Teacher Support Materials 2009

Maths GCE

Paper Reference MPC4

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Question 1

(a) Use the Remainder Theorem to find the remainder when $3x^3 + 8x^2 - 3x - 5$ is divided by $3x - 1$. *(2 marks)*

(b) Express $\dfrac{3x^3 + 8x^2 - 3x - 5}{3x - 1}$ in the form $ax^2 + bx + \dfrac{c}{3x-1}$, where a, b and c are integers. *(3 marks)*

Student Response

Question number

Leave blank

1 (a)

$$\begin{array}{r} x^2 + 3x \\ 3x-1 \overline{)\, 3x^3 + 8x^2 - 3x - 5} \\ 3x^3 - x^2 \\ \hline 9x^2 - 3x \\ 9x^2 - 3x \\ \hline -5 \end{array}$$

division

M0 0

(b) $\dfrac{3x^3 + 8x^2 - 3x - 5}{3x - 1}$

$\Rightarrow Ax^2(3x-1) + Bx(3x-1) + C \Rightarrow 3x^3 + 8x^2 - 3x - 5$

$x = \frac{1}{3}$, $\boxed{C = -5}$ ✓, $\boxed{a = 2 \quad b = 4 \quad c = -5}$

B1

$x = 1$, $A2 + B2 - 5 = 3 + 8 - 3 - 5$

$A2 + B2 = 8$ ✓

M1

$x = 2$, $20A + 10 - 5 = 45$

$20A + 10B = 40$

A0 2

$A2 + B2 = 8$

$20A + 10B = 40$

$\times 10$ $A20 + B20 = 80$

$- \; A20 + 10B = 40$

$\Rightarrow B10 = 40$ $\boxed{A = 2}$

$\boxed{B = 4}$

2

Commentary

Part (a): The candidate has done this correctly by algebraic long division, but scored no marks. This is because the question asked for the use of the remainder theorem so they didn't do the question by the prescribed method. However, having done the division, they could now write down the answers to part (b) immediately, but they didn't realise this.

Part (b): The candidate chose to do this by clearing the fraction and substituting values of *x* into the equation. This is a longer method than long division, but valid. They could have equated coefficients, which would have given the values of *a* and *c* immediately, from the x^3 and constant terms respectively. Many candidates made an error in handling the algebra in using this type of approach. This candidate's method is correct in setting up the simultaneous equations, but they then made an arithmetic error in solving them. They scored the method mark for a correct approach. They found the value of *c* correctly and so scored the B mark, but apparently they didn't associate this with the value of the remainder from part (a).

Mark scheme

Q	Solution	Marks	Total	Comments
1(a)	$f\left(\frac{1}{3}\right)=3\times\frac{1}{27}+8\times\frac{1}{9}-3\times\frac{1}{3}-5$	M1		Use $\frac{1}{3}$ in evaluating $f(x)$
	$=-5$	A1	2	No ISW Evidence of Remainder Theorem
(b)	$3x-1\overline{)3x^3+8x^2-3x-5}$ quotient x^2+3x $3x^3 \quad -x^2$ $9x^2-3x$ $9x^2-3x$	M1		Division with x^2 and an x term seen; x^2+px
	$a=1 \quad b=3$ or $x^2+3x+\dfrac{c}{3x-1}$	A1		Explicit or in expression
	$c=-5$	B1		Condone $+\dfrac{-5}{3x-1}$
	Alternative $\dfrac{(3x-1)(x^2+px)}{3x-1}-\dfrac{5}{3x-1}$	(M1)		Split fraction and attempt factors
	$x^2+3x \qquad -\dfrac{5}{3x-1}$	(A1) (B1)		$a=1 \quad b=3$ $c=-5$
	Alternative $f(x)=3ax^3+(3b-a)x^2-bx+c$	(M1)		Multiply by $(3x-1)$ and attempt to collect terms
	$a=1 \quad b=3$ $c=-5$	(A1) (B1)		
	Alternative $f(x)=(ax^2+bx)(3x-1)+c$	(M1)		Multiply by $(3x-1)$ and attempt to find a, b, c: substitute 3 values of x and form 3 simultaneous equations, and attempt to solve; or substitute 3 values of x into given equation
	$x=0\Rightarrow c=-5$ $x=1\Rightarrow 2a+2b+c=3$ $x=2\Rightarrow 20a+10b+c=45$	(B1)		
	$a=1 \quad b=3$	(A1)	3	
	Total		**5**	

Question 2

A curve is defined by the parametric equations

$$x = \frac{1}{t}, \qquad y = t + \frac{1}{2t}$$

(a) Find $\frac{dy}{dx}$ in terms of t. *(4 marks)*

(b) Find an equation of the normal to the curve at the point where $t = 1$. *(4 marks)*

(c) Show that the cartesian equation of the curve can be written in the form

$$x^2 - 2xy + k = 0$$

where k is an integer. *(3 marks)*

Student Response

Question number

Leave blank

2a) $x = \frac{1}{t}$ $\quad t^{-1}$ $\quad y = t + \frac{1}{2t}$ $\quad 2t^{-1}$ B1

$\frac{dx}{dt} = -1t^{-2}$ $\quad \frac{dy}{dt} = -2t^{-2}$ B0

$\left(\frac{1}{-1t^{-2}} \times -2t^{-2}\right) = \frac{dy}{dx}.$ M1 A0 2

$\equiv k$

b) when $t = 1$ $\quad x = 1$ and $y = 3/2$

gradient $= -1/2$ B1 M0

$y - 1.5 = -1/2\,(x-1)$

$y - 1.5 = \frac{-x}{2} + 1/2$ B1F

$\left(y = \frac{-x}{2} + 2\right)$ or $2y = -x + 1.$ A0 2

c) $x^2 - 2xy + k = 0.$

$(x+y)^2(x-y)$

① $t = \frac{1}{x}$

② $y = \left(\frac{1}{x}\right) + \frac{1}{2(1/x)}$ $\quad = \frac{1}{x^2} + \frac{x^2}{4} = 0.$ M1

