A crash course in C1 Maths

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Abstract

The aim of this piece of work is to summarise key points of the C1 syllabus. This is not a comprehensive document. Its possible that on the exam there will be some techniques not covered in these notes, but each of the topics listed here has been examined within the last 3 years.

Finally, whilst every effort has been made to ensure these notes are free of errors, they are not guaranteed to be so. If any errors are found, please email them to *errata@maths-tutors.com*.

1 C1 - What do we need to know?

In this module, you have studied the following areas of mathematics

1. Differentiation
2. Straight line graphs, tangents and normals
3. Completing the square and factorization
4. The quadratic equation and the discriminant
5. The points of intersection between quadratics and straight lines
6. Division of polynomials, the factor theorem and the remainder theorem
7. Integration
8. Circles

Each of these topics will be discussed in a seprate section with summaries of key formulae and some examples

2 Differentiation

It may seem like something of a suprise to find that the first topic to be discussed here is differentiation since this was almost certainly not the topic you studied first in class. However, of all the topic discussed in C1, none will be more important over the course of your A-Level since many later subjects build upon the foundations of calculus you discuss in C1.

Let us jump straight into the deep end. Formally, the derivative, $(f'(x))$, of a function, $f(x)$, at a point $x = x_0$ is given by

$$\left.\frac{df}{dx}\right|_{x=x_0} = f'(x_0) = \lim_{h\to 0}\left(\frac{f(x_0+h)-f(x_0)}{h}\right). \tag{1}$$

Geometrically, this formula is calculating the gradient of a curve by taking two point on a curve and calculating the gradient of the chord between them and then calculating what happens as these two points get closer together.

Using this formula is know as differentiation through first principles. We can use this method to *prove* the following rules:

1. $\frac{d}{dx}(c) = 0$, for any constant, c.
2. $\frac{d}{dx}(x) = 1$.
3. $\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$.
4. $\frac{d}{dx}(cf(x)) = c\frac{d}{dx}(f(x))$, for any function f and any constant c.
5. $\frac{d}{dx}(x^n) = nx^{n-1}$ for all $n \neq 0$.

The rules listed above are all that are required for differentiation in C1 — YOU NEED TO KNOW THEM!!!!!.

Example 1 *Let $f(x) = x^3$. Then using differentiation by first principles $f'(x_0)$ is given by*

$$\begin{aligned} f'(x_0) &= \lim_{h\to 0}\frac{(x_0+h)^3 - x_0^3}{h} && (2)\\ &= \lim_{h\to 0}\left(\frac{3x_0^2h + 3x_0h^2 + h^3}{h}\right) && (3)\\ &= \lim_{h\to 0}\left(3x_0^2 + 3x_0h + h^2\right) && (4)\\ &= 3x_0^2 && (5)\end{aligned}$$

Example 2 *Let $f(x) = x^3$, then using the rules derived from first principles, listed above, we get*

$$\frac{d}{dx}x^3 = 3x^2$$

We see immediately that it is far easier to differentiate using the list of rules rather than to use the method of first principles.

Question 1 *These questions are for the reader to try.*

- *Differentiate the following functions using first principles*
- *Differentiate the same functions using the list of rules above*

1. $f(x) = x$

2. $f(x) = x^2$

3. $f(x) = 2x^2 + 6x + 2$

4. $f(x) = \frac{1}{x}$

2.1 Stationary points

There are three types of stationary points that A-Level students need to be aware of. All of which occur when $\frac{dy}{dx} = 0$.

- Maximum
- Minimum
- A point of inflection

The type of stationary point can be determined by calculating the value of the second derivative $\frac{d^2y}{dx^2}$ at the stationary point.

1. $\frac{d^2y}{dx^2} < 0$, then the stationary point is a maximum.
2. $\frac{d^2y}{dx^2} > 0$, then the stationary point is a minimum.
3. $\frac{d^2y}{dx^2} = 0$, then the stationary point is a point of inflection.

