Teacher Support Materials 2008

Maths GCE

Paper Reference MFP4

Question 1

Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{bmatrix} 7 & 12 \\ 12 & 0 \end{bmatrix}$.	*(6 marks)*

Student Response

1. $\begin{vmatrix} 7-\lambda & 12 \\ 12 & -\lambda \end{vmatrix} = \lambda^2 - 7\lambda - 144$ $= 0$ ✓

$\lambda = \dfrac{7 \pm \sqrt{(-7)^2 - (4 \times 1 \times -144)}}{2}$

$\lambda = \dfrac{7 \pm \sqrt{625}}{2} = \dfrac{7 \pm 25}{2}$

$\lambda = \dfrac{32}{2} = 16$ ✓ $\quad \lambda = \dfrac{-18}{2} = -9$ ✓

$\begin{bmatrix} 7-16 & 12 \\ 12 & -16 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -9x + 12y \\ 12x - 16y \end{bmatrix} = \underline{0}$

$-9x + 12y = 0 \Rightarrow 3x = 4y$

$12x - 16y = 0 \Rightarrow 3x = 4y$

vector for $\lambda = 16$: $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ ✓

$\begin{bmatrix} 7+9 & 12 \\ 12 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 16x + 12y \\ 12x + 9y \end{bmatrix} = \underline{0}$

$16x + 12y = 0 \Rightarrow 4x = -3y$

$12x + 9y = 0 \Rightarrow 4x = -3y$

Vector for $\lambda = -9$: $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ ✓

Commentary

This candidate's work is (almost) exemplary. He has first written down the quadratic characteristic equation, albeit without the "= 0" that makes it an equation, then solved it to find the two eigenvalues. Substituting each in turn into the matrix-vector equation $(\mathbf{M} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{0}$ then yields a cartesian equation (actually, two equations, but they are in fact the same line) satisfied by the components of any representative eigenvector, and then any suitable eigenvector has been written down. [Note that any (non-zero) multiple of the given answers would also have been a valid answer.]

Mark scheme

1	Attempt at char eqn $\lambda^2 - 7\lambda - 144 = 0$ Solving quadratic to find evals $\lambda = 16$ or -9 $\lambda = 16 \Rightarrow -9x + 12y = 0 \Rightarrow y = \frac{3}{4}x$ $\Rightarrow$ evecs $\alpha\begin{bmatrix}4\\3\end{bmatrix}$ $\lambda = -9 \Rightarrow 16x + 12y = 0 \Rightarrow y = -\frac{4}{3}x$ $\Rightarrow$ evecs $\beta\begin{bmatrix}3\\-4\end{bmatrix}$	M1 M1 A1 M1 A1 A1	 6	Any suitable method Ignore missing "= 0" Any method CAO Either λ substituted back CAO (for any non-zero α) CAO (for any non-zero β)
	Total		**6**	

Question 2

The vectors **a**, **b** and **c** are given by

$$\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \quad \mathbf{b} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \mathbf{c} = -2\mathbf{i} + t\mathbf{j} + 6\mathbf{k}$$

where t is a scalar constant.

(a) Determine, in terms of t where appropriate:

(i) $\mathbf{a} \times \mathbf{b}$; *(2 marks)*

(ii) $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$; *(2 marks)*

(iii) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$. *(2 marks)*

(b) Find the value of t for which **a**, **b** and **c** are linearly dependent. *(2 marks)*

(c) Find the value of t for which **c** is parallel to $\mathbf{a} \times \mathbf{b}$. *(2 marks)*

Student response

Question number

Leave blank

2. a) i) $\underline{a} = \underline{i} + 2\underline{j} + 3\underline{k}$ $\qquad \underline{b} = 2\underline{i} + \underline{j} + \underline{k}$

$\underline{a} \times \underline{b} = \underline{k} - \underline{i} - 2\underline{k} + 2\underline{i} + 6\underline{i} - 3\underline{j}$ bod M1

$= -\underline{i} + 5\underline{j} - \underline{k}$ A0

ii) $(\underline{a} \times \underline{b}) \cdot \underline{c} = \begin{bmatrix} -1 \\ 5 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ t \\ 6 \end{bmatrix} = 2 + 5t - 6 = 5t - 4$ ✓ft. M1 A1

iii) $(\underline{a} \times \underline{b}) \times \underline{c} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & 5 & -1 \\ -2 & t & 6 \end{vmatrix} = (30 + t)\underline{i} + 8\underline{j} + (10 - t)\underline{k}$ M1 A0.