$= \frac{1}{x} + \frac{1}{\frac{2}{x}}.$ A0

$\cancel{(x+y) \times (x-y)}$

$\left(\frac{1}{x}\right) \times \left(\frac{1}{2t} - y\right)$ A0 1

$y = \left(\frac{1}{x} + \frac{x}{2}\right)^2$

$\left(\frac{1}{2xt} \times - \frac{1}{xy}\right)^2$

5

Commentary

Part (a): Like most candidates, this candidate differentiated the expression for *x* correctly and scored a B mark but they made an error when differentiating the expression for *y*. They have written down the given expression for *y*, but then they seemed to ignore the *t* and focused on the index -1, with $\frac{1}{2}$ incorrectly becoming 2 in the process. They used the chain rule correctly using their derivatives and so scored the method mark, but didn't see that their t^{-2} should cancel, and that in fact they haven't got $\frac{dy}{dx}$ in terms of *t*, as requested in the question; their value is a constant equal to 2.

Part (b): As such they can't gain the method mark for substituting for $t = 1$ in their expression for $\frac{dy}{dx}$. They gained two marks in part (b); one for finding the correct values of *y* and *x*, and the other for finding the gradient of the normal from their gradient of the tangent. The final accuracy mark was only awarded if the whole solution was correct, so they can't earn it.

Part (c): The candidate made a correct algebraic approach and has substituted for *t* and obtained a correct expression in *x*, which they now needed to simplify to the required form given in the question. They dealt with the fraction within a fraction correctly, and had they now not squared their result they would have scored the first accuracy mark. However, they chose to try and square their expression, presumably because there is an x^2 term in the given answer. Their attempt at squaring is incorrect and they seemed to realise they were not getting the required form and left the question. Had they not squared, but multiplied through by 2*x* as expected, they might well have scored the second accuracy mark as well.

Mark Scheme

Q	Solution	Marks	Total	Comments
2(a)	$\frac{dx}{dt}=-\frac{1}{t^2}$ $\quad\frac{dy}{dt}=1-\frac{1}{2t^2}$	B1B1		
	$\frac{dy}{dx}=\frac{1-\frac{1}{2t^2}}{-\frac{1}{t^2}}$ $\quad\left(=\frac{2t^2-1}{-2}\right)$	M1		Their $\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$; condone 1 slip
		A1		CSO; ISW
	Alternative			
	$y=\frac{1}{x}+\frac{x}{2}$	(B1)		
	$\frac{dy}{dx}=-\frac{1}{x^2}+\frac{1}{2}$	(B1)		
	Substitute $x=\frac{1}{t}$	(M1)		
	$\frac{dy}{dx}=-t^2+\frac{1}{2}$	(A1)	4	CSO
(b)	$t=1$ $\quad\frac{dy}{dx}=-\frac{1}{2}$	M1		Substitute $t=1$ in $\frac{f(t)}{g(t)}\neq k$
	$m_T=-\frac{1}{2}\Rightarrow m_n=2$	B1F		F on $m_T\neq 0$; if in $t\rightarrow$numerical later
	$(x,y)=(1,\frac{3}{2})$	B1		PI $\frac{3}{2}=m(\times 1)+c$
	$(y-\frac{3}{2})=2(x-1)$ **or** $y=2x+c,\ c=-\frac{1}{2}$	A1	4	ISW, CSO (a) and (b) all correct
(c)	$y=\frac{1}{\frac{1}{t}}+\frac{1}{2}\times\frac{1}{t}$	M1		Attempt to use $t=\frac{1}{x}$ to eliminate t t, or equivalent
	$=\frac{1}{x}+\frac{x}{2}$	A1		
	$2xy=2+x^2\Rightarrow x^2-2xy+2=0$	A1		Correct algebra to AG with $k=2$ allow $k=2$ stated $k=2$, no working or from $\left(1,\frac{3}{2}\right)$: 0/3
	Alternative **or**			
	$\left(\frac{1}{t}\right)^2-2\left(\frac{1}{t}\right)\left(t+\frac{1}{2t}\right)$ | $xy=\frac{1}{t}\left(t+\frac{1}{2t}\right)$	(M1)		Substitute and multiply out
	$=-2$ | $=1+\frac{x^2}{2}$	(A1)		Eliminate t
	$\Rightarrow x^2-2xy+2=0$	(A1)	3	Conclusion, $k=2$
			11	

Question 3

(a) Find the binomial expansion of $(1-x)^{-1}$ up to and including the term in x^2. *(2 marks)*

(b) (i) Express $\dfrac{3x-1}{(1-x)(2-3x)}$ in the form $\dfrac{A}{1-x}+\dfrac{B}{2-3x}$, where A and B are integers. *(3 marks)*

(ii) Find the binomial expansion of $\dfrac{3x-1}{(1-x)(2-3x)}$ up to and including the term in x^2. *(6 marks)*

(c) Find the range of values of x for which the binomial expansion of $\dfrac{3x-1}{(1-x)(2-3x)}$ is valid. *(2 marks)*

Student Response

3a) $(1-x)^{-1}$

$$\left(1+(-x)(-1)+\frac{(-1)(-2)}{2!}(-x)^2\right.$$

$$\underline{\underline{1+x+x^2}}$$ ✓ 2

bi) $$\frac{3x-1}{(1-x)(2-3x)} = \frac{A}{1-x}+\frac{B}{2-3x}.$$

$$3x-1 = A(2-3x)+B(1-x)$$ ✓

$x=1$

$$3(1)-1 = A(2-3(1))$$

$$2=-A \qquad A=-2.$$ ✓

$x=2/3$

$$3(2/3)-1 = B(1-2/3)$$

$$1=\tfrac{1}{3}B \qquad B=3$$ ✓ 3

Leave blank

$$-\frac{2}{1-x} + \frac{3}{2-3x}$$ ✓

ii) $$\frac{3x-1}{(1-x)(2-3x)} = \frac{①\ 2}{-(1-x)} + \frac{②\ 3}{(2-3x)}$$

① $$\frac{-2}{1-x} = -2(1-x)^{-1}$$ ✓

$$-2^{-1}\left(1+(-x)(-1)+\frac{(-1)(-2)}{2!}(-x)^2\right)$$

$$-\tfrac{1}{2}(1+x+x^2) = (-\tfrac{1}{2} - \tfrac{1}{2}x - \tfrac{1}{2}x^2)$$ B0