Example 3 *Let $y = -x^2 + 6x + 8$. Then $\frac{dy}{dx} = -2x + 6$. We see that the function has a stationary point when $\frac{dy}{dx} = 0$ which gives $-2x + 6 = 0$. We see that the turning point occurs when $x = 3$. The second derivative is given by*

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-2x + 6) = -2 < 0$$

Example 4 *Let $y = x^3 - 6x^2 + 9x - 12$. Then $\frac{dy}{dx} = 3x^2 - 12x + 9$. We see that the function has a stationary point when $\frac{dy}{dy} = 0$. This gives $3x^2 - 12x + 9 = 0$. We get that the turning points occur when $x^2 - 4x + 3 = 0$. Solving this quadratic gives*

$$x = \frac{4 \pm \sqrt{4^2 - 4 * 3}}{2} = \frac{4 \pm 2}{2} = 2 \pm 1 = 1 \text{ or } 3$$

To work out what type of turning point each x value corresponds to, we take the second derivative.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(3x^2 - 12x + 9\right) = 6x - 12.$$

When $x = 1$, we get $\frac{d^2y}{dx^2} = -6 < 0$ therefore we have a maximum.
When $x = 3$, we get $\frac{d^2y}{dx^2} = 6 > 0$ therefore we have a minimum.

Example 5 *Let $y = x^3 - 3x^2 + 3x - 7$. Then $\frac{dy}{dx} = 3x^2 - 6x + 3$. We see that the function has a stationary point when $\frac{dy}{dy} = 0$. This gives $3x^2 - 6x + 3 = 0$. We get that the turning points occur when $x^2 - 2x + 1 = 0$. Solving this quadratic gives a single solution, $x = 1$.*

To work out what type of turning point corresponds to $x = 1$, we take the second derivative.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(3x^2 - 6x + 3\right) = 6x - 6.$$

When $x = 1$, we get $\frac{d^2y}{dx^2} = 6 \times 1 - 6 = 0$. This gives a point of inflection. (draw the graph!)

Question 2 *Calculate the x-values and hence the coordinates of the stationary points for the following curves.*

1. $y = x^3 + 10$
2. $y = x^3 + 9x^2 + 7x + 15$
3. $y = x^5 - x^3$
4. $y = 12 - x^2$

Using the second derivative, identify the type of each stationary point found above.

3 Straight line graphs, tangents and normals

A straight line can be represented by an equation of the form

$$y = mx + c$$

where c is the y-intercept, and m is the gradient.

Given two points, (x_0, y_0) and (x_1, y_1), we can construct the straight line linking them.

We first get the gradient, m by using the formula

$$m = \frac{y_1 - y_0}{x_1 - x_0}.$$

Notice that this formula is simply the change in y divided by the change in x and is the prototype for differentiation.

We then need to calculate the y-intercept. To do this, we use the formula

$$y - y_0 = m(x - x_0)$$

and then rearrange to give

$$y = mx - mx_0 + y_0$$

consequently, $c = y_0 - mx_0$.

Example 6 *Calcualate the equation of the line linkning (1,2) and (4,8).*

We get the gradient is given by

$$m = \frac{8-2}{4-1} = 3.$$

The y-intercept c is given by

$$c = 2 - (3 \times 1) = -1$$

so the equation of the line is

$$y = 3x - 1$$

Question 3 *Calculate the equation of the lines through the following pairs of points*

1. $(1, 2)$ *and* $(12, 6)$
2. $(10, 10)$ *and* $(-4, 3)$
3. $(100, 50)$ *and* $(0, 0)$
4. $(0, 0)$ *and* $(10, 11)$
5. $(3, -4)$ *and* $(6, 8)$

3.1 Normal line

A normal line is a line that is 90^o to a tangent line. The gradient of the normal is given by $n = -\frac{1}{m}$.

$$n = \frac{x_0 - x_1}{y_1 - y_0}.$$

$$\begin{aligned} y - y_0 &= n(x - x_0) & (6) \\ y &= nx - nx_0 + y_0 & (7) \\ && (8) \end{aligned}$$

This gives a y-intercept of

$$c = -\frac{x_0 - x_1}{y_1 - y_0} x_0 + y_0 \qquad (9)$$

Example 7 *Calculate the line normal to the line $y = 3x - 12$ at the point $A = (4, 0)$.*