Leave blank

b) 3 vectors are only linearly dependent if the scalar triple product is 0.

$\therefore \quad 5t - 4 = 0$

$5t = 4$

$t = \frac{4}{5}$ ✓ft 2

c) Two vectors are parallel if the cross product is 0.

$\begin{bmatrix} 30 - t \\ 8 \\ 10 - t \end{bmatrix} = \underline{0}$ M0

This not possible, as it would require $8 = 0$

$\therefore \underline{c}$ cannot be parallel to $\underline{a} \times \underline{b}$

(6)

Commentary

This is a very interesting script, and there are several worthwhile points to be drawn from it. In (a) (i), this candidate has attempted the vector product using the *Distributive Law* – as one would if multiplying out two brackets – then applying the properties that **i** × **i = 0**, **i** × **j = k** and **i** × **k = – j** (etc.). However, by employing just about the longest possible method, the whole thing has gone badly wrong, and none of the components in the answer vector is correct.

Note, next, that the working in (ii) is correct on a follow-through basis and both marks are scored as a result. It is then strange to see that the next vector product attempted by this candidate, in (iii), uses a much more concise method than he employed in (i); however, by this point, follow-through of possibly one or two previous follow-throughs has not been considered suitable in the mark-scheme and, although correct on this basis, only the method mark has been allowed. In (b), however, full follow-through of (a) (ii)'s answer has been permitted.

Finally, there are a couple of issues that arise in part (c). The first is that (a) (iii)'s answer has mysteriously changed. In cases of this sort of carelessness, follow-through rules cannot apply, and this candidate would necessarily find himself penalised for inconsistent working and/or answers. The much greater point at stake here, however, is that the candidate *HAS* actually recognised what should be going on ….. *BUT* has then failed to realise that this clearly shows that an error has occurred earlier and, rather than making an obviously incorrect statement, he should have gone back and checked his previous working in order to put it right.

Mark Scheme

2(a)(i)	$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&3\\2&1&2\end{vmatrix}=\begin{bmatrix}1\\4\\-3\end{bmatrix}$	M1 A1	 2	Genuine vector product attempt CAO
(ii)	$(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c}=\begin{bmatrix}1\\4\\-3\end{bmatrix}\bullet\begin{bmatrix}-2\\t\\6\end{bmatrix}=4t-20$	M1 A1	 2	Must get a scalar answer ft
(iii)	$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&4&-3\\-2&t&6\end{vmatrix}=\begin{bmatrix}3t+24\\0\\t+8\end{bmatrix}$	M1 A1	 2	Either using (a)(i) or starting again CAO
(b)	$(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c}=0 \;\Rightarrow\; t=5$	M1A1	2	ft from (a)(ii)
(c)	$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\mathbf{0}$ or $\mathbf{c}=$ mult. of $(\mathbf{a}\times\mathbf{b})$ $\Rightarrow t=-8$	M1 A1	 2	Use of any non-zero row to find some value of t CAO – allow unseen check
	Total		**10**	

Question 3

The matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 4 & 3 & k \end{bmatrix}$, where k is a constant.

Determine, in terms of k where appropriate,

(a) det **A**; *(2 marks)*

(b) $\mathbf{A}^{-1}$. *(5 marks)*

Student Response

Leave blank

b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 4 & 3 & k \end{bmatrix}$

Matrix of cofactors

$$\begin{bmatrix} k-9 & k-12 & -1 \\ k-3 & k-4 & -1 \\ 2 & 2 & 0 \end{bmatrix}$$

Change signs $\begin{bmatrix} k-9 & 12-k & -1 \\ 3-k & k-4 & 1 \\ 2 & -2 & 0 \end{bmatrix}$

Transpose $\begin{bmatrix} 0 & 3-k & -1 \\ 12-k & k-4 & -2 \\ 2 & 1 & k-9 \end{bmatrix}$!

$A^{-1} = \frac{1}{2} \begin{bmatrix} 0 & 3-k & -1 \\ 12-k & k-4 & -2 \\ 2 & 1 & k-9 \end{bmatrix}$

B1 M1 M0 A0 A0 (4)

Commentary

[Note: this candidate's (a) was fully correct and has not been included here.]

The marks for this part of the question are split between the various types. There are two method (**M**) marks; the first for the general idea of attempting the use of the "transposed matrix of co-factors" approach – which this candidate has clearly attempted – and the second for the use of the alternating signs within the matrix of co-factors **and** the transposition. This second method mark has not been earned as this candidate has reflected the matrix's elements about the *non-leading* (top-right-to-bottom-left) diagonal, instead of the leading diagonal (top-left-to-bottom-right). If you look at the mark-scheme, you will see that the first of the two accuracy (**A**) marks is awarded for at least five correct entries. Thus, forgetting to change the signs in alternate places can still lead to the acquisition of this first **A** mark. However, messing up the transposition throws both of these marks away.