② $$\frac{3}{2-3x} = 3(2-3x)^{-1} \quad 6(1-\tfrac{3}{2}x)^{-1}$$ B0

$$6^{-1}\left(1+(-\tfrac{3}{2}x)(-1)+\frac{(-1)(-2)}{2!}(-\tfrac{3}{2}x)^2\right)$$

$$\tfrac{1}{6}(1+\tfrac{3}{2}x+\tfrac{9}{4}x^2$$ M1 A1

$$(\tfrac{1}{6}+\tfrac{1}{4}x+\tfrac{3}{8}x^2)$$

① × ② M0 A0

$$(-\tfrac{1}{2}-\tfrac{1}{2}x-\tfrac{1}{2}x^2)(\tfrac{1}{6}+\tfrac{1}{4}x+\tfrac{3}{8}x^2)$$

$$-\tfrac{1}{12}-\tfrac{1}{8}x-\tfrac{3}{16}x^2 - \tfrac{1}{12}x - \tfrac{1}{8}x^2$$

$$-\tfrac{1}{12}x^2$$

$$-\tfrac{1}{12}-\tfrac{5}{24}x-\tfrac{1}{48}x^2$$ × 2

Leave blank

c) $|1-x| < 1 \quad |2-3x| < 1$

$|-x| < 1 \quad |-3x| < \tfrac{1}{2}$

$|-x| < 1 \quad |-x| < \tfrac{1}{6}$

$|x| > -1 \quad |x| > -\tfrac{1}{6}$ × M0 A0 0

7

Commentary

Part(a): The candidate gave a correct expansion of $(1-x)^{-1}$, although many candidates made a sign or coefficient error when simplifying their expressions.

Part (b)(i): They made the conventional approach to finding partial fractions substituting two appropriate values of *x*, with the working set out clearly leading to the correct values of *A* and *B*. Some candidates made an error in finding *A* and *B*, this often occurring when the working was not kept tidy. This candidate turned the page to write the partial fractions down and he did it correctly; often candidates dropped the minus sign, or made a similar copying error, when they wrote down the partial fractions to use in part (b)(ii).

Part (b)(ii): The candidate started from the correct partial fractions, but went on to make several common errors in using the partial fractions to find a binomial expansion of this type. To start with, they should have just multiplied their expansion from part (a) by his value of *A*, -2, but apparently influenced by the index -1, they chose to invert it and multiplied by $-\frac{1}{2}$ and so lost the B mark. They now had to deal with 2 in $(2-3x)$. They factorised out the 2 correctly in that $-\frac{3}{2}$ remained in the bracket, but where now they should have inverted the 2, due to the -1 index, they didn't and so didn't score the second B mark. . Multiplication by 6 was a common error, but this candidate went furttheir, and consistent with what they did with the -2, they now inverted their 6 to become $\frac{1}{6}$. Their expansion of $\left(1-\frac{3}{2}x\right)^{-1}$ is correct and so they scored the M1A1 marks for that; many candidates made an error there either not using $-\frac{3}{2}x$ or making a sign or coefficient error. The candidate now has their two parts of the binomial expansion and just had to add them and simplify to the score the final method mark; they can't score the final accuracy mark as they can't get the right answer. However, they chose to multiply their two expressions so lost the method mark.

Part (c): The candidate shows they know that the range of validity is something to do with moduli and manipulating the expressions $(1-x)$ and $(2-3x)$ and 1. Their opening statement is wrong, but the next statement mod $|-x|<1$ is in fact correct, although they follow this up with $|x|>-1$, which although technically correct as a statement, for the validity condition it should have been $|x|<1$. However, the other term $(2x-3)$ provides the stronger condition on validity and there was no credit for dealing with $(1-x)$. They had a similarly wrong opening line, and their algebraic thinking is then difficult to follow. Had they started from $|-3x|<2$ they would have gained the method mark. They end up with two moduli, both greater than negative numbers, and although technically this is correct, any modulus is >0 by definition. The accuracy mark required a clear conclusion of $|x|<\frac{2}{3}$ although $|-x|<\frac{2}{3}$ was condoned.

