Teacher Support Materials 2008

Maths GCE

Paper Reference MPC4

Question 1

1 The polynomial $f(x)$ is defined by $f(x) = 27x^3 - 9x + 2$.

(a) Find the remainder when $f(x)$ is divided by $3x + 1$. *(2 marks)*

(b) (i) Show that $f\left(-\frac{2}{3}\right) = 0$. *(1 mark)*

(ii) Express $f(x)$ as a product of three linear factors. *(4 marks)*

(iii) Simplify

$$\frac{27x^3 - 9x + 2}{9x^2 + 3x - 2}$$

(2 marks)

Student Response

number

1

a) $3x+1=0$

$x=-\frac{1}{3}$

$f(-\frac{1}{3}) = 27(-\frac{1}{3})^3 - 9(-\frac{1}{3}) + 2$

$= 4$ ✓ 2

b) i) $f(-\frac{2}{3}) = 27(-\frac{2}{3})^3 - 9(-\frac{2}{3}) + 2$

$= 0$ B0 0

Question number

b) ii)

~~$3x+1 \,|\, 27x^3 - 9x + 2$~~

✓

$(3x+1)(3x+2)(3x-2) = 27x^3 - 9x + 2$ B1

$$\begin{array}{r} 9x^2 - 3x + 2 \\ 3x+1 \enclose{longdiv}{27x^3 - 9x + 2} \\ 27x^3 - 9x^2 \end{array}$$

$$\begin{array}{r} 3x - 1 \\ 3x+2 \enclose{longdiv}{9x^2 - 3x + 2} \\ 9x^2 + 6x \\ \hline 3x - 2 \end{array}$$

✗ M0 A0

A0 1

iii) $\frac{27x^3 - 9x + 2}{9x^2 + 3x - 2} = \frac{(3x+1)(9x^2 - \cancel{3x} + 2)}{9x^2 + \cancel{3}x - 2}$

M0

$= \frac{3x+1}{9x^2+3x-2} x - 1$

$= \frac{3\cancel{x+1}}{(3\cancel{x+1})(3x-2)} x - 1$

$= \frac{1}{(3x-2)}$ ✗ ✓A0 0

3

Commentary

(a) The answer for the remainder was not given in the question. The candidate is correct.
(b) (i) The answer is given in the question. The candidates simply writes down $f(-\frac{2}{3})$ with no details of any evaluation shown. Thus the mark is not awarded.
(b) (ii) The candidates correctly notes from (i) that $(3x+2)$ is a factor so gets a mark. He fails to note from (a) that $(3x+1)$ cannot be a factor as there is a non-zero remainder, and attempts to divide $f(x)$ by it, but doesn't recover the remainder from (a). He now divides what he believes to be the result by $(3x+2)$ without any apparent conclusion. Had he divided $f(x)$ by $(3x+2)$ he probably would have gained at least one more mark and might well have completed successfully.
(b)(iii) The candidate doesn't have three linear factors for $f(x)$ but nor does he factorise the quadratic expression correctly; he can make no further progress and scores no marks.

Mark scheme

Q	Solution	Marks	Total	Comments
1(a)	$f\left(-\frac{1}{3}\right)=27\times\left(-\frac{1}{3}\right)^3-9\times\left(-\frac{1}{3}\right)+2$	M1		Use of $\pm\frac{1}{3}$ or complete division with integer remainder M1
	$=-1+3+2=4$	A1	2	remainder = 4 indicated A1
(b)(i)	$f\left(-\frac{2}{3}\right)=-8+6+2=0$	B1	1	AG
(b)(ii)	$f(x)=(3x+2)(ax^2+bx+c)$	B1		$(3x+2)$ or $\left(x+\frac{2}{3}\right)$ is a factor PI
	$a=9$ $c=1$	M1		quadratic factor; find coefficients; 2 correct
	x^2 term $3b+2a=0$ or x term $3c+2b=-9$			equate coefficients and solve for b
	$b=-6$ or (could be shown as) $9x^2-6x+1$	A1		correct quadratic factor or a, b, and c correct or use division or factor theorem to seek another factor (see alternative methods at end of scheme)
	$f(x)=(3x+2)(3x-1)(3x-1)$	A1	4	SC (see alternative methods at end of scheme)
(b)(iii)	$9x^2+3x-2=(3x-1)(3x+2)$	M1		factorise denominator correctly or complete division
	$\frac{27x^3-9x+2}{9x^2+3x-2}=3x-1$	A1	2	simplified result indicated
	Total		**9**	

Question 2

2 A curve is defined, for $t \neq 0$, by the parametric equations

$$x = 4t + 3, \quad y = \frac{1}{2t} - 1$$

At the point P on the curve, $t = \frac{1}{2}$.