The final mark is a **B** mark, which is a stand-alone mark for correct application of a single result, idea or method, often one that can be done at any stage of a solution.Here, the transposed matrix of co-factors has to be multiplied by the reciprocal of the determinant, which was the answer to part (a). In this solution, this is the factor of ½ that appears on the very last line. Even had the candidate got this wrong in (a), this would have followed-through here (except for the obviously problematic answer of zero).

Mark Scheme

3(a)	Det $\mathbf{A} = k + 3 + 12 - 4 - 9 - k = 2$	M1 A1	2	CAO
(b)	$\mathbf{A}^{-1} = \frac{1}{\text{Det A}}(\text{adj } \mathbf{A})$ $= \frac{1}{2}\begin{bmatrix} k-9 & 3-k & 2 \\ 12-k & k-4 & -2 \\ -1 & 1 & 0 \end{bmatrix}$	B1 M1 M1 A1 A1	5	Correct use of the determinant (any value) Attempt at matrix of cofactors Use of transposition **and** signs At least 5 entries correct (even if 2nd M1 not earned) CAO – ft det only
	Total		7	

Question 4

Two planes have equations

$$\mathbf{r}\,.\begin{bmatrix}5\\1\\-1\end{bmatrix}=12 \quad\text{and}\quad \mathbf{r}\,.\begin{bmatrix}2\\1\\4\end{bmatrix}=7.$$

(a) Find, to the nearest 0.1°, the acute angle between the two planes. *(4 marks)*

(b) (i) The point $P(0, a, b)$ lies in both planes. Find the value of a and the value of b. *(3 marks)*

(ii) By using a vector product, or otherwise, find a vector which is parallel to both planes. *(2 marks)*

(iii) Find a vector equation for the line of intersection of the two planes. *(2 marks)*

Student Response

Question number

Leave blank

(4) $\underline{r} \cdot \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix} = 12$ $\quad$ $\underline{r} \cdot \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = 7$

a) ~~or~~ $\underline{a} \cdot \underline{b} = |a||b|\cos\theta$.

$\begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = 10 + 1 - 4 = 7$ ✓

$\left|\begin{matrix} 5 \\ 1 \\ -1 \end{matrix}\right| = \sqrt{25+1+1} = \sqrt{27}$ ✓ $\quad \left|\begin{matrix} 2 \\ 1 \\ 4 \end{matrix}\right| = \sqrt{4+1+16} = \sqrt{21}$ ✓

$\cos\theta = \dfrac{7}{\sqrt{27}\sqrt{21}}$

$\theta = \cos^{-1}\left(\dfrac{7}{\sqrt{27}\sqrt{21}}\right) = 72.90°$ ✓ between normals of planes. 3

$\theta = 90 - 72.90 = \underline{\underline{17.1°}}$ ✗ wrong angle A0.

b)i) $r_1]\ 5x + y - z = 12$ $\quad$ $r_2]\ 2x + y + 4z = 7$

$P(0, a, b)$ $\quad$ ~~$\theta=$~~ $x = 0$.

$y - z = 12$ ✓ $\quad$ $a - b = 12$ ①

$2y + 4z = 7$ $\quad$ $a + 4b = 7$ ✓ ②

②−① $\quad 5b = -5$

$\underline{\underline{b = -1}}$ ✓ $\quad a + 1 = 12$

$\underline{\underline{a = 11}}$ ✓ 3

6.

Question number

Leave blank

ii) $\begin{bmatrix}5\\1\\-1\end{bmatrix} \times \begin{bmatrix}2\\1\\4\end{bmatrix} = \begin{vmatrix} i & j & k \\ 5 & 1 & -1 \\ 2 & 1 & 4 \end{vmatrix} = i(4+1) - j(20+2) + k(5-2)$ 2