Mark Scheme

Q	Solution	Marks	Total	Comments
3(a)	$(1-x)^{-1}=1+(-1)(-x)+\frac{1}{2}(-1.-2)(-x)^2$	M1		$1\pm x+kx^2$
	$=1+x+x^2$	A1	2	Fully simplified
(b)(i)	$3x-1=A(2-3x)+B(1-x)$	M1		
	$x=1 \quad x=\frac{2}{3}$	m1		Use 2 values of x or equate coefficients and solve $-3A-B=3 \quad 2A+B=-1$ condone coefficient errors
	$A=-2 \quad B=3$	A1	3	Both values NMS 3/3 if both correct, 1/3 if one correct
(ii)	$\left(\frac{3x-1}{(1-x)(2-3x)}=\frac{-2}{1-x}+\frac{3}{2-3x}\right)$			
	$\frac{-2}{1-x}=-2-2x-2x^2$	B1F		F on $(1-x)^{-1}$ and A
	$\frac{1}{2-3x}=\frac{1}{2}\left(1-\frac{3}{2}x\right)^{-1}$	B1		
	$=(p)\left(1+kx+(kx)^2\right)$	M1		$p,\ k$ = candidate's $\frac{1}{2},\frac{3}{2},\ k\neq\pm1$
	$=(p)\left(1+\frac{3}{2}x+\frac{9}{4}x^2\right)$	A1		Use (a) or start binomial again; condone missing brackets, and one sign error
	$\frac{3x-1}{(1-x)(2-3x)}=-2(1-x)^{-1}+3(2-3x)^{-1}$	M1		Valid combination of both expansions
	$=-\frac{1}{2}+\frac{1}{4}x+\frac{11}{8}x^2$	A1		CSO
	Alternative			
	$(2-3x)^{-1}=\frac{1}{2}\left(1-\frac{3}{2}x\right)^{-1}$	(B1)		k = candidate's $\frac{3}{2} \quad k\neq\pm1$
	$(1-kx)^{-1}=1+kx+(kx)^2$	(M1)		Use (a) or start binomial again; condone missing brackets and one error
	$=1+\frac{3}{2}x+\frac{9}{4}x^2$	(A1)		
	$\frac{3x-1}{(1-x)(2-3x)}=(3x-1)(1-x)^{-1}(2-3x)^{-1}$	(M1)		$(3x-1)\times$ both expansions
	$\frac{3x-1}{(1-x)(2-3x)}=-\frac{1}{2}+\frac{1}{4}x+\frac{11}{8}x^2$	(m1) (A1)	6	Multiply out; collect terms to form $a+bx+cx^2$ CSO
	Alternative for $(2-3x)^{-1}$			Using $(a+bx)^n$
	$2^{-1}+(-1)(2)^{-2}(-3x)+\frac{(-1)(-2)(2)^{-3}(-3x)^2}{2}$	(M1)		Condone missing brackets, and 1 error
	$=\frac{1}{2}+\frac{3}{4}x+\frac{9}{8}x^2$	(A1) (A1)		First two terms x^2 term
(c)	$-2<3x<2$	M1		PI, or any equivalent form
	$\Rightarrow -\frac{2}{3}<x<\frac{2}{3}$	A1	2	Condone $\leq$; accept $\pi\geq\frac{2}{3}$ or $x\geq-\frac{2}{3}$ CSO; allow $\lvert\pm x\rvert\leq\frac{2}{3}$, or $x<\frac{2}{3}$ and $x>-\frac{2}{3}$
	Total		**13**	

Question 4

A car depreciates in value according to the model

$$V = Ak^t$$

where £V is the value of the car t months from when it was new, and A and k are constants. Its value when new was £12 499 and 36 months later its value was £7000.

(a) (i) Write down the value of A. *(1 mark)*

(ii) Show that the value of k is 0.984 025, correct to six decimal places. *(2 marks)*

(b) The value of this car first dropped below £5000 during the nth month from new. Find the value of n. *(3 marks)*

Student Response

4. a) i) $A = £12\,499$ ✓ 1

a) ii) $V = Ak^t$ $V = 7000$ $A = 12\,499$ $t = 36.$

$7000 = 12\,499\,k^{36}$

$= \frac{7000}{12499} = k^{36.}$ ✓ $= \ln\left(\frac{7000}{12499}\right) = \ln(k^{36})$

$36\ln(k).$

$= \frac{\ln\left(\frac{7000}{12499}\right)}{36} = \ln(k) \rightarrow = e^{-0.016103847} = e^{\ln(k)}$

-0.016103847 ✓

$\underline{k = 0.984025\cancel{126\,7}}$ (6.d.p) 2

4. b) want to find t

$5000 = 12\,499 \times k^{(t)}.$

$= \frac{5000}{12499} = \cancel{1229}\;k^t$ $\ln k^t$ M1

$= \ln\frac{5000}{12499} = t\ln(k)$ $t = \frac{\ln\left(\frac{5000}{12499}\right)}{\ln(k)}.$ M1

$= \cancel{\ln\left(\frac{5000}{12499} - k\right) = t.}$

$\cancel{t = \ln(0.40003200026\ -0.95)}\,4025\,1267)$

$t = \ln(-0.583993121)$

$t = 56.89390424.$ ✓ A0 2

it first reached this value towards the end of the 56^{th} month. ✗

5

Commentary

Part (a): Most candidates got this correct. This candidate has crossed out the £ sign, although it was condoned if left in. Strictly the value of *A* is 12499

Part (b): This candidate chose to find the value of *k* by logarithms, which is a valid method although finding the 36th root of 7000/12499 on a calculator is quicker. The candidate hasn't set out his work very clearly but it is possible to follow his thinking and the key value of –0.016103847 is seen and the final value of *k* can be seen correctly to more than 6 decimal places, even if the 7th place onwards have been deleted. They scored both marks.

Part (c): The candidate's thinking can again be followed, and their little heuristic type notes probably helped him. They clearly had a correct expression for *t* with an incorrect attempt deleted, so they scored the 2 method marks. They evaluated *t* correctly, and tried to interpret the value, but didn't realise that 56.89… actually means it is now the 57th month. A quick check with the calculator would have shown with $n = 56$ the value is £5072 and with $n = 57$ it is £4991, confirming 57 as the answer.

Mark Scheme

4(a)(i)	$A = 12499$	B1	1	Stated in (i) or (ii)
(ii)	$k^{36} = \dfrac{7000}{\text{their } A}$	M1		$p = \dfrac{7000}{12499} = 0.560044803$
	$k = \sqrt[36]{0.56(00448...)} = 0.9840251(26)$ or $\left(0.56(00448...)\right)^{\frac{1}{36}}$ or $k = \sqrt[36]{\dfrac{7000}{12499}}$ $k = 0.984025$	A1	2	Correct expression for k or 7th dp seen. $k = 10^{\frac{1}{36}\log p}$ or $k = 10^{-0.00699...}$ $k = e^{\frac{1}{36}\ln p}$ or $k = e^{-0.016103...}$ AG
(b)	$k^t = \dfrac{5000}{\text{their } A}$	M1		$\dfrac{5000}{12499} = 0.400032...$; condone 4999
	$t\log(k) = \log\left(\frac{5000}{A}\right)$ $(t = 56.89)$	m1		Correct use of logs
	$n = 57$	A1		n integer; $n = 57$ CAO
	Alternative ; trial and improvement on $5000 = 12499 \times 0.984025^t$			
	2 values of $t \geq 40$	(M1)		
	1 value of t $50 < t < 60$	(m1)		
	$n = 57$	(A1)	3	
	Special case, answer only $n = 57$ 3/3 $n = 56$ 0/3 $n = 56.9$ 2/3			
	Total		**6**	

Question 5

A curve is defined by the equation $4x^2 + y^2 = 4 + 3xy$.