(a) Find the gradient of the curve at the point P. *(4 marks)*

(b) Find an equation of the normal to the curve at the point P. *(3 marks)*

(c) Find a cartesian equation of the curve. *(3 marks)*

Student response

(2) $x = 4t + 3$ $\quad y = \frac{1}{2t} - 1$ $\quad t \neq 0$ $\quad$ 9

$P.\ t = \frac{1}{2}$ $\quad 2t^{-1} - 1$

a) $dx/dt = 4$ ✓ $\quad dy/dt = -2t^{-2}$ $\quad$ M1

when $t = \frac{1}{2}$ $\quad$ A0

$dx/dt = 4$ $\quad dy/dt = -1$

$dy/dx = \frac{1}{4} \times -1$ $\quad$ M1

$= -\frac{1}{4}$ ✗ $\quad$ A0 $\quad$ 2

b) normal gradient $= 4$ ✓ $(m_1 m_2 = -1)$

when $t = \frac{1}{2}$ $\quad$ B1F

$x = 5$ $\quad y = 0$

$y - y_1 = m(x - x_1)$

$y - 0 = 4(x - 5)$ $\quad$ M1

$y = 4x - 20$ ✓ $\quad$ A1F $\quad$ 3

Question number $\quad$ Leave blank

c) $y = \frac{1}{2t} - 1$ $\quad x = 4t + 3$

$\frac{x - 2}{2} = 2t$ $\quad$ B0

$y = \frac{1}{x - 2/2} - 1$ $\quad$ M1

$= \frac{2}{x - 2} - 1$

$= \frac{2}{x} - 1 - 1$ $\quad$ 1

$= \frac{2}{x} - 2$ ✗ $\quad$ ✓A0 $\quad$ 6

Commentary

(a) In differentiating the equation for y, the candidate has treated $\frac{1}{2}$ as if it were 2, although her derivative of $\frac{1}{t}$ is correct. She uses the chain rule correctly, but cannot get the right answer for the gradient, so also loses the final mark.
(b) She however uses her gradient to find the gradient of the normal and continues to find an equation of the normal correctly, so is awarded full marks for part (b).
(c) In principle, the candidate understands what is required, but makes an error in finding her expression for $2t$ so loses this mark. Her approach has merit, but had she gone through the simpler $x-3=4t$ stage first, she might well not have made the error and gone onto score full marks. As it is she cannot get a correct form for the cartesian equation so loses the final accuracy mark as well.

Mark Scheme

Q	Solution	Marks	Total	Comments
2(a)	$\frac{dx}{dt}=4 \quad \frac{dy}{dt}=-\frac{1}{2t^2}$	M1 A1		differentiate. 4; at^{-2} seen both derivatives correct
	$\frac{dy}{dx}=-\frac{1}{2t^2}\times\frac{1}{4}$	M1		use chain rule candidates' $\frac{dy}{dt} \Big/ \frac{dx}{dt}$
	$t=\frac{1}{2} \quad \frac{dy}{dx}=-\frac{1}{2}$	A1	4	CSO
(b)	gradient of normal = 2 $(x,y)=(5,0) \quad \frac{y}{x-5}=2$	B1F M1 A1F	3	F if gradient $\neq \pm 1$ calculate and use (x, y) on normal F on gradient of normal ACF
(c)	$x-3=4t \quad$ or $\quad y+1=\frac{1}{2t}$	B1		or $t=\frac{x-3}{4}$ or $\frac{1}{t}=2(y+1)$
	$(x-3)(y+1)=2$	M1 A1	3	eliminate t; allow one error accept $y=\dfrac{1}{\frac{2(x-3)}{4}}-1$ ACF SC allow marks for part (c) if done in part (a)
	Total		10	

Question 3

3	(a)	By writing $\sin 3x$ as $\sin(x+2x)$, show that $\sin 3x = 3\sin x - 4\sin^3 x$ for all values of x.	*(5 marks)*
	(b)	Hence, or otherwise, find $\int \sin^3 x\,dx$.	*(3 marks)*

