2}} - (1+u)^{\frac{1}{2}}$	M1
	Limits (0 to 9)	M1
	(ii) Let $x = \sinh\theta$ $\quad \frac{\mathrm{d}x}{\mathrm{d}\theta} = \cosh\theta$	M1
	$\int \frac{\sinh^3\theta}{\cosh\theta} \cdot \cosh\theta\,\mathrm{d}\theta = \int \sinh\theta(\cosh^2\theta - 1)\mathrm{d}\theta$	M1
	Then, as in the alternative solution,	
	$\int \frac{1}{3}\sinh\theta(\cosh^2\theta - 1)\mathrm{d}\theta = \frac{1}{3}\left[\frac{\cosh^3\theta}{3} - \cosh\theta\right]$	M1
	Limits (0 to arsinh3)	M1

Question number		Scheme	Marks
7.	(iii)	Let $u = \tan\theta \quad \frac{du}{d\theta} = \sec^2\theta$	M1
		$\int \frac{\tan^3\theta}{\sec\theta}\cdot\sec^2\theta\,d\theta = \int \tan\theta\sec\theta(\sec^2\theta - 1)d\theta$	M1
		$= \int \sec^2\theta(\sec\theta\tan\theta)d\theta - \int(\sec\theta\tan\theta)d\theta = \frac{\sec^3\theta}{3} - \sec\theta$	M1
		Limits $(\sec\theta = 1 \text{ to } \sec\theta = \sqrt{10})$	M1
	(iv)	(By parts… must be the 'right way round', not integrating x^2)	
		$u = x^2,\ \frac{du}{dx} = 2x \qquad \frac{dv}{dx} = \frac{x}{\sqrt{1+x^2}},\ v = \sqrt{1+x^2}$	M1
		$x^2\sqrt{1+x^2} - \int 2x\sqrt{1+x^2}\,dx$	M1
		$x^2\sqrt{1+x^2} - \frac{2}{3}(x^2+1)^{\frac{3}{2}}$	M1
		Limits	M1
	(v)	(By parts)	
		$u = x^3,\ \frac{du}{dx} = 3x^2 \qquad \frac{dv}{dx} = \frac{1}{\sqrt{1+x^2}},\ v = \text{arsinh}\,x$ No progress	M0
	(vi)	$\frac{x^3}{\sqrt{1+x^2}} = \frac{x(x^2+1) - x}{\sqrt{1+x^2}} = \frac{x(x^2+1)}{\sqrt{1+x^2}} - \frac{x}{\sqrt{1+x^2}}$	M1
		$\int x\sqrt{1+x^2}\,dx - \int \frac{x}{\sqrt{1+x^2}}dx$	M1
		$= \frac{1}{3}(1+x^2)^{\frac{3}{2}} - (1+x^2)^{\frac{1}{2}}$	M1
		Limits	M1

Question number	Scheme	Marks
8.	(a) $\int x^n \cosh x\,dx = x^n \sinh x - \int nx^{n-1} \sinh x\,dx$	M1 A1
	$= x^n \sinh x - nx^{n-1}\cosh x + n(n-1)\int x^{n-2}\cosh x\,dx$	M1
	$I_n = x^n \sinh x - nx^{n-1}\cosh x + n(n-1)I_{n-2}$ (*)	A1 (4)
	(b) $I_4 = x^4 \sinh x - 4x^3 \cosh x + 12I_2$	M1
	$I_4 = x^4 \sinh x - 4x^3 \cosh x + 12\left(x^2 \sinh x - 2x\cosh x + 2I_0\right)$	M1
	(This M may also be scored by finding I_2 by integration.)	
	$I_0 = \int \cosh x\,dx = \sinh x + k$	B1
	$I_4 = \left(x^4 + 12x^2 + 24\right)\sinh x,\ + \left(-4x^3 - 24x\right)\cosh x \quad (+C)$	A1, A1 (5)
	(c) $\left[\left(x^4 + 12x^2 + 24\right)\sinh x + \left(-4x^3 - 24x\right)\cosh x\right]_0^1$	
	$= 37\sinh 1 - 28\cosh 1$ M: $x = 1$ substituted throughout (at some stage)	M1
	$= 37\left(\frac{e - e^{-1}}{2}\right) - 28\left(\frac{e + e^{-1}}{2}\right)$	M1
	M: Use of exp. Definitions (can be in terms of x)	
	$= \frac{1}{2}\left(9e - 65e^{-1}\right)$	A1 (3)
		12
	(b) Integration constant missing <u>throughout</u> loses the B mark.	

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<tr><th>Question number</th><th>Scheme</th><th>Marks</th></tr>
<tr><td>9.</td><td>(a) $\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$ $b^2x^2+a^2(mx+c)^2=a^2b^2$
$(b^2+a^2m^2)x^2+2a^2mcx+a^2(c^2-b^2)=0$ (*)
(b) $(2a^2mc)^2=4(b^2+a^2m^2)a^2(c^2-b^2)$
$4a^4m^2c^2=4a^2(b^2c^2-b^4+a^2m^2c^2-a^2m^2b^2)$
$c^2=b^2+a^2m^2$ (*)
(c) Find height and base of triangle (perhaps in terms of c).
$OB=c\ \left(=\sqrt{b^2+a^2m^2}\right)$ and $AO=\frac{c}{m}\ \left(=\frac{\sqrt{b^2+a^2m^2}}{m}\right)$
Area of triangle OAB = $\frac{c^2}{2m}=\frac{b^2+a^2m^2}{2m}$ M: Find area and subs. for c.
(d) $\Delta=\frac{b^2+a^2m^2}{2m}=\frac{b^2}{2}m^{-1}+\frac{a^2}{2}m$
$\frac{d\Delta}{dm}=-\frac{b^2}{2}m^{-2}+\frac{a^2}{2}=0$ $\frac{b^2}{m^2}=a^2$ $m=\frac{b}{a}$
$\Delta=\left(\frac{b^2}{2}\right)\left(\frac{a}{b}\right)+\left(\frac{a^2}{2}\right)\left(\frac{b}{a}\right)=ab$ (*)
(e) Root of quadratic: $x=\frac{-a^2mc}{b^2+a^2m^2}$ (Should be <u>correct</u> if quoted directly)
Using $m=\frac{b}{a}$ and $c=\sqrt{b^2+a^2m^2}$: $x=-\frac{a}{\sqrt{2}}$
(The 2nd M is dependent on using the quadratic equation).</td><td>M1
A1 (2)
M1

A1 (2)
M1
A1
M1 A1 (4)

M1 A1
A1 (3)
M1
M1 A1 (3)

14</td></tr>
<tr><td></td><td>(d) <u>Alternative:</u> $b^2+a^2m^2\ge 2bam$ (since $(b-am)^2\ge 0$) [M1]
$\frac{b^2+a^2m^2}{2m}\ge ab$ [A1]   …Conclusion [A1]
(e) <u>Alternative:</u> Begin with full eqn. $(b^2+a^2m^2)x^2+2a^2mcx+a^2(c^2-b^2)=0$.
In the eqn., use conditions $m=\frac{b}{a}$ and $c=\sqrt{b^2+a^2m^2}$ $(=b\sqrt{2})$ [M1]
Simplify and solve eqn., e.g. $2x^2+2a\sqrt{2}x+a^2=0$ $x=-\frac{a}{\sqrt{2}}$</td><td></td></tr>
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