Find the gradient at the point $(1, 3)$ on this curve. *(5 marks)*

Student Response

5 $4x^2 + y^2 = 4 + 3xy$

$8x\frac{dy}{dx} + 2y\frac{dy}{dx} = 3y + 3x$ A B0 B1 M0 A0

$\frac{dy}{dx}(8x+2y) = 3y + 3x$

$\frac{dy}{dx} = \frac{3(y+x)}{8x+2y}$

$\frac{dy}{dx} = \frac{3(y+x)}{2(4x+y)}$

$\frac{dy}{dx} = \frac{3(3+1)}{2(4+3)}$

$= \frac{12}{14}$

$\frac{dy}{dx} = \frac{6}{7}$ ✗ A0

1

1

Commentary

Although most candidates got this question correct, this candidate has made typical mistakes. The differentiation is with respect to *x*, so there should not be a $\frac{dy}{dx}$ attached to 8*x*, but there should be to 2*y*; thus they scored B0 B1. The 4 has correctly gone, but in differentiating the product, they didn't attach $\frac{dy}{dx}$ to either term, so scored M0A0. The $\frac{dy}{dx}$ should be attached to the 3*x* term because *y* has been differentiated. If the candidate had attached $\frac{dy}{dx}$ to one of the two terms in their attempt to differentiate the product, they would have scored the method mark. They unnecessarily solved their equation to make $\frac{dy}{dx}$ the subject; the question requested a numerical value of the gradient, which can be found following the second line of working by substituting $x = 1$ and $y = 3$. The question is correct solution only for the final A1 accuracy mark and this candidate cannot get the right answer, and so scored 1 mark for the question.

Mark Scheme

5	$8x+2y\dfrac{dy}{dx}=3y+3x\dfrac{dy}{dx}$			
	$8x$ and $4\to 0$	B1		
	$2y\dfrac{dy}{dx}$	B1		
	$3y+3x\dfrac{dy}{dx}$	M1 A1		Two terms with one $\dfrac{dy}{dx}$
	at $(1,3)$ (gradient) $\dfrac{dy}{dx}=\dfrac{1}{3}$	A1	5	CSO
	Total		**5**	

Question 6

(a)	(i)	Show that the equation $3\cos 2x + 7\cos x + 5 = 0$ can be written in the form $a\cos^2 x + b\cos x + c = 0$, where a, b and c are integers.	*(3 marks)*
	(ii)	Hence find the possible values of $\cos x$.	*(2 marks)*
(b)	(i)	Express $7\sin\theta + 3\cos\theta$ in the form $R\sin(\theta + \alpha)$, where $R > 0$ and α is an acute angle. Give your value of α to the nearest $0.1°$.	*(3 marks)*
	(ii)	Hence solve the equation $7\sin\theta + 3\cos\theta = 4$ for all solutions in the interval $0° \leqslant \theta \leqslant 360°$, giving θ to the nearest $0.1°$.	*(3 marks)*
(c)	(i)	Given that β is an acute angle and that $\tan\beta = 2\sqrt{2}$, show that $\cos\beta = \frac{1}{3}$.	*(2 marks)*
	(ii)	Hence show that $\sin 2\beta = p\sqrt{2}$, where p is a rational number.	*(2 marks)*

Student Response

Q6, $3\cos 2x + 7\cos x + 5 = 0$

$a\cos^2 x + b\cos x + c = 0$

$\cos 2x = 2\cos^2 x + 1$ B0

$3(2\cos^2 x + 1) + 7\cos x + 5 = 0$ M1

$6\cos^2 x + 3 + 7\cos x + 5 = 0$

$6\cos^2 x + 7\cos x + 8 = 0$ A0 1

where $a = 6$, $b = 7$, $c = 8$

ii let $\cos x = y$ so $6y^2 + 7y + 8 = 0$ M0

$y = 0.9965$ or $y = -0.5833$ A0 0

Leave blank

$\cos x = 0.9965$ and $\cos x = -0.5833$

$x = \cos^{-1}(0.9965)$ $\quad x = \cos^{-1}(-0.5833)$

$x = 4.79$ to 3sf $\quad x = 125.$ to 3sf

ISW

b) $7\sin\theta + 3\cos\theta = R\sin\theta\cos\alpha + R\sin\alpha\cos\theta$

$\sin\theta: \quad 7 = R\cos\alpha$
$\cos\theta: \quad 3 = R\sin\alpha$

$R = \sqrt{3^2 + 7^2}$
$R = \cancel{\sqrt{88}}, \cancel{\sqrt{410}}, \sqrt{58}$ ✓

$\frac{\cancel{R}\sin\alpha}{\cancel{R}\cos\alpha} = \frac{3}{7}$ ✓

$\tan\alpha = \frac{3}{7} \quad \therefore \alpha = \tan^{-1}\frac{3}{7}$

$\alpha = 23.2$ ✓

3

$\therefore 7\sin\theta + 3\cos\theta = \sqrt{58}\sin(\theta + 23.2)$ ✓

ii) $\sqrt{58}\sin(\theta + 23.2) = 4$

$\sin(\theta + 23.2) = \cancel{\frac{4}{\sqrt{116}}} \; \frac{4}{\sqrt{58}}$ M1 ✓

$\theta + 23.2 = \sin^{-1}\left(\frac{4}{\sqrt{58}}\right)$ $\quad \theta + 23.2 = 31.68, 211.68$
148.32, 328.32

Question number

Leave blank

so $\theta = 8.5, 118.5, 125.1, 305.1$ ✓ ✗ A1 A0 2

c, $\tan\beta = 2\sqrt{2}$ show $\cos\beta = \frac{1}{3}$

$\cancel{\tan\beta = \frac{\sin\beta}{\cos\beta} \quad \frac{1}{\tan\beta} = \frac{\cos\beta}{\sin\beta}}$