Student Response

3a) $\sin(x+2x)$ $\quad (x-3)$ $\quad$ 9

$\sin 3x = \sin(x+2x)$

$\sin x\cos 2x + \cos x \sin 2x$ M1

$= \sin x(1-2\sin^2 x) + \cos x(\cancel{2\cos x\sin x})$

$= \sin x - 2\sin^3 x + \cos x(2\sin x\cos x)$

B1 B1

$= \sin x - 2\sin^3 x + 2\sin x\cos x + \cos^2 x.$

$= \sin x - 2\sin^3 x + 2\sin x\cos x + (\cos 2x + 1)$

$= \sin x - 2\sin^3 x + 2\sin x\cos x + (1-\sin^2 x + 1)$ A0

$\cos 2x = \cos^2 x - 1$

$\cos 2x + 1 = \cos^2 x$

$= \sin x - 2\sin^3 x + 2\sin x\cos x$ A0 $\quad$ 3

$= \sin x - 2\sin^3 x + \sin 2x + 2 - \sin^2 x$

$= 3\sin x -$

b) $\int \sin^3 x\, dx.$

$= \cancel{-\dfrac{\cos^3 x}{3}}$

$\cancel{\sin 3x} = 3\sin x - 4\sin^3 x.$

$4\sin^3 x = 3\sin x$

$\sin^3 x = \dfrac{3\sin x}{4}.$ M0

A0 $\quad$ 0

A0

$\int \sin^3 x\, dx$

$= \int \dfrac{3\sin x}{4}\, dx.$

$= \dfrac{3}{4}\int \sin x\, dx \rightarrow \dfrac{3}{4}(-\cos x) + c.$ $\quad$ 3

$-3/4\cos x + c.$

Commentary

(a) The candidate has started correctly and continued to use correct double angle formulae for $\cos 2x$ and $\sin 2x$. However, she makes an expansion error in the fifth line and continues to carry the $2\sin x\cos x$ term until it is just "lost" in the last line. Also rather than using $\sin^2 x+\cos^2 x=1$ to eliminate $\cos^2 x$ she returns to the double angle cosine, and this time makes an error in replacing it in terms of $\sin x$. It is sensible in a proof of an identity question such as this to write down formulae likely to be of use, as she has done in the highlighted area. Unfortunately for this candidate, this version of $\cos 2x$ is not correct. (b) The candidate seems uncertain as to how to approach the integral. She seems to think she should use the identity from (a) but inexplicably replaces the $\sin 3x$ with 0. She then solves for $\sin^3 x$ not realising that she now has an equation in $\sin x$ and attempts the integral. She has the integral of $\sin x$ correct. but this gains no marks out of the context of using the identity. Had she attempted to solve the identity for $\sin^3 x$ and then integrated she might well have scored 2 marks.

Mark Scheme

3(a)	$\sin(x+2x)=\sin x\cos 2x+\cos x\sin 2x$	M1		
	$=\sin x(1-2\sin^2 x)+\cos x(2\sin x\cos x)$	B1B1		double angles; ACF ISW condone missing x
	$=\sin x(1-2\sin^2 x)+2\sin x(1-\sin^2 x)$	A1		all in $\sin x$, correct expression
	$=3\sin x-2\sin^3 x-2\sin^3 x$ $=3\sin x-4\sin^3 x$	A1	5	CSO AG
(b)	$\sin^3 x=a\sin x+b\sin 3x$	M1		attempt to solve for $\sin^3 x$ where $a\neq 0$ and $b\neq 0$
	$\int \sin^3 x\,dx=-a\cos x-\frac{b}{3}\cos 3x$	A1F		either integral correct F on a, b
	$\int \sin^3 x\ dx=\frac{1}{4}\left(-3\cos x+\frac{1}{3}\cos 3x\right)\ (+C$	A1	3	CAO alternative method by parts (see end of mark scheme)
	Total		**8**	

Question 4

4 (a) (i) Obtain the binomial expansion of $(1-x)^{\frac{1}{4}}$ up to and including the term in x^2.

(2 marks)

(ii) Hence show that $(81-16x)^{\frac{1}{4}} \approx 3 - \frac{4}{27}x - \frac{8}{729}x^2$ for small values of x.

(3 marks)

(b) Use the result from part (a)(ii) to find an approximation for $\sqrt[4]{80}$, giving your answer to seven decimal places.