The gradient of the normal line is given by $\frac{-1}{m} = \frac{-1}{3}$. Therefore the equation is given by

$$y = \frac{-1}{3}x + c.$$

We need to work out the value of c. To do so we plug in the values of the point $A = (4, 0)$. This gives

$$0 = -\frac{4}{3} + c$$

so finally the equation of the normal line is given by

$$y = \frac{-1}{3}x + \frac{4}{3}.$$

Question 4 *Calculate the normals to the following lines*

1. $y = 2x - 1$ *at the point* $A = (2, 3)$
2. $y = -x + 4$ *at the point* $A = (3, 1)$
3. $y = 10x - 26$ *at the point* $A = (3, 4)$

3.2 Curves, Tangents and Normals

Example 8 *A curve is defined by the equation $y = \sqrt{x}$ calculate both the tangent and the normal to this curve at $x = 4$.*

The Solution We first calculate the coordinates of the point on the curve at $x = 4$.

$$x = 4 \Rightarrow y = \sqrt{4} = 2 \Rightarrow (4, 2) \text{ is on the curve.}$$

Next we differentiate the equation $y = \sqrt{x} = x^{1/2}$.

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}$$

To calculate the gradient of the tangent line, we evaluate $\frac{dy}{dx}$ at $x = 4$

$$\left.\frac{dy}{dx}\right|_{x=4} = \frac{1}{2} \times \frac{1}{\sqrt{4}} = \frac{1}{4}.$$

The tangent line is given by

$$y = \frac{1}{4}x + c,$$

substituting the point $(4, 2)$ gives

$$\begin{aligned} 2 &= \frac{1}{4} \times 4 + c \\ c &= 1. \end{aligned}$$

This gives the final equation of the tangent as

$$y = \frac{1}{4}x + 1.$$

Now that we have the tangent line, it is possible to calculate the normal line. The gradient of the normal is given by $\frac{-1}{m} = -4$. This gives a normal line with equation

$$y = -4x + c$$

substituting in the point (4,2) gives us the value of c.

$$2 = -4 \times 4 + c \Rightarrow c = 18.$$

This gives us the normal line

$$y = -4x + 18$$

Question 5 *Calculate the tangents and normals to the following curves*

1. $y = x^3 + 4x^2 - 12x - 2$ *at* $x = 2$.
2. $y = \frac{-8}{x^3} + \frac{2}{x}$ *at* $x = 1$.
3. $y = 6x^2 - 18x$ *at* $x = 4$

4 Completing the square and factorization

4.1 Completing the square

The aim of completing the square is to rewrite a quadratic equation $y = ax^2 + bx + c$ in the form $y = a(x-p)^2 - q$. The method of rewriting is very simple. All of the transformations are performed on the terms $ax^2 + bx$ and we simply carry the constant c along for the ride. Consider the following example

$$\begin{aligned} x^2 + 10x - 40 &= (x^2 + 10x) - 40 && (10)\\ &= \left((x+5)^2 - 5^2\right) - 40 && (11)\\ &= (x+5)^2 - (5^2 + 40) && (12)\\ &= (x+5)^2 - 65 && (13) \end{aligned}$$

We see that it is ALWAYS possible to write $ax^2 + bx$ in the following manner

Example 9

$$\begin{aligned} ax^2 + bx &= a\left(x^2 + \frac{b}{a}x\right) && (14)\\ &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} && (15) \end{aligned}$$

Let us look at a series of examples.

Example 10

$$\begin{aligned} x^2 + 4x &= (x+2)^2 - 2^2 && (16)\\ x^2 - 4x &= (x-2)^2 - 2^2 && (17)\\ x^2 + 6x &= (x+3)^2 - 3^2 && (18)\\ x^2 - 6x &= (x-3)^2 - 3^2 && (19)\\ 10x^2 - 60x &= 10(x-3)^2 - \frac{60^2}{40} && (20) \end{aligned}$$

We have seen that it is always possible to complete the square on expressions of the form

$$ax^2 + bx$$

It should be clear then that this means we can complete the square on *any* quadratic since we can just add a constant at the end to take care of the c.