$\cancel{\beta = 70.5}$

$\beta = \tan^{-1} 2\sqrt{2}$

$\beta = 70.5$ ✗

$\cos(70.5) = \frac{1}{3}$ M0 A0 0

$\therefore \cos\beta = \frac{1}{3}$ as required.

ii $\sin 2\beta = 2\sin^2\beta\cos^2\beta$ ✓ M1 A0 1

$2\sin\left(\frac{1}{3}\right) = \sin 2\beta$

7

Commentary

Part (a)(i): This candidate has tried to remember the formula for $\cos 2x$, but has a plus where a minus should be. If unsure, candidates are advised to work out double angle formulae from the compound angle formulae given in the formula book. However, they did use their expression for $\cos 2x$ and so gained the method mark, but they can't get the correct quadratic equation.

Part (a)(ii): In fact they wrote down a quadratic equation that is insoluble. ($b^2 - 4ac = -143$) and had they realised this they might have checked and found their mistake. As it is they 'solved' their equation in that they gave two solutions but there is no evidence as to how they got them. A seen attempt to factorise or use the quadratic formula could have resulted in the method mark being awarded. Having got two values for $\cos x$ they then spent some time unnecessarily finding the angles; these were not requested in the question.

Part (b)(i): This was done correctly with the working shown clearly.

Part (b)(ii): The candidate started to use the result from part (i) correctly, and has a correct value, 31.68 for the inverse sine. However, they had three other values as well, a 'solution' in each quadrant, only one other of which 148.32, is correct. In completing this part of the question they scored 1 accuracy mark for a correct answer, but lost the other accuracy mark for the extra, wrong solutions, in the required range.

Part (c)(i): This question requested candidates to show that the exact value of $\cos\beta = \frac{1}{3}$. The term "exact" implies it cannot be done with a calculator. That is what this candidate has done as evidenced by $\beta = 70.5°$ and so they scored no marks. Some recognition of acute angles and use of Pythagoras theorem was expected.

Part (c)(ii): The candidate wrote down a correct version of the double angle formula for sine, having deleted the squares they at first included. They were given credit for indicating the correct approach, although their attempted use of the formula is confused, and they appear to have replaced angle β with the value of its cosine in writing down $\sin\left(\frac{1}{3}\right)$ and they left the question incomplete. They didn't realise that the given exact values of $\tan\beta$ and $\cos\beta$ can be used to find the exact value of $\sin\beta$, which can then be substituted in the double angle formula to get the required result.

Mark Scheme

6(a)(i)	$\cos 2x = 2\cos^2 x - 1$	B1		Seen in question, in consistent variable
	$3\left(2\cos^2 x - 1\right) + 7\cos x + 5$	M1		Substitute candidate's $\cos 2x$ in terms of $\cos x$
	$6\cos^2 x + 7\cos x + 2 (= 0)$	A1	3	
(ii)	$(2\cos x + 1)(3\cos x + 2)$	M1		Attempt factors; formula ('a' and 'c' correct; allow one slip)
	$\cos x = -\frac{1}{2}$ $\quad \cos x = -\frac{2}{3}$	A1	2	Accept $-0.5, -0.67$ $x = \cos^{-1}\left(-\frac{1}{2}\right); \cos^{-1}\left(-\frac{2}{3}\right)$
(b)(i)	$R = \sqrt{58}$	B1		Accept 7.6 or better
	$\alpha = \sin^{-1}\left(\frac{3}{\text{their } R}\right)$	M1		OE $\alpha = \sin^{-1}\left(\frac{3}{7}\right)$
	$= 23.2°$	A1	3	AWRT 23.2° (23.1985…)
(ii)	$\alpha + \theta = \sin^{-1}\left(\frac{4}{\text{their } R}\right)$	M1		Candidate's R, α
	$\theta = 8.5°$	A1F		F on α, AWRT, condone 8.6
	$\theta = 125.1°$	A1	3	Two solutions only, but ignore out of range
(c)(i)	$h^2 = 1 + \left(2\sqrt{2}\right)^2$	M1		Pythagoras with h or $\sec x$
	$h = 3 \Rightarrow \cos\beta = \frac{1}{3}$	A1	2	AG
(ii)	$\sin 2\beta = 2\sin\beta\cos\beta$	M1		
	$\sin 2\beta = \frac{4}{9}\sqrt{2}$	A1	2	CSO; accept $p = \frac{4}{9}$ (not 0.444…)
	Total		**15**	

Question 7

The points A and B have coordinates $(3,\ -2,\ 5)$ and $(4,\ 0,\ 1)$ respectively.

The line l_1 has equation $\mathbf{r} = \begin{bmatrix} 6 \\ -1 \\ 5 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$.

(a) Find the distance between the points A and B. *(2 marks)*

(b) Verify that B lies on l_1. *(2 marks)*

(c) The line l_2 passes through A and has equation $\mathbf{r} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ 3 \\ -8 \end{bmatrix}$.

The lines l_1 and l_2 intersect at the point C. Show that the points A, B and C form an isosceles triangle. *(6 marks)*

Student Response

7, $A = \cancel{B}\ (3,-2,5)$ $\quad B = (4,0,1)$

$l_1 = \quad r = \begin{bmatrix} 6 \\ -1 \\ 5 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

a, $\vec{AB} = B - A = \begin{bmatrix} 4-3 \\ 0--2 \\ 1-5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$

distance $= \sqrt{1^2 + 2^2 + (-4)^2}$

$= \sqrt{21} = 4.58$ ✓ 2

b, ~~6+2λ~~ $\begin{bmatrix} 6+2\lambda \\ -1-1\lambda \\ 5+4\lambda \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$

Leave blank

$2\lambda = -2$
$-\lambda = 1$
$4\lambda = -4.$

$\lambda = -1,$
as this works for all b must lie on line l_1 ✓ 2

c, l_2 passes through A.