(2 marks)

Student Response

Question number

Leave blank

4. a) (i) $(1-x)^{\frac{1}{4}}$

$1+\left(\frac{1}{4}\right)(-x)+\frac{\left(\frac{1}{4}\right)\left(\frac{1}{4}-1\right)}{2!}(-x)^2$

$1+-\frac{1}{4}x+\frac{\left(-\frac{3}{16}\right)}{2!}\times -x^2$

$=1\cancel{z}-\frac{x}{4}+\frac{-3}{32}\times -x^2$ ✗

M1

$=1-\frac{x}{4}+\frac{3x^2}{32}$ A0

(ii) $(81-16x)^{\frac{1}{4}}$

$=81\left(1-\frac{16}{81}x\right)^{\frac{1}{4}}$ B0

$81\left[1+\left(\frac{1}{4}\right)\left(\frac{-16}{81}x\right)+\frac{\left(\frac{1}{4}\right)\left(\frac{1}{4}-1\right)}{2!}\cancel{x}\left(-\frac{16}{81}\right)x^2\right]$ M1

$81\left(1+-\frac{4}{81}x+\frac{-\left(\frac{3}{16}\right)\left(-\frac{16}{81}\right)}{2!}x^2\right)$

$81\left(1-\frac{4}{81}x+-\frac{1}{54}x^2\right)$

$=81-4x-\frac{3}{2}x^2$

$\div 27$

$=3-\frac{4}{27}x-\left(\frac{8}{729}x^2\right)$ A0 cso

Question number

Leave blank

4. b) $(81-16x)^{\frac{1}{4}}$

$= \sqrt[4]{81-16x}$

when $x = \left(\frac{1}{16}\right)$ ✓ $\approx \sqrt[4]{80}$

$\Rightarrow 3 - \frac{4}{27}x - \frac{8}{729}x^2$

$= 3 - \frac{4}{27}\left(\frac{1}{16}\right) - \frac{8}{729}\left(\frac{1}{16}\right)^2$ M1

$= 3 - \frac{1}{108} - \frac{1}{23328}$

$= 2.9906$ ✗ A0 1

$=$ approx 3

Commentary

(a) (i) The candidate sets out his opening line of the binomial expansion very clearly but drops the brackets on his $-x^2$ term and thus makes a sign error.
(a) (ii) The candidate knows in principle what to do, but makes an error in taking out the 81; it should be $81^{\frac{1}{4}}$, the fourth root of 81. He continues his expansion by using his opening line from part (a) but fails to indicate that the $\left(\frac{16}{81}\right)$ term should be squared. He now cannot get the given answer, but he divides by 27 for no reason other than this does give the first two terms. He gains only 1 mark for attempting to use his expansion from (a).
(b) The candidate understands what to do and substitutes $x = \frac{1}{16}$ correctly. His evaluation looks to be correct, but he hasn't rounded to seven decimal places, so loses the final mark. His comment of "approx 3" suggest he didn't read the question carefully.

Mark Scheme

Q	Solution	Marks	Total	Comments
4(a)(i)	$(1-x)^{\frac{1}{4}}=1+\frac{1}{4}(-x)+\frac{1}{2}\times\frac{1}{4}\left(-\frac{3}{4}\right)(-x)^2$	M1		$1\pm\frac{1}{4}x+kx^2$
	$=1-\frac{1}{4}x-\frac{3}{32}x^2$	A1	2	equivalent fractions or decimals
(a)(ii)	$(81-16x)^{\frac{1}{4}}=81^{\frac{1}{4}}\left(1-\frac{16}{81}x\right)^{\frac{1}{4}}$	B1		
	$=k\left(1-\frac{1}{4}\times\frac{16}{81}x-\frac{3}{32}\left(\frac{16}{81}x\right)^2\right)$	M1		x replaced by $\frac{16}{81}x$
				or start binomial again condone one error (missing bracket; x or x^2 ; sign error)
	$=3(\qquad\qquad)$ $=3-\frac{4}{27}x-\frac{8}{729}x^2$	A1	3	CSO AG
				use of $(a+bx)^n$ ignoring hence (see end of mark scheme)
(b)	$3-\frac{4}{27}\times\frac{1}{16}-\frac{8}{729}\left(\frac{1}{16}\right)^2$	M1		use $x=\frac{1}{16}$
	$=2.9906979$	A1	2	seven decimal places only
	Total		**7**	

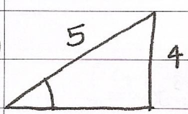
Question 5

5 (a)		The angle α is acute and $\sin\alpha = \frac{4}{5}$.	
	(i)	Find the value of $\cos\alpha$.	*(1 mark)*
	(ii)	Express $\cos(\alpha - \beta)$ in terms of $\sin\beta$ and $\cos\beta$.	*(2 marks)*
	(iii)	Given also that the angle β is acute and $\cos\beta = \frac{5}{13}$, find the exact value of $\cos(\alpha - \beta)$.	*(2 marks)*
(b)	(i)	Given that $\tan 2x = 1$, show that $\tan^2 x + 2\tan x - 1 = 0$.	*(2 marks)*
	(ii)	Hence, given that $\tan 45° = 1$, show that $\tan 22\frac{1}{2}° = \sqrt{2} - 1$.	*(3 marks)*

