General Certificate of Education
January 2005
Advanced Level Examination

MATHEMATICS (SPECIFICATION A) **MAP4**
Unit Pure 4

Friday 21 January 2005 Afternoon Session

In addition to this paper you will require:
- an 8-page answer book;
- the AQA booklet of formulae and statistical tables.

You may use a standard scientific calculator **only**.

Time allowed: 1 hour 20 minutes

Instructions
- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MAP4.
- Answer **all** questions.
- All necessary working should be shown; otherwise marks for method may be lost.
- The **final** answer to questions requiring the use of tables or calculators should normally be given to three significant figures.
- Tie loosely any additional sheets you have used to the back of your answer book before handing it to the invigilator.

Information
- The maximum mark for this paper is 60.
- Mark allocations are shown in brackets.

Advice
- Unless stated otherwise, formulae may be quoted, without proof, from the booklet.

Answer **all** questions.

1 The cubic equation

$$x^3 + px^2 + qx + 30 = 0\,,$$

where p and q are real, has a root $\alpha = 1 + 2\mathrm{i}$.

(a) Write down the other non-real root, β, of the equation. *(1 mark)*

(b) Find:

(i) $\alpha\beta$; *(1 mark)*

(ii) the third root, γ, of the equation. *(2 marks)*

(c) Hence, or otherwise, find the values of p and q. *(3 marks)*

2 (a) Show that

$$r^2(r+1)^2 - (r-1)^2 r^2 = 4r^3.$$ *(2 marks)*

(b) Use the method of differences to find the value of $\sum_{r=50}^{100} r^3$. *(4 marks)*

3 (a) Express $1 + \mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *(2 marks)*

(b) Show that

$$(1+\mathrm{i})^{21} - (1-\mathrm{i})^{21} = k\mathrm{i}\,,$$

where k is an integer to be found. *(5 marks)*

4 (a) On the **same** Argand diagram, sketch the loci of points satisfying:

(i) $|z + 3 + \mathrm{i}| = 5$; *(2 marks)*

(ii) $\arg(z + 3) = -\frac{3}{4}\pi$. *(2 marks)*

(b) (i) From your sketch, explain why there is only one complex number satisfying both equations. *(1 mark)*

(ii) Verify that this complex number is $-7 - 4\mathrm{i}$. *(4 marks)*

5 (a) (i) Use the definitions $\cosh x = \frac{1}{2}(e^x + e^{-x})$ and $\sinh x = \frac{1}{2}(e^x - e^{-x})$ to show that the equation

$$p \cosh x + q \sinh x = r,$$

may be written as

$$(p+q)e^{2x} - 2re^x + (p-q) = 0.$$

(3 marks)

(ii) Given that p, q and r are positive and that

$$p^2 = q^2 + r^2,$$

show that the equation

$$p \cosh x + q \sinh x = r$$

has just one solution. *(5 marks)*

(b) Find the solution of the equation

$$13 \cosh x + 5 \sinh x = 12,$$

giving your answer in the form $\ln k$, where k is a rational number. *(2 marks)*

6 (a) Given that

$$f(n) = 4 \times 7^n + 3 \times 5^n + 5,$$

show that

$$f(n+1) - f(n) = 24 \times 7^n + 12 \times 5^n.$$

(4 marks)

(b) Prove by induction that

$$4 \times 7^n + 3 \times 5^n + 5$$

is a multiple of 12 for all $n \geqslant 1$. *(4 marks)*

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7 (a) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sinh^{-1} x + x\sqrt{1+x^2}\right) = 2\sqrt{1+x^2}. \qquad \textit{(4 marks)}$$

(b) The arc of the curve $y = \mathrm{e}^x$ between the points where $x = \ln\left(\frac{3}{4}\right)$ and $x = \ln\left(\frac{4}{3}\right)$ is rotated through 2π radians about the x-axis.

(i) Show that S, the surface area generated, is given by

$$S = 2\pi \int_{\ln\left(\frac{3}{4}\right)}^{\ln\left(\frac{4}{3}\right)} \mathrm{e}^x \sqrt{1+\mathrm{e}^{2x}}\,\mathrm{d}x\,. \qquad \textit{(2 marks)}$$

(ii) Use the substitution $u = \mathrm{e}^x$ to evaluate S, giving your answer in the form

$$\pi(a + \ln b),$$

where a and b are rational numbers. *(7 marks)*

END OF QUESTIONS