Question 6 *Complete the square on each of the following expressions. Where possible, solve the quadratic. When it is not possible to solve the quadratic state why.*

1. $x^2 + 6x - 4 = 0$
2. $-x^2 - 12x + 16 = 0$
3. $4x^2 - 24x + 32 = 0$
4. $9x^2 - 27x = 0$
5. $2x^2 + 2x + 2 = 0$
6. $4x^2 + 8x - 4 = 0$

4.2 Factorization

The aim of factorization is to express a quadratic equation $y = x^2 + bx + c$ as the product of two linear factors

$$x^2 + bx + c = (x + p)(x + q)$$

It is important to remember that when factorizing a polynomial, the two numbers, p and q that we are seeking satisfy

$$\begin{aligned} c &= p \times q && (21) \\ b &= p + q && (22) \\ & && (23) \end{aligned}$$

The method of factorization requires guesswork, and will only work quickly if the solutions of the equation are "nice". Essentially, factorization should only be used in the following two scenarios:

1. The question you are doing *explicitly* states that you are to factorize.
2. You happen, through merely glancing at a question, to spot the factorization.

Example 11

$$x^2 + 9x + 18 = 0$$

We work out the possible factorizations of 18 : $(18, 1), (9, 2), (6, 3)$. We then calculate the sums of these to pairs to see if they equal 9.

$$18 + 1 = 19, \;\; 9 + 2 = 11, \;\; 6 + 3 = 9$$

Question 7 *Factorize and hence solve the following quadratics*

1. $x^2 + x - 2 = 0$
2. $x^2 + 11x - 60 = 0$
3. $x^2 + 50x + 96 = 0$
4. $2x^2 - 7x - 4 = 0$
5. $12x^2 - 14x - 6 = 0$
6. $3x^2 + 9x + 6 = 0$

5 The quadratic equation and the discriminant

Given a quadratic polynomial,

$$y = ax^2 + bx + c$$

$$\begin{aligned} ax^2 + bx + c &= a(x^2 + \frac{b}{a}x) + c && (24) \\ &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c && (25) \end{aligned}$$

We can use this completed the square form to solve the quadratic.

$$\begin{aligned} ax^2 + bx + c &= 0 && (26) \\ a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c &= 0 && (27) \\ a\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a} - c && (28) \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} && (29) \\ x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} && (30) \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. && (31) \end{aligned}$$

We can use this to solve any quadratic. For example,

$$x^2 + 10x - 24 = 0$$

for this polynomial, the coefficients a, b, c are given by

$$a = 1,\ b = 10,\ c = -24$$

$$\begin{aligned} x &= \frac{-10 \pm \sqrt{100 + 96}}{2} && (32) \\ &= -5 \pm \sqrt{49} && (33) \\ &= -5 \pm 7 && (34) \end{aligned}$$

This gives $x = -12$ or $x = 2$

then the discriminant is given by

$$b^2 - 4ac.$$

The value of the discriminat determines the number of real roots of the polynomial.

$$\begin{aligned} b^2 - 4ac &< 0 \Rightarrow && \text{No solutionts} && (35) \\ b^2 - 4ac &= 0 \Rightarrow && \text{A single solution} && (36) \\ b^2 - 4ac &> 0 \Rightarrow && \text{Two solutions} && (37) \end{aligned}$$

Question 8 *How many roots do the following polynomials have?*

1. $x^2 - 2x + 2$

2. $x^2 + 2x + 1$

3. $4x^2 - 6x + 16$

4. $4x^2 + 8x - 16$

5. $-2x^2 + 2x - 8$

6. $-3x^2 - 18x + 9$

for those polynomials with one or more roots, use the quadratic formula to solve them.

6 The points of intersection between quadratics and straight lines

Applications of the discriminant has been examined alot in recent years. A standard way in which this has been done is by asking students to calculate the intersection points betwen a line and a quadratic. These questions have tended to involve a parameter m. Consider the following example

Example 12 *The equation $y = mx + 1$ describes a line of gradient m with y-intercept, 1. The equation $y = x^2 - 2x + 2$ describes a quadratic.*

1. *Calculate the value of m for which there is a single point of intersection between the line and curve.*

2. *write down the point of intersection for this value of m*

3. *explain the geometric consequences of the single point of intersection*

To solve this question, we first note that the x-value of any point of intersection are governed by

$$mx + 1 = x^2 - 2x + 2$$

Rearranging this equation, we get

$$x^2 - (2 + m) + 1 = 0 \tag{38}$$

As a consequence, we see that the x-value of any point of intersection between the line and curve is a solution of equation (38) Therefore, if there is to exist only a single point of intersection between the line and the curve, then the equation given in (38) must have a single solution.