Student Response

SOHCAHTOA

5. a) i)

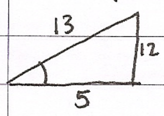


$\cos\alpha = \frac{3}{5}$ ✓ 1

(7)

ii) $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ ✓ M1 A0 1

iii)



$\sin\beta = \frac{12}{13}$ $\left(\cos\frac{3}{5}\right)\left(\cos\frac{5}{13}\right) + \left(\sin\frac{4}{5}\right)\left(\sin\frac{12}{13}\right)$ $\frac{48}{65}$ B0

$\cos^2\frac{3}{13} + \sin^2\frac{48}{65}$ ✗

$1 - \sin^2 x = \cos^2 x$

$\cos^2 x + 1 - \cos^2 x$ B0 0

b) i) $\tan 2x = 1$ ~~1-2tanx~~ $\frac{2\tan x}{1-\tan^2 x} = 1 \Rightarrow 2\tan x = (1 - \tan^2 x)$

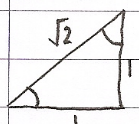
$\tan^2 x + 2\tan x = 1$

$\tan^2 x + 2\tan x - 1 = 0$ ✓ M1 A0 2

ii) $\tan 45° = \tan 2(22\frac{1}{2})° = 1$ $\tan^2(22\frac{1}{2}) + 2\tan(22\frac{1}{2}) - 1 = 0$ $= \sqrt{2} - 1$

$a=1$ $b=2$ $c=-1$ $(\tan(22\frac{1}{2}))(\tan(22\frac{1}{2}))$

$\frac{\tan A + \tan B}{1 - \tan A\tan B}$



$\frac{-2 \pm \sqrt{4+4\times1\times1}}{2}$ ✓ $\frac{-2 \pm \sqrt{8}}{2}$ $\frac{-2 + 4\sqrt{2}}{}$ M1

$\frac{-2 + 4\sqrt{2}}{2} = \sqrt{2}$ A0 1

A0

(5)

Commentary

(a) (i) The candidate makes a sensible start with the sketch of the 3 4 5 triangle.
(a) (ii) She expands $\cos(\alpha-\beta)$ correctly but loses a mark through not substituting for $\sin\alpha$ and $\cos\alpha$ with the now known values.
(a) (iii) The sketch of the 5 12 13 triangle is again helpful and she has written $\sin\beta=\frac{12}{13}$.
However, in substituting in her expansion from part (ii), the angles have become confused with the values of their sines and cosines and the subsequent line is meaningless. Thus these 2 marks are denied.
(b) (i) The identity for $\tan 2x$ is used clearly in obtaining the given quadratic equation in $\tan x$.
(b) (ii) She replaces x with $22\frac{1}{2}$, which is acceptable, and knows she is to solve this quadratic equation but there is a lack of confidence in her approach. An apparent attempt to factorise is sensibly abandoned given there is a $\sqrt{2}$ in the final answer, but the attempt to use the quadratic formula is not clear. She scores 1 mark for a correct opening line only.

Mark Scheme

Q	Solution	Marks	Total	Comments
5(a)(i)	$\cos\alpha=\frac{3}{5}$	B1	1	ACF
(a)(ii)	$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$	M1		
	$=\frac{3}{5}\cos\beta+\frac{4}{5}\sin\beta$	A1	2	ACF
(a)(iii)	$\sin\beta=\frac{12}{13}$	B1		
	$\cos(\alpha-\beta)=\frac{63}{65}$	B1	2	$\frac{63}{65}$ NMS B1B1
(b)(i)	$\tan 2x=\frac{2\tan x}{1-\tan^2 x}$	M1		
	$2\tan x=1-\tan^2 x$ $\tan^2 x+2\tan x-1=0$	A1	2	CSO AG
(b)(ii)	$\tan x=\frac{-2\pm\sqrt{4+4}}{2}$	M1		must solve quadratic equation by formula or by completing the square condone one slip
	$=-1\pm\sqrt{2}$	A1		$\pm\sqrt{2}$ required
	$2x=45°\Rightarrow x=22\frac{1}{2}^{\circ}$ is acute			
	$\Rightarrow\tan 22\frac{1}{2}^{\circ}=\sqrt{2}-1$	E1	3	explain selection of positive root
	Total		**10**	