$r = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ 3 \\ -8 \end{bmatrix}$

l_1 l_2 intersect at C.

~~3 + - 1μ~~ $\begin{bmatrix} 3-1\mu \\ -2+3\mu \\ 5-8\mu \end{bmatrix} = \begin{bmatrix} 6+2\lambda \\ -1-1\lambda \\ 5+4\lambda \end{bmatrix}$

$3-\mu = 6+2\lambda$ ①
$-2+3\mu = -1-1\lambda.$ ②
①$+2\times$② $-1+5\mu = 4$
$5\mu = 5$ $\mu = 1$ ✓
$3-(1) = 6+2\lambda$
$2 = 6+2\lambda$ $2\lambda = -4$ $\lambda = -2$ ✓
so they intersect at $\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = C$ ✓

Question number

Leave blank

angle between
AB and AC (BAC)

$\vec{AB} = B - A = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$

$\vec{AC} = C - A = \begin{bmatrix} 2 & & 3 \\ 1 & - & -2 \\ -3 & & 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -8 \end{bmatrix}$

angle. $\cos\theta = \cancel{\frac{A \cdot B}{|A||B|}} \quad \frac{a \cdot b}{|a||b|}$

$a = \vec{AB}$

$b = \vec{AC} \qquad a \cdot b = (1 \times -1) + (2 \times 3) + (-4 \times -8) = 37$ ✓

$|a| \times |b| = \sqrt{21} \times \sqrt{74}$.

$\cos\theta = \frac{37}{\sqrt{21}\sqrt{74}}$

$\theta = \cos^{-1}\left(\frac{37}{\sqrt{21}\sqrt{74}}\right) = 20 \cdot 18°$ ✓

~~angle between BA and BC.~~

$\cancel{\vec{BA} = A - B = \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix}} \qquad \cancel{\vec{BC} = C - B = \begin{bmatrix} -2 \\ 1 \\ -4 \end{bmatrix}}$

$\cancel{\frac{a \cdot b}{|a||b|} = \frac{(-1 \times 2) + (-2 \times 1) + (4 \times -4)}{\sqrt{21} \times \sqrt{21}}}$

$\cancel{\theta = \cos^{-1} \frac{-20}{\sqrt{21}\sqrt{21}}}$

$\cancel{\theta = 162 \cdot 25}$ angle between CA and CB. (BCA)

$\vec{CA} = A - C = \begin{bmatrix} 3 - & 2 \\ -2 - & 1 \\ 5 - & -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 8 \end{bmatrix}$

$\vec{CB} = B - C = \begin{matrix} 4 - 2 \\ 0 - 1 \\ 1 - -3 \end{matrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ ✓

Question number

Leave blank

7c) $\frac{a.b}{|a||b|} = \frac{(1\times 2)+(-3\times -1)+(8\times 4)}{\sqrt{74}\times\sqrt{21}}$

$\cos\theta = \frac{37}{\sqrt{74}\sqrt{21}}$

$\theta = \cos^{-1}\frac{37}{\sqrt{74}\sqrt{21}} =$

$= 20.18^\circ$

as angles BAC and BCA are the same, the triangle must be isosceles.

6

10

Commentary

Part (a): The candidate first found the vector $\overrightarrow{AB}$ rather than using the distance formula. This is fine as they then went on to find the modulus, correctly calling it the distance. Candidates who stopped after finding vector $\overrightarrow{AB}$ scored no marks.

Part (b): The candidate has written down three correct component equations derived from the vector equation and notes that $\lambda = -1$. Had they stopped there they would only score the method mark. However, their comment whilst not being the most elegant way of saying it, does imply $\lambda = -1$ is a consistent solution for all three equations so they gained the accuracy mark as well.

Part (c): The candidate set up and correctly solved the simultaneous equations derived from equating the vectors equations of the two lines and went on to find the intersection point *C* correctly. They now know the coordinates of the three vertices *A B* and *C*, so could have calculated the lengths of the sides having already found *AB* in part (a). The question could have been competed quickly and successfully. However, they were thinking about two angles being the same in an isosceles triangle rather than two sides having the same length, and started to calculate angles. This was all done correctly and they gained full marks for the question. However, at the stage of their deleted work they might have noticed that their vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ have components whose sum of squares is the same, and therefore they must be the same length. They made an arithmetic error in calculating the scalar product, which should be –16, and would lead to an angle of 139.6º, giving the third angle as 20.2º and they could have stopped there. So they spent an unnecessarily long time solving this problem; their solution is commendable, but they paid a time penalty in other questions.

Mark Scheme

7(a)	$(AB^2=)(4-3)^2+(0--2)^2+(1-5)^2$	M1		Condone one sign error in one bracket
	$AB=\sqrt{21}$	A1	2	Accept 4.58 or better
(b)	$4=6+2\lambda \Rightarrow \lambda=-1$	M1		$\lambda=-1$
	$0=-1+(-1)\times(-1)$ $1=5+(-1)\times 4$	A1	2	$\lambda=-1$ confirmed in other two equations Accept for M1A1 $\begin{bmatrix}6\\-1\\5\end{bmatrix}-\begin{bmatrix}2\\-1\\4\end{bmatrix}=\begin{bmatrix}4\\0\\1\end{bmatrix}$
	Special case $\begin{bmatrix}6\\-1\\5\end{bmatrix}+\lambda\begin{bmatrix}2\\-1\\4\end{bmatrix}=\begin{bmatrix}4\\0\\1\end{bmatrix}, \quad \lambda\begin{bmatrix}2\\-1\\4\end{bmatrix}=\begin{bmatrix}-2\\1\\-4\end{bmatrix}$ $\lambda=-1$	(B2)		M1 condone 1 slip
(c)	$\begin{bmatrix}3\\-2\\5\end{bmatrix}+\mu\begin{bmatrix}-1\\3\\8\end{bmatrix}=\begin{bmatrix}6\\-1\\5\end{bmatrix}+\lambda\begin{bmatrix}2\\-1\\4\end{bmatrix}$	M1		Equate vector equations PI by two equations in λ or μ
	$\left.\begin{array}{r}3-\mu=6+2\lambda\\-2+3\mu=-1-\lambda\end{array}\right\}$ eliminate λ *or* μ	m1		Form (any) two simultaneous equations and solve for λ or μ
	$\lambda=-2$ or $\mu=1$	A1		
	C has coordinates $(2, 1, -3)$	A1		CAO condone $\begin{bmatrix}2\\1\\-3\end{bmatrix}$
	$BC^2=(2-4)^2+(0-1)^2+(1--3)^2$ $BC=\sqrt{21}$	M1		Use C to find BC or AC or to find two angles
	$AB=BC \quad (=\sqrt{21})$	A1	6	$AB=BC$ or $\angle A=\angle C$ $(=20.2°)$ stated
	Total		**10**	