If the equation 38 has a single solution, then its discriminant must equal 0. *This piece of information tells us that the parameter m satisfies the equation*

$$b^2 - 4ac = (2 + m)^2 - 4 = 0.$$

Expanding this gives us

$$m^2 + 4m = 0$$

and consequently, we find that if $m = 0$ or $m = -4$, then the line and curve have a single point of intersection.

To rewrite this in a more meaningful way, we see that the line $y = -4x + 1$ *and the quadratic curve* $y = x^2 - 2x + 2$ *have a single point of intersection and we see that the line* $y = 1$ *and the quadratic curve* $y = x^2 - 2x + 2$ *have a single point of intersection.*

For the second part of the question, we substitute our values of m *back into the equation 38 to give us*

1. *In the case where* $m = 0$,

$$x^2 - 2x + 1 = 0$$

   Solving this quadratic gives us an x-value of $x = 1$ *(exactly one solution as required). The point of interection is* $(1, 1)$ *since the line was given by the equation* $y = 1$.

2. *In the case where* $m = -4$,

$$x^2 + 2x + 1 = 0$$

   Solving this quadratic gives us an x-value of $x = -1$ *(again exactly one solution as expected). The point of intersection in this case is given by (-1,5) since the line was given by the equation* $y = -4x + 1$.

Geometrically, these two lines are in fact both tangents to the quadratic curve. REMEMBER THIS ANSWER.

7 Division of polynomials, the factor theorem and the remainder theorem

In this section we discuss how to "divide" polynomials and we see some of the consequences of this algorithmic method of division. In each case, we will only ever be dividing by a linear polynomial. (Although the method works in general).

Suppose we wish to divide the polynomial $P(z) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$ by the linear polynomial $Q(z) = b_1 z + b_0$, then we proceed as follows...

1. Work out what the highest term of $P(z)$ is divided by the highest term of $Q(z)$ and call the coefficient of this term m_{n-1}. ie $\frac{a_n x^n}{b_1 z} = \frac{a_n}{b_1} x^{n-1} = m_{n-1} x^{n-1}$
2. Work out $Q(z) \times m_{n-1} x^{n-1}$
3. Subtract from $P(z)$. ie. work out $P_{New}(z) = P(z) - m_{n-1} x^{n-1}$.
4. Repeat.

This will become much clearer with a few examples.

$$P(x) = x^3 + 6x^2 + 11x + 6,\ Q(x) = x + 1$$

On the first execution of the algorithm, we get that $x^3/x = x^2$. This term goes on the top line of the table (just like in the division of numbers). We then multiply $x + 1$ by x^2. This term then goes into the third line of the table

	x^2							Comments
$x+1$	x^3	$+$	$6x^2$	$+$	$11x$	$+$	6	$\frac{x^3}{x} = x^2$
	x^3	$+$	x^2					$Q(x) \times x^2$
	$0x^3$	$+$	$5x^2$	$+$	$11x$	$+$	6	$P(x) - (Q(x) \times x^2) = P_1(x)$

We now repeat this process with the final line of the table above now playing the role of $P(x)$.

			$5x$			Comments
$x+1$	$5x^2$	$+$	$11x$	$+$	6	$\frac{5x^2}{x} = 5x$
	$5x^2$	$+$	$5x$			$Q(x) \times 5x$
	$0x^2$	$+$	$6x$	$+$	6	$P_1(x) - (Q(x) \times 5x) = P_2(x)$

Again, because the final line is not just a constant, we repeat the process, with the term $6x + 6$ now playing the role of $P(x)$.