$= 5\underline{i} - 22\underline{j} + 3\underline{k}$

iii) $5x + y - z = 12$ ①   ①−②: $3x - 5z = 5$

$2x + y + 4z = 7$ ②   ~~$3x = 5 + 5z$~~

$x = \lambda$.   ~~$5\lambda + y$~~ $3\lambda - 5z = 5$

~~5λ~~ $3\lambda - 5 = 5z$

$z = \frac{3}{5}\lambda - 1$

①×4 = $20x + 4y - 4z = 48$ +

$2x + y + 4z = 7$

$22x + 5y = 55$

$22\lambda + 5y = 55$

$5y = 55 - 22\lambda$

$y = 11 - \frac{22}{5}\lambda$

$\underline{r} = \begin{bmatrix}0\\11\\-1\end{bmatrix} + \lambda \begin{bmatrix}1\\-\frac{22}{5}\\3/5\end{bmatrix}$ $\underline{r} = \begin{bmatrix}0\\11\\-1\end{bmatrix} + \mu \begin{bmatrix}5\\-22\\3\end{bmatrix}$ 2

(10)

Commentary

Although there is very little explanation of what is going on here, this candidate's answers are clearly set out and obviously attempting to do the right things. In (a), however, you can see that a lack of initial explanation has led to their getting carried away; having actually found the correct angle, they then proceed to find another one. You may have noticed the marking principle of ***ISW*** – "ignore subsequent working" – which is often applied to candidates' working when, having gained an acceptable correct form of an answer, they then proceed to "tidy it up" incorrectly. ***ISW*** applies in such cases. It, sadly, *doesn't* apply when the working clearly proceeds in an incorrect way; in this case, to the finding of another angle altogether.

The rest of the question runs very smoothly, although this candidate has clearly failed to notice that the required answers to (b) (i) and (ii) are actually the bits of the answer needed for (iii), and has started all over again using a different method. As it is correct, full marks have been gained, but some precious time may have been used up in this way.

Mark Scheme

4(a)	Use of $\cos\theta = \dfrac{\text{scalar product}}{\text{product of moduli}}$	M1		**Must** be $\begin{bmatrix}2\\1\\4\end{bmatrix}$ and $\begin{bmatrix}5\\1\\-1\end{bmatrix}$ "sin θ=" scores M0 at this stage
	Numerator = 7 Denominator = $\sqrt{21}\sqrt{27}$	B1,B1		Allow denominator unsimplified
	$\theta = 72.9°$	A1	4	CAO (but A0 if candidate proceeds to find its complement)
(b)(i)	$a + 4b = 7$ and $a - b = 12$	B1 M1		At least one correctly stated Solving simultaneously
	$a = 11$ and $b = -1$	A1	3	CAO
(ii)	$\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\5 & 1 & -1\\2 & 1 & 4\end{bmatrix} = \begin{bmatrix}5\\-22\\3\end{bmatrix}$	M1		For any valid, complete method for finding a suitable direction **vector**, eg finding a 2nd common point, eg (2½ , 0, ½) or (1⅔, 3⅔, 0), and then **dv** = difference
		A1	2	CAO
(iii)	$\mathbf{r} = \begin{bmatrix}0\\11\\-1\end{bmatrix} + \lambda\begin{bmatrix}5\\-22\\3\end{bmatrix}$	M1		**Must** be a line equation and use their (b)(ii)
	or other equivalent line form eg $(\mathbf{r} - \mathbf{a}) \times \mathbf{d} = \mathbf{0}$	A1	2	ft their (b)(i) point, or any other correct point on the line A0 if no **r** = or l = etc
	Total		**11**	

Question 5

A plane transformation is represented by the 2×2 matrix **M**. The eigenvalues of **M** are 1 and 2, with corresponding eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively.

(a) State the equations of the invariant lines of the transformation and explain which of these is also a line of invariant points. *(3 marks)*

(b) The diagonalised form of **M** is $\mathbf{M} = \mathbf{U}\,\mathbf{D}\,\mathbf{U}^{-1}$, where **D** is a diagonal matrix.

(i) Write down a suitable matrix **D** and the corresponding matrix **U**. *(2 marks)*

(ii) Hence determine **M** ; *(4 marks)*

(iii) Show that $\mathbf{M}^n = \begin{bmatrix} 1 & \mathrm{f}(n) - 1 \\ 0 & \mathrm{f}(n) \end{bmatrix}$ for all positive integers n, where $\mathrm{f}(n)$ is a function of n to be determined. *(3 marks)*

Student Response

Leave blank

5. $\lambda = 1, 2, \quad v = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}$

a) ~~line of~~ invariant lines = $y = x$, and ~~y = 0 (or the x axis)~~ all ~~the line of y = x is also~~ lines perpendicular to the x axis.

The line of $y = -x$ is also ~~then~~ a line of invariant points.

Why?