Question 6

6	(a)	Express $\dfrac{2}{x^2-1}$ in the form $\dfrac{A}{x-1}+\dfrac{B}{x+1}$.	*(3 marks)*
	(b)	Hence find $\displaystyle\int \frac{2}{x^2-1}\,\mathrm{d}x$.	*(2 marks)*
	(c)	Solve the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2y}{3(x^2-1)}$, given that $y=1$ when $x=3$. Show that the solution can be written as $y^3=\dfrac{2(x-1)}{x+1}$.	*(5 marks)*

Student Response

6. a)

$$\frac{2}{x^2-1}=\frac{A}{x-1}+\frac{B}{x+1}$$

(PART B LOCATED ON EXTRA SHEET)

$$2=A(x+1)+B(x-1)$$

when $x=1$, $2=2A$ when $x=-1$

$A=1$ $2=-2B$

$B=-1$

$$\Rightarrow \frac{2}{x^2-1}=\frac{1}{x-1}-\frac{1}{x+1}$$

on M12

(4)

3

Question number

Leave blank

b) $\int \frac{2}{x^2-1}\,dx$

$\int \frac{1}{x-1} - \frac{1}{x+1}\,dx = \int (x-1)^{-1} - (x+1)^{-1}\,dx$

$= (x-1)^{0} - (x+1)^{0}$

$= 1-1$

$= 0$

M0
A0
×

c) $\frac{dy}{dx} = \frac{2y}{3(x^2-1)}$

$2y\,dx = 3(x^2-1)\,dy$

$\frac{2y}{dy} = \frac{3x^2-1}{dx}$

$2y\,dy = 3x^2-1\,dx$

$\int 2y\,dy = \int 3x^2-1\,dx$

$y^2 = x^3 - x$

M1
A0
A0
1

$\Rightarrow y^3 = \frac{2(x-1)}{x+1}$

$1^3 = \frac{2(2)}{4}$

$1 = 1$

$\therefore y^3 = \frac{2(x-1)}{x+1}$

M0
A0
4

Commentary

(a) The candidate has used a conventional partial fractions approach to find the values of A and B.
(b) The candidate does not apparently know these are standard ln integrals and he makes an error in the integration which leads to a result which is nonsense as this is an indefinite integral. However, it doesn't occur to him to check any of his working.
(c) The candidate knows he is to separate the variables, but just manipulates the expression to what he believes is an integrable form, making algebraic errors and using poor calculus notation. He scores 1 mark for the attempt but both integrals are incorrect. He fails to add an arbitrary constant so can score no further marks. However, he takes the given answer and just demonstrates that it is satisfied by $(3,1)$, apparently believing this shows the given result is true.

Mark Scheme

Q	Solution	Marks	Total	Comments
6(a)	$\dfrac{2}{(x^2-1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1}$			
	$2 = A(x+1) + B(x-1)$	M1		
	$x = 1 \qquad x = -1$	m1		use two values of x or equate coefficients and solve $A + B = 0$ and $A - B = 2$
	$A = 1 \qquad B = -1$	A1	3	both A and B
(b)	$\int \dfrac{2}{x^2-1}\,\mathrm{d}x = p\ln(x-1) + q\ln(x+1)$	M1		ln integrals
	$= \ln(x-1) - \ln(x+1)$	A1F	2	F on A and B condone missing brackets
(c)	$\int \dfrac{\mathrm{d}y}{y} = \int \dfrac{2}{3(x^2-1)}\,\mathrm{d}x$	M1		separate and attempt to integrate on one side
	$\ln y = \frac{1}{3}(\ln(x-1) - \ln(x+1)) \quad (+C)$	A1 A1F		left hand side F from part (b) on right hand side
	$(3,1) \qquad \ln 1 = \frac{1}{3}(\ln 2 - \ln 4) + C$	m1		use (3, 1) to attempt to find a constant
	$3\ln y = \ln(x-1) - \ln(x+1) - (\ln 2 - \ln 4)$			
	$3\ln y = \left(\ln\left(\dfrac{x-1}{x+1}\right) + \ln 2\right)$			
	$\ln y^3 = \ln\left(\dfrac{2(x-1)}{x+1}\right)$			
	$y^3 = \dfrac{2(x-1)}{x+1}$	A1	5	CSO AG
	Total		**10**	

Question 7

7 The coordinates of the points A and B are $(3, -2, 1)$ and $(5, 3, 0)$ respectively.

The line l has equation $\mathbf{r} = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$.