			6	Comments
$x+1$	$6x$	$+$	6	$\frac{6x}{x} = 6$
	$6x$	$+$	6	$Q(x) \times 6$
	$0x$	$+$	0	$P_2(x) - (Q(x) \times 6) = 0$

At this point we stop at the final answer is just a constant (0 in this case). Normally, all of this calculation is done in a single table

								Comments
			x^2	$+$	$5x$	$+$	6	
$x+1$	x^3	$+$	$6x^2$	$+$	$11x$	$+$	6	$\frac{x^3}{x} = x^2$
	x^3	$+$	x^2					$Q(x) \times x^2$
	$0x^3$	$+$	$5x^2$	$+$	$11x$	$+$	6	$P(x) - (Q(x) \times x^2) = P_1(x)$
$x+1$			$5x^2$	$+$	$11x$	$+$	6	$\frac{5x^2}{x} = 5x$
			$5x^2$	$+$	$5x$			$Q(x) \times 5x$
			$0x^2$	$+$	$6x$	$+$	6	$P_1(x) - (Q(x) \times 5x) = P_2(x)$
$x+1$					$6x$	$+$	6	$\frac{6x}{x} = 6$
					$6x$	$+$	6	$Q(x) \times 6$
					$0x$	$+$	0	$P_2(x) - (Q(x) \times 6) = 0$

As we have got 0 as an answer, then we can conclude that

$$Q(x) \times (x^2 + 5x + 6) = P(x) \Rightarrow P(x)/Q(x) = x^2 + 5x + 6.$$

Let us try another example.

$$P(x) = x^3 + 7x^2 + 14x + 8,\ Q(x) = x + 1$$

								Comments
			x^2	$+$	$6x$	$+$	8	
$x+1$	x^3	$+$	$7x^2$	$+$	$14x$	$+$	8	$\frac{x^3}{x} = x^2$
	x^3	$+$	x^2					$Q(x) \times x^2$
	$0x^3$	$+$	$6x^2$	$+$	$14x$	$+$	6	$P(x) - (Q(x) \times x^2) = P_1(x)$
$x+1$			$6x^2$	$+$	$14x$	$+$	6	$\frac{6x^2}{x} = 6x$
			$6x^2$	$+$	$6x$			$Q(x) \times 6x$
			$0x^2$	$+$	$8x$	$+$	6	$P_1(x) - (Q(x) \times 6x) = P_2(x)$
$x+1$					$8x$	$+$	6	$\frac{8x}{x} = 8$
					$8x$	$+$	8	$Q(x) \times 8$
					$0x$	$+$	-2	$P_2(x) - (Q(x) \times 6) = -2$

We see that the final term here is again a constant. This constant is called the *remainder*. If the remainder is zero, then $Q(x)$ divides $P(x)$ *exactly*. If the remainder is non-zero, then $Q(x)$ does not divide $P(x)$ *exactly*.

Question 9 *Calculate the remainder for each of the following pairs of polynomials. In each case state whether $Q(x)$ divides $P(x)$ exactly.*

$Q(x)$	$P(x)$
$x+1$	x^2+2x+1
$x-1$	x^2+0x-1
$x+2$	$x^4+3x^3+3x^2+6x+4$
$x-10$	$x^3+11x^2+11x+10$

8 Integration

The following rules are all that are required for integration in C1.

1. $\int 0dx = c$
2. $\int 1dx = x + c$
3. $\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx$
4. $\int cf(x)dx = c\int f(x)dx$
5. $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$

Integration can be thought of in two ways, either as the opposite operation to differentiation, or as a method of calculating areas underneath curves. In some sense, indefinite integration covers the first case (since we expect to get a function), and definite integrals cover the second (since area is represented by a numeric value).

8.1 Indefinite Integration

When no bounds are specified on the integral ($\int$), then we say we have an indefinate integral. *The answer we expect for an indefinite integral is a function.* To calculate an indefinite integral, we simply apply the rules listed above. For example

Example 13

$$\begin{aligned}\int 4x^2+3x+2dx &= \int 4x^2dx + \int 3xdx + \int 2dx \text{ \textit{(by rule 3)}}\\ &= 4\int x^2dx + 3\int xdx + 2\int 1dx \text{ \textit{(by rule 4)}}\\ &= 4\frac{x^3}{3} + 3\frac{x^2}{2} + 2x + c \text{ \textit{(by rules 2 and 5)}}\end{aligned}$$

Every integral you are faced with in C1 can be dealt with through applying these rules. Admittedly, you may need to use the rules of indices first to get the question into an appropriate form.