B0

B0.

b) i) $D = \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$ $U = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$

2

~~b)~~ ~~c)~~ ii) $M = UDU^{-1} \Rightarrow \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix} = \begin{bmatrix}1 & 2\\0 & 2\end{bmatrix}$

$U^{-1} = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$

$\begin{bmatrix}1 & 2\\0 & 2\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix} = \begin{bmatrix}3 & 2\\2 & 2\end{bmatrix} = M$

M1 A1

B0

A0.

iii) $M^n = UD^nU^{-1}$

$D^n = \begin{bmatrix}1 & 0\\0 & 2^n\end{bmatrix}$

$M^n = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 2^n\end{bmatrix} = \begin{bmatrix}1 & 2^n\\0 & 2^n\end{bmatrix}$

$\begin{bmatrix}1 & 2^n\\0 & 2^n\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix} = \begin{bmatrix}1+2^n & 2^n\\2^n & 2^n\end{bmatrix}$

$\rightarrow \begin{bmatrix}1 & 2^n\\0 & 2^n\end{bmatrix}$

2

A0.

6

Commentary

This candidate has made a real mess of part (a), where the invariant lines of the transformation were to be deduced from the information given about eigenvalues and eigenvectors. Also, it is really no use guessing which of these invariant lines is a line of invariant points, as no marks are given for lucky guessers, and so it is essential to give a valid reason in these cases. He then proceeds correctly into part (b), but falls down at the point when he has to write down the inverse of a 2×2 matrix. This is strange, because this candidate had previously gained full marks on Q3, where the question asked for the inverse of a much more difficult, 3×3 matrix.

This silly mistake has also cost him an accuracy mark in the final part. Moreover, his final answer clearly doesn't match his answer for $\mathbf{M}^1$ a few lines above. Surely this should flag up that a bit of careful checking is needed somewhere; at the least a check that his $\mathbf{U}$ and $\mathbf{U}^{-1}$ multiply to give the identity matrix, **I**, would have been worth the time and effort

Mark Scheme

5(a)	$y = 0$ (or "x-axis") and $y = x$	B1,B1		or $\mathbf{r} = a\begin{bmatrix}1\\0\end{bmatrix}$, $\mathbf{r} = b\begin{bmatrix}1\\1\end{bmatrix}$
	$y = 0$ is a line of invariant points since $\lambda = 1$	B1	3	Allow if proven from $(x', y') = (x, y)$ or ft from their line corresponding to $\lambda = 1$
(b)(i)	$\mathbf{D} = \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$, $\mathbf{U} = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$,	B1,B1	2	ft **U** from **D**
(ii)	$\mathbf{U}^{-1} = \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}$	B1		ft from **U** (provided non-singular)
	$\mathbf{M} = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}$	M1		Attempt
	$= \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}$ or $\begin{bmatrix}1 & 2\\0 & 2\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}$	A1		ft first multiplication
	$= \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}$	A1	4	CAO NMS $\Rightarrow$ 0
(iii)	$\mathbf{D}^n = \begin{bmatrix}1 & 0\\0 & 2^n\end{bmatrix}$	B1		Noted or used
	$\mathbf{M}^n = \mathbf{U}\,\mathbf{D}^n\,\mathbf{U}^{-1}$	M1		Used; **must** actually do some multiplying
	$= \begin{bmatrix}1 & 2^n - 1\\0 & 2^n\end{bmatrix}$	A1	3	
	Total		**12**	

Question 6

Three planes have equations

$$\begin{aligned} x + y - 3z &= b \\ 2x + y + 4z &= 3 \\ 5x + 2y + az &= 4 \end{aligned}$$

where a and b are constants.

(a) Find the coordinates of the single point of intersection of these three planes in the case when $a = 16$ and $b = 6$. *(5 marks)*

(b) (i) Find the value of a for which the three planes do not meet at a single point. *(3 marks)*

(ii) For this value of a, determine the value of b for which the three planes share a common line of intersection. *(5 marks)*

Student Response

6 a) $x + y - 3z = 6$ (i)

$2x + y + 4z = 3$ (ii)

$5x + 2y + 16z = 4$ (iii)

To eliminate y: (iii) − 2×(ii): $x + 8z = -2$ (iv) ✓

(iii) − 2×(i): $3x + 22z = -8$ (v) ✓

(v) − 3×(iv): $-2z = -2$, $z = 1$ ✓

Subbing $z = 1$ into (iv) gives $x = -2 - 8 = -10$ ✓

Subbing $z = 1$ and $x = -10$ into (i) gives $y = 6 + 3 - -10 = 19$ ✓

So single point is $(-10, 19, 1)$

5

b) i) System is represented as $\begin{vmatrix} 1 & 1 & -3 \\ 2 & 1 & 4 \\ 5 & 2 & a \end{vmatrix}$