(a) Find the distance between A and B. *(2 marks)*

(b) Find the acute angle between the lines AB and l. Give your answer to the nearest degree. *(5 marks)*

(c) The points B and C lie on l such that the distance AC is equal to the distance AB. Find the coordinates of C. *(5 marks)*

Student Response

Leave blank

7) $A(3, -2, 1)$ $B(5, 3, 0)$

$$\underline{r} = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$

a) $\text{distance} = \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}$

$= \sqrt{(3-5)^2 + (-2-3)^2 + (1-0)^2}$

$= \sqrt{4 + 25 + 1}$

$= \sqrt{30}$ units. ✓

2

b) $\cos\theta = \dfrac{a \cdot b}{|a||b|}$

$a = AB = \sqrt{30}$

$b = \ell = \sqrt{1^2 + 0^2 + -3^2}$

$= \sqrt{10}$

$= \dfrac{9}{\sqrt{30} \times \sqrt{10}}$ ✗

$a \cdot b = AB \cdot \ell$ ✗

$= (3 \times 5) + (-2 \times 3) + (1 \times 0)$

$= 15 - 6$ M0

$= 9$ A0

~~answer~~

B0

$\theta = 58.69355\ldots$

$\theta = 59°$ (nearest degree) ✗ M0 A0

0

(P.t.o)

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c) $\sqrt{30} = \sqrt{(3-x)^2 + (-2-y)^2 + (1-z)^2}$ $\quad (-2-y)(-2-y)$

$4 + 2y + 2y + y^2$

$= \sqrt{(9 - 6x + x^2) + (4 + 4y + y^2) + (1 - 2z + z^2)}$ $\quad (1-z)(1-z)$

$1 - 2z + z^2$

$5 + \lambda 1 = 3 + tx$

$3 + \cancel{3}0 = -2 + ty$

$0 - 3\lambda = 1 + tz$

$AC: \underline{r} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + t \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$

M0

2

Commentary

(a) The candidate starts with a very clear and correct use of the distance formula. Had she also written down the vector $\overrightarrow{AB}$ it might have helped her in part (b).
(b) She knows the formula she should use to find the required angle, but she uses the wrong vectors in the scalar product. Despite writing down *AB.l* she actually uses the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$,the given point *B* on line *l* rather than its direction. Thus she scores no marks for the scalar product. She finds the moduli of two vectors, but has now changed to $\overrightarrow{AB}$ and the direction of line I , so her moduli are inconsistent with her scalar product and she thus scores no marks for attempted use of the formula.
(c) The candidates opening line suggests she understands the question with point C written as (x, y, z). She seems to realise that expanding the brackets is not fruitful, and seems to know she should make use of the given fact that point C lies on line l, as she has written it down but what she has on the right hand side isn't clear, although she now seems to think x, y and z form a direction vector. She apparently gives up in confusion and scores no marks. Had she just substituted her expressions in λ from the line for x, y and z into her opening line she would have scored at least 1 mark and quite possibly more.

Mark Scheme

Q	Solution	Marks	Total	Comments
7(a)	$AB^2 = (5-3)^2 + (3--2)^2 + (0-1)^2$	M1		use $\pm\left(\overrightarrow{OB}-\overrightarrow{OA}\right)$ in sum of squares of components allow one slip in difference
	$AB = \sqrt{30}$	A1	2	accept 5.5 or better
(b)	$\begin{bmatrix}2\\5\\-1\end{bmatrix}\bullet\begin{bmatrix}1\\0\\-3\end{bmatrix} = 2+3=5$	M1		$\pm\overrightarrow{AB}\bullet$ direction l evaluated condone one component error
		A1		5 or – 5
	$\cos\theta = \dfrac{5}{\sqrt{30}\sqrt{10}}$	B1F M1		F on either of candidates' vectors use $\lvert a\rvert\lvert b\rvert\cos\theta = a\bullet b$; values needed
	$\theta = 73°$	A1	5	CAO (condone 73.2, 73.22 or 73.22…)
(c)	$\overrightarrow{AC} = \begin{bmatrix}5+\lambda\\3\\-3\lambda\end{bmatrix} - \begin{bmatrix}3\\-2\\1\end{bmatrix} = \begin{bmatrix}2+\lambda\\5\\-1-3\lambda\end{bmatrix}$	M1 A1		for $\overrightarrow{OC}-\overrightarrow{OA}$ or $\overrightarrow{OA}-\overrightarrow{OC}$ with $\overrightarrow{OC}$ in terms of λ condone one component error
	$(2+\lambda)^2 + 5^2 + (-1-3\lambda)^2 = 30$	m1		
	$10\lambda^2 + 10\lambda = 0$			
	$(\lambda = 0 \text{ or})\ \lambda = -1$	A1		
	$(\lambda = 0 \Rightarrow (5,3,0) \text{ is } B)$			
	$\lambda = -1 \Rightarrow C$ is $(4,3,3)$	A1	5	condone $\begin{bmatrix}4\\3\\3\end{bmatrix}$
	Total		**12**	