Example 14

$$\begin{aligned}
\int \sqrt{25x} + \frac{6}{x^2} dx &= \int \sqrt{25x} dx + \int \frac{6}{x^2} dx \\
&= \int 5\sqrt{x} dx + 6 \int \frac{1}{x^2} dx \\
&= 5 \int x^{1/2} dx + 6 \int x^{-2} dx \\
&= 5\frac{x^{3/2}}{3/2} + 6\frac{x^{-1}}{-1} + c \\
&= \frac{10}{3} x^{3/2} - 6x^{-1} + c
\end{aligned}$$

8.2 Definite Integrals

Suppose we are given a function $f(x)$ and we are told we want to work out the curve under the graph $y = f(x)$ between $x = a$ and $x = b$. Then we can formulate this question as an integral.

$$\text{Area} = \int_a^b f(x) dx$$

To solve this problem, we "pretend" that the limits are not there, and work out what the indefinite integral is, calling the answer $F(x)$.

$$F(x) = \int f(x) dx$$

We then evaluate this new function $F(x)$, at $x = a$ and $x = b$. The area is then given by

$$\text{Area} = F(b) - F(a)$$

This is sometimes written as

$$\text{Area} = [F(x)]_a^b$$

Notice that this will simply give us a numerical answer as we would expect.

Example 15

$$\begin{aligned}
\int_1^3 4x^2 + 3x + 2dx &= \left[4\frac{x^3}{3} + 3\frac{x^2}{2} + 2x\right]_1^3 \\
&= \left(4\frac{3^3}{3} + 3\frac{3^2}{2} + 2 \times 3\right) - \left(4\frac{1^3}{3} + 3\frac{1^2}{2} + 2\right) \\
&= \left(36 + \frac{27}{2} + 6\right) - \left(\frac{4}{3} + \frac{3}{2} + 2\right) \\
&= 50\frac{2}{3} \qquad (39)
\end{aligned}$$

Question 10 *Calculate the following indefinite integrals.*

1. $\int_0^{10} x^2 - xdx$
2. $\int_2^4 \sqrt{t^3} + t^2 dt$
3. $\int_0^{\pi} x - \frac{x^3}{3} + \frac{x^5}{5} dx$
4. $\int_{-1}^{1} x^6 - x^4 + x^2 dx$
5. $\int_0^5 5dx$

9 Circles

The equation of a circle can be expressed in the form

$$(x-a)^2 + (y-b)^2 = r^2 \tag{40}$$

It is easy to draw the circle when it is expressed in this form. We get the centre and radius of the cirlce from this equation:

$$\text{Centre} = (a, b)$$

$$\text{Radius} = r$$

Question 11 *Write down the centre and radius for the following circles*

$$\begin{aligned}
(x-5)^2 + (y-7)^2 &= 6^2 && (41)\\
(x-12)^2 + (y-17)^2 &= 144 && (42)\\
(x-2)^2 + y^2 &= 1 && (43)\\
(x+2)^2 + y^2 &= 1 && (44)\\
(x+10)^2 + (y+10)^2 &= 15 && (45)
\end{aligned}$$

Usually, however, the examiners are not so kind as to present the circle in such a simple manner. Instead, the formula is presented in the form

$$ax^2 + bx + c + dy^2 + ey = f \tag{46}$$

and the student is expected to rewrite this in the same form as is given in equation (40). To acheive this, we need to complete the square, *twice.* We rewrite the equation (46) as

$$(ax^2 + bx) + (dy^2 + ey) = f - c$$

and then complete the square for the x's and the y's. This then gives us an equation similar to that in (40). (*Note that if $a \neq d$ then in fact we could have an ellipse or hyperbola - these are discussed further in C3 and C4*)

Question 12 *Work out the centre and radius for each of the following circles.*

1. $x^2 - 4x + y^2 - 3x = 10$
2. $x^2 + 6x + 10 + y^2 + 12y = -18$
3. $x^2 + 8x + y^2 + 4y = 2$
4. $x^2 - 32x + y^2 - 64x = -100$