No single solution when this determinant $= 0$ ✓

$\begin{vmatrix} 1 & 1 & -3 \\ 2 & 1 & 4 \\ 5 & 2 & a \end{vmatrix} = 1\times(a\times1 - 2\times4) - 1\times(2\times a - 4\times5) + -3\times(2\times2 - 1\times5)$ ✓

$= (a - 8) - (2a - 20) + 3 = 15 - a$ ✓

Question number

Leave blank

When $a = 15$, $|M| = 0$, so no unique point.

ii) When there is a common line, upon elimination of a variable the remaining two equations are of the same form.

$x + y - 3z = b$ (i)

$2x + y + 4z = 3$ (ii)

$5x + 2y + 15z = 4$ (iii)

To eliminate y: (iii) − 2×(ii) : $x + 7z = -2$ (iv)

(iii) − 2×(i) : $3x + 21z = 4 - 2b$ (v)

Dividing eq^n (v) by 3 gives : $x + 7z = \frac{4-2b}{3}$

and we know $x + 7z = -2$.

So $\frac{4-2b}{3} = -2$

$4 - 2b = -6$

$b = 5$

3

5

13

Commentary

I have included this candidate's solution to this question as it is absolutely fantastic. Every step has been explained, and every bit of working is clearly laid out. Now, if only all candidates presented their working like this!

6(a)	eg **(2) – (1)** $\Rightarrow x + 7z = -3$ **(3)** $- 2 \times$ **(2)** $\Rightarrow x + 8z = -2$ Solving 2×2 system $x = -10,\ y = 19,\ z = 1$	M1A1 A1 M1 A1	5	Eliminating first variable
(b)(i)	$\begin{vmatrix} 1 & 1 & -3 \\ 2 & 1 & 4 \\ 5 & 2 & a \end{vmatrix} = 15 - a$ Setting = to zero and solving for a $a = 15$	B1 M1 A1	3	Determinant **Must** get a numerical answer ft
(ii)	$x + y - 3z = b$ $2x + y + 4z = 3$ $5x + 2y + 15z = 4$ eg **(2) – (1)** $\Rightarrow x + 7z = 3 - b$ **(3)** $- 2 \times$ **(2)** $\Rightarrow x + 7z = -2$ Equating the two RHSs $b = 5$	M1A1 A1 M1 A1	5	Eliminating first variable CAO NB Eliminating x : $-y + 10z = 3 - 2b$ $-3y + 30z = 4 - 5b$ $-y + 10z = -7$ NB Eliminating z : $10x + 7y = 4b + 9$ $10x + 7y = 5b + 4$ $10x + 7y = 29$
	Total		**13**	
6(a)	**Alternate Schemes** **Cramer's Rule** $\Delta = \begin{vmatrix} 1 & 1 & -3 \\ 2 & 1 & 4 \\ 5 & 2 & 16 \end{vmatrix},\ \Delta_x = \begin{vmatrix} 6 & 1 & -3 \\ 3 & 1 & 4 \\ 4 & 2 & 16 \end{vmatrix},$ $\Delta_y = \begin{vmatrix} 1 & 6 & -3 \\ 2 & 3 & 4 \\ 5 & 4 & 16 \end{vmatrix},\ \Delta_z = \begin{vmatrix} 1 & 1 & 6 \\ 2 & 1 & 3 \\ 5 & 2 & 4 \end{vmatrix}$ $= -1,\ 10,\ -19$ and -1 respectively $x = \frac{\Delta_x}{\Delta},\ y = \frac{\Delta_y}{\Delta},\ z = \frac{\Delta_z}{\Delta}$ $x = -10,\ y = 19,\ z = 1$	M1 A1 M1 A1 A1	(5)	Attempt at any two Any one correct At least one attempted numerically Any 2 correct ft All 3 correct CAO