Question 8

8 (a) The number of fish in a lake is decreasing. After t years, there are x fish in the lake. The rate of decrease of the number of fish is proportional to the number of fish currently in the lake.

(i) Formulate a differential equation, in the variables x and t and a constant of proportionality k, where $k > 0$, to model the rate at which the number of fish in the lake is decreasing. *(2 marks)*

(ii) At a certain time, there were 20 000 fish in the lake and the rate of decrease was 500 fish per year. Find the value of k. *(2 marks)*

(b) The equation

$$P = 2000 - A\mathrm{e}^{-0.05t}$$

is proposed as a model for the number of fish, P, in another lake, where t is the time in years and A is a positive constant.

On 1 January 2008, a biologist estimated that there were 700 fish in this lake.

(i) Taking 1 January 2008 as $t = 0$, find the value of A. *(1 mark)*

(ii) Hence find the year during which, according to this model, the number of fish in this lake will first exceed 1900. *(4 marks)*

Student Response

Question number

Leave blank

8a) i)	$x \propto t$		
	$x = kt.$	MO	
(ii)	$\frac{dy}{dx} = Ak^{*}$		
	$500 = 20000k^{*}$	MO	
	$k = 0.025.$	csO	0
b) i)	$P = 2000 - Ae^{-0.05t}$		
	$700 = 2000 - Ae^{-0.05\times0}$		
	$700 = 2000 - A$		
	$A = 2000 - 700$		
	$A = 1300.$	B1	1
ii).	$1900 \leq 2000 - 1300e^{-0.05t}$		
	$1900 - 2000 = -1300e^{-0.05t}$		
	$100 = 1300e^{-0.05t}$	M1	
	$\ln 100 = -0.05t \ln 1300$	mO	
	$\frac{\ln 100}{\ln 1300} = -0.05t$		
	$\frac{4.60517}{7.1701} = -0.05t$		
	$t = -12.845$		
	$t = -13$ ~~yeas~~. years.	AO	
	The year is $2008 - 13 = 1995.$	AO	1
			2

Commentary

(a) (i) The candidate appears not to understand what "formulate a differential equation" means as he has written down a relationship between x and t, without a derivative present.
(a) (ii) However, here he does seem to realise a derivative is involved but changes one of the variables to *y.* he also has a product of two constants on the right hand side. His next line is in fact correct, and he gets the correct value for *k*, but there is no evidence here that he knows 20 000 is the value of x. He thus scores no marks. He would have scored 1 mark had he included an x which could be seen to become 20 000 in the way his derivative is seen to become 500.
(b) (i) He finds the value of *A* correctly.
(b) (ii) He starts the solution of the equation for t correctly, but makes a mistake in taking logarithms in omitting the + sign; he would probably have done better had he divided by 1300 first. However, he proceeds and deduces that t is negative. He doesn't query his answer in the context of the question and the given equation, which in fact makes it nonsense, and he simply deducts it from 2008.

Mark Scheme

Q	Solution	Marks	Total	Comments
8(a)(i)	$p\frac{\mathrm{d}x}{\mathrm{d}t}=q$	M1		where p and q are functions
	$\frac{\mathrm{d}x}{\mathrm{d}t}=-kx$	A1	2	in any correct combination
(a)(ii)	$-500=-k\,20000$ or $500=k\,20000$	M1		condone sign error or missing 0 k can be on either side of the equation
	$k=\frac{5}{200}\quad(=0.025)$	A1	2	CSO both (a)(i) and (a)(ii)
(b)(i)	$A=1300$	B1	1	
(b)(ii)	$100>Ae^{-0.05t}$	M1		condone = for >; condone 99 for 100
	$\ln\left(\frac{100}{A}\right)>-0.05t$	m1		take logs correctly condone 0.5
	$t>51.3$	A1		or by trial and improvement (see end of mark scheme)
	population first exceeds 1900 in 2059	A1F	4	F if M1 m1 earned and t>0 following A
	Total		**9**	
	TOTAL		**75**	