Question 8

(a) Solve the differential equation

$$\frac{dx}{dt} = \frac{150 \cos 2t}{x}$$

given that $x = 20$ when $t = \frac{\pi}{4}$, giving your solution in the form $x^2 = f(t)$. *(6 marks)*

(b) The oscillations of a 'baby bouncy cradle' are modelled by the differential equation

$$\frac{dx}{dt} = \frac{150 \cos 2t}{x}$$

where x cm is the height of the cradle above its base t seconds after the cradle begins to oscillate.

Given that the cradle is 20 cm above its base at time $t = \frac{\pi}{4}$ seconds, find:

(i) the height of the cradle above its base 13 seconds after it starts oscillating, giving your answer to the nearest millimetre; *(2 marks)*

(ii) the time at which the cradle will first be 11 cm above its base, giving your answer to the nearest tenth of a second. *(2 marks)*

Student Response

Question number

Leave blank

8. (a) $\frac{dx}{dt} = \frac{150 \cos 2t}{x} \Rightarrow$ ~~$\int x \, dx = \int 150 \cos 2t \, dt$~~

~~$= \ln x = 75 \sin 2t + c$~~

$\int x \, dx = \int 150 \cos 2t \, dt \Rightarrow x^2 = 75 \sin 2t + C$ B1

B0 B1

$x = 20,\ t = \pi/4 \Rightarrow 20^2 = 75 \sin 2\,\pi/4 + C$ M1

$20^2 = 75 \sin \pi/2 + C$

$20^2 = 75(1) + C \quad \therefore C = 325$ A1F A0 4

$\Rightarrow x^2 = 75 \sin 2t + 325$

(b) $x^2 = 75 \sin 2t + 325$

$x^2 = 75 \sin 2(13) + 325$ M1

$x^2 = 75 \sin 26 + 325 \quad x^2 = 357.8778\ldots$ A0 1

$\therefore x = 18.91765.. \quad x = 19$ millimetres.

(c) $x^2 = 75 \sin 2t + 325$

$11^2 = 75 \sin 2t + 325$ M0

$121 - 325 = 75 \sin 2t$ A0 0

$-204 = 75 \sin 2t$

$-2.72 = \sin 2t$

$\sin^{-1} -2.72 = 2t = -0.047455\ldots = 2t$

$= t = -0.023727\ldots$

time cannot be negative $\therefore t = 0.24$ (2d.p) seconds

5

Commentary

Part (a): The candidate separated the variables correctly but they missed the required $\frac{1}{2}$ in integrating *x* and so lost a B mark. They included a constant and proceeed to find it by the correct method, and they used $\sin\left(\frac{\pi}{2}\right)=1$, and so their calculated constant scores the follow through accuracy mark. They can't however, get the final answer correct so scored A0 as this accuracy mark was for a fully correct solution only.

Part (b): The candidate correctly understood that they had been given a time and substituted correctly in their answer to part (a) and thus scored the method mark. However, they found the value of sin26 with their calculator in degrees mode rather than the required radians. They can't get the right answer so did not score the accuracy mark.

Part (c): Using their solution to part (a) the candidate understood that they had been given the distance *x* and proceeded correctly to solve for $\sin 2t$. However, they didn't realise that their $\sin 2t = -2.72$ is impossible and that they must have made a mistake. They are apparently determined to get to an answer and just swapped the roles of $2t$ and –2.72, and with their calculator still in degrees mode they got a negative value and so decided to ignore the negative sign. Had they realised that they must have made an error and checked back and found the error in the integration of *x*, they might have scored 2 more marks, and had they thought further that a problem like this must be solved in radians, they might have got a full marks solution. However, most of the candidates who did correctly get to $2t=-1.035$ in part (c) also just ignored the negative sign, instead of considering the next, positive, solution.

Mark Scheme

8(a)	$\int x\ \mathrm{d}x = \int 150\cos 2t\ \ \mathrm{d}t$	B1		Correct separation; condone missing $\int$ signs; must see d*x*, d*t*
	$\frac{1}{2}x^2 = 75\sin 2t \quad (+C)$	B1B1		Correct integrals Accept $\frac{1}{2}\times 150$
	$\left(20,\frac{\pi}{4}\right) \quad \frac{1}{2}\times 20^2 = 75\sin\left(2\times\frac{\pi}{4}\right)+C$	M1		*C* present. Use $\left(20,\frac{\pi}{4}\right)$ to find *C*
	$C=125$	A1F		F on $x^2 = k\sin 2t$
	$x^2 = 150\sin 2t + 250$	A1	6	Correct integrals and evaluation of *C*
(b)(i)	$t=13 \quad x^2 = 150\sin 26 + 250 \ (=364.38)$	M1		Evaluate $x^2 = \mathrm{f}(13)$; $x^2 = k\sin 2t + c$ with numerical *k* and *t*
	$x = 19.1\ (\mathrm{cm})$	A1	2	AWRT
(ii)	$x=11 \quad \sin 2t = -\frac{129}{150} \quad (=-0.86)$ or $\quad 2t = -1.035..,\ 4.176...$	M1		
	$t = 2.1(\text{seconds})$	A1	2	AWRT
	Total		**10**	