6(a)	**Inverse matrix method** $C^{-1} = \frac{1}{-1}\begin{bmatrix} 8 & -22 & 7 \\ -12 & 31 & -10 \\ -1 & 3 & -1 \end{bmatrix}$	M1 A1		M0 if no inverse matrix is given
	$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = C^{-1}\begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -10 \\ 19 \\ 1 \end{bmatrix}$	M1 A1 A1	(5)	Any 2 correct ft All 3 correct CAO
6(all)	$\left[\begin{array}{ccc\|c} 1 & 1 & -3 & b \\ 2 & 1 & 4 & 3 \\ 5 & 2 & a & 4 \end{array}\right]$			
	$\rightarrow \left[\begin{array}{ccc\|c} 1 & 1 & -3 & b \\ 0 & -1 & 10 & 3-2b \\ 0 & -3 & a+15 & 4-5b \end{array}\right]$			$R_2' = R_2 - 2R_1$ $R_3' = R_3 - 5R_1$
	$\rightarrow \left[\begin{array}{ccc\|c} 1 & 1 & -3 & b \\ 0 & 1 & -10 & 2b-3 \\ 0 & 0 & a-15 & b-5 \end{array}\right]$		(4)	$R_3' = R_3 + 3R_2$
	(b)(i) For non-unique solutions, $a = 15$		(2)	
	(ii) For consistency, $4 - 5b = 3(3 - 2b) \Rightarrow b = 5$		(2)	
	(a) When $a = 16, b = 6$ $\left[\begin{array}{ccc\|c} 1 & 1 & -3 & 6 \\ 0 & 1 & -10 & 9 \\ 0 & 0 & 1 & 1 \end{array}\right]$ $\Rightarrow z = 1,\ y = 19,\ x = -10$		(5)	

Question 7

A transformation T of three-dimensional space is given by the matrix $\mathbf{W} = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}$.

(a) (i) Evaluate det **W**, and describe the geometrical significance of the answer in relation to T. *(2 marks)*

(ii) Determine the eigenvalues of **W**. *(6 marks)*

(b) The plane H has equation $\mathbf{r} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 0$.

(i) Write down a cartesian equation for H. *(1 mark)*

(ii) The point P has coordinates (a, b, c). Show that, whatever the values of a, b and c, the image of P under T lies in H. *(4 marks)*

Student Response

Leave blank

7a) $\det W = \begin{vmatrix} 3 & -1 & 1 \\ 2 & 0 & 2 \\ -1 & 1 & 1 \end{vmatrix} = 3(-2) + 1(2+2) + 1(2)$

$= -6 + 4 + 2$

$= \underline{0}$ ✓ B1

~~Area~~ scale factor of transformation is 1. ✗ B0.

ii) $\begin{vmatrix} 3-\lambda & -1 & 1 \\ 2 & -\lambda & 2 \\ -1 & 1 & 1-\lambda \end{vmatrix} = 0.$

$(3-\lambda)\left(-\lambda(1-\lambda)-2\right) + 1\left(2(1-\lambda)+2\right) + 1\left(2-(-\lambda(-1))\right)$

$\lambda^2 - \lambda - 2$

$= (3-\lambda)(-\lambda+\lambda^2-2) + (~~2-\lambda~~ ~~2-\lambda~~ 4-2\lambda) + (2-\lambda)$

$= (3-\lambda)~~(\lambda+1)(\lambda-2)~~(\lambda^2-\lambda-2+4-~~2\lambda~~+2-\lambda)$ M1

$= (3-\lambda)(\lambda^2-4\lambda+4)$ A0 ....

$= (3-\lambda)(\lambda-2)^2$ M1

A0

eigenvalues are 3 and 2 (twice). ✗

b) i) $x - y + z = 0.$ ✓ 1

ii) $\begin{pmatrix} 3 & -1 & 1 \\ 2 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a-b+c \\ 2a+2c \\ -a+b+c \end{pmatrix}.$ ✓✓✓

$\begin{pmatrix} 3a-b+c \\ 2a+2c \\ -a+b+c \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 3a-b+c-2a-2c-a+b+c$ ✓

$= 0$

$\therefore$ image of P lie under T lies in H. ✓ 4

(8)

Commentary

Despite scoring the majority of the marks on this question, this candidate has made small, but costly, mistakes which could have been discovered with either some more careful thought, or by a bit of checking. His first answer in (a) is correct, but he is then unsure how to interpret it. At first, he refers to "area"; then elects to hedge his bets by crossing it out (it should refer to volume of course). Having got in a muddle about it, he then suggests that the scale factor (of something unspecified) is 1, which doesn't even match his answer of 0.

To find the (cubic) characteristic equation for the given matrix, one could elect to expand this (correct) determinant fully and then work with the resulting expression algebraically, or to undertake some row/column operations first in order to simplify it. This candidate chooses the first approach, but then "spots" a common factor of (3 – λ) which doesn't even appear in the final term of his expansion. Factorising the following incorrect quadratic term, or solving the equation by (say) the quadratic formula, is now as much a matter of luck as anything, and this candidate has earned only the two method marks available, but **none** of the accuracy marks. This despite the fact that the zero determinant in (a) should have flagged up the fact that (at least) one eigenvalue is zero.

The final part, though not explained, is fully correct.

Mark Scheme

7(a)(i)	det **W** = 0 Transformed shapes have zero volume	B1 B1	2	Or equivalent statement ft volume statements
(ii)	Char eqn is $\lambda^3 - 4\lambda^2 + 4\lambda = 0$	M1A3		One A mark for each coefficient (not the λ^3)
	Solving the cubic eqn $\lambda = 0, 2, (2)$	M1 A1	6	
7(a)(ii)	**Alternative:** $\text{Det}(\mathbf{W} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & -1 & 1 \\ 2 & -\lambda & 2 \\ -1 & 1 & 1-\lambda \end{vmatrix}$			
	$= \begin{vmatrix} 2-\lambda & 0 & 2-\lambda \\ 2 & -\lambda & 2 \\ -1 & 1 & 1-\lambda \end{vmatrix}$	M1		Use of R/C ops. $R_1' \to R_1 + R_3$
	$= (2-\lambda)\begin{vmatrix} 1 & 0 & 1 \\ 2 & -\lambda & 2 \\ -1 & 1 & 1-\lambda \end{vmatrix}$	A1		Factor of $(2-\lambda)$ correctly extracted
	$= (2-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 2 & -\lambda & 0 \\ -1 & 1 & 2-\lambda \end{vmatrix}$			$C_1' \to C_3 - C_1$
	$= (2-\lambda)^2\begin{vmatrix} 1 & 0 & 0 \\ 2 & -\lambda & 0 \\ -1 & 1 & 1 \end{vmatrix}$	A1A1		
	$= (2-\lambda)^2(-\lambda)$ giving eigenvalues 0 and 2 (twice)	M1 A1	(6)	Complete factorisation attempt
(b)(i)	$x - y + z = 0$	B1	1	
(ii)	$\begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3a-b+c \\ 2a+2c \\ -a+b+c \end{bmatrix}$	M1A1		
	$x' - y' + z' = 3a - b + c - 2a - 2c - a + b + c$ $= 0 \Rightarrow P'$ in H also	M1 A1	4	Shown carefully
	Total		**13**	

Question 8

By considering the determinant $\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}$, show that $(x + y + z)$ is a factor of $x^3 + y^3 + z^3 - kxyz$ for some value of the constant k to be determined. *(3 marks)*

Student Response

8) $\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}$

$x(x^2 - zy) - y(zx - y^2) + z(z^2 - xy)$

$x^2 - xyz - xyz + y^3 + z^3 - xyz$

$x^2 + y^3 + z^3 - 3xzy$ ✓

$\underline{K = 3}$ B1

$\begin{vmatrix} x & y & z \\ z+x+y & x+y+z & y+z+x \\ y & z & x \end{vmatrix}$ R_1, R_2, R_3

$R_2 + R_1 + R_3$ M1

$(x+y+z)\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ y & z & x \end{vmatrix}$

↑ factor ✓

no conclusion

A0 (2)

Commentary

There are essentially two parts to this question. Firstly, expand the determinant fully as it stands to get the expression $x^3 + y^3 + z^3 - 3xyz$. Then use row/column operations to extract the rather obvious factor of $(x + y + z)$. This candidate has done both of these things, and yet not gained the final mark. This final mark is for putting the two halves together, and stating what must seem the obvious that, since A is a factor of Δ(the determinant), and $\Delta = B$, then A is a factor of B. Alternatively, any line of working that began with $AC = \ldots$ and ended up with $\ldots = B$ would also have done the trick. However, since the question actually gave you this up front, you need to be careful to ensure that your working dots all the i's and crosses all the t's in the right way. Leaving the two sides of a line of reasoning un-matched will usually lose you a mark. Especially on a further maths paper.

8	Expanding fully: $\Delta = x^3 + y^3 + z^3 - 3xyz$ Using row/column operations: eg $R_1' = R_1 + (R_2 + R_3)$ $\Rightarrow \Delta = (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ y & z & x \end{vmatrix}$ NB Any line of argument that leads correctly from $(x+y+z)\,\mathrm{f}(x, y, z)$ to $x^3 + y^3 + z^3 - 3xyz$ scores full marks	B1 M1 A1	3	With conclusion that $(x+y+z)$ is a factor of the required expression when $k = 3$
	Total		**3**	