Teacher Support Materials

Maths GCE

Paper Reference MM03

Question 1

1 The magnitude of the gravitational force, F, between two planets of masses m_1 and m_2 with centres at a distance x apart is given by

$$F = \frac{Gm_1m_2}{x^2}$$

where G is a constant.

(a) By using dimensional analysis, find the dimensions of G. *(3 marks)*

(b) The lifetime, t, of a planet is thought to depend on its mass, m, its initial radius, R, the constant G and a dimensionless constant, k, so that

$$t = km^{\alpha}R^{\beta}G^{\gamma}$$

where α, β and γ are constants.

Find the values of α, β and γ. *(5 marks)*

Student Response

Question number

1)a) $F = \dfrac{G m_1 m_2}{x^2}$

$\Rightarrow [F] = \left[\dfrac{G m_1 m_2}{x^2}\right]$

$\Rightarrow MLT^{-2} = \dfrac{[G] \times M \times M}{L^2}$ ✓

Leave blank

$$\Rightarrow \quad MLT^{-2} = [G]\, M^2 L^{-2}$$

$$\Rightarrow \quad [G] = M^{-1} L^3 T^{-2}$$

3

1)b)

$$t = k\, m^{\alpha} R^{\beta} G^{\gamma}$$

$$\Rightarrow \quad [t] = [m]^{\alpha} [R]^{\beta} [G]^{\gamma}$$

$$\Rightarrow \quad T = M^{\alpha} L^{\beta} (M^{-1} L^3 T^{-2})^{\gamma}$$

$\Rightarrow$ by comparing indicies

$$(T): \Rightarrow \quad 1 = -2\gamma$$

$$\Rightarrow \quad \gamma = \frac{-1}{2}$$

$$(L): \Rightarrow \quad 0 = \beta + 3\gamma$$

$$\Rightarrow \quad \beta = \frac{3}{2}$$

$$(M): \Rightarrow \quad 0 = \alpha - \gamma$$

$$\Rightarrow \quad \alpha = \frac{-1}{2}$$

5

8

Commentary

The candidate is well familiar with the concept of dimensional analysis. The square brackets are used for dimensions of the physical quantities. The candidate uses the dimensions of force, mass, length squared and time correctly. Algebraic manipulation and solution of simultaneous equations are correct and lead to the results required for full marks in parts (a) and (b) of the question.

Mark scheme

MM03

Q	Solution	Marks	Total	Comments
1(a)	$\text{MLT}^{-2} = \dfrac{[G]\text{MM}}{\text{L}^2}$	M1 A1		
	$[G] = \text{L}^3\text{M}^{-1}\text{T}^{-2}$	A1F	3	
(b)	$t = km^{\alpha}R^{\beta}G^{\gamma}$			
	$\text{T} = \text{M}^{\alpha}\text{L}^{\beta}\text{M}^{-\gamma}\text{L}^{3\gamma}\text{T}^{-2\gamma}$	M1 A1F		L, M, T for G are needed to gain M1
	$-2\gamma = 1 \quad \Rightarrow \quad \gamma = -\dfrac{1}{2}$			
	$\alpha - \gamma = 0 \quad \Rightarrow \quad \alpha = -\dfrac{1}{2}$	m1 m1		Getting 3 equations Solution
	$\beta + 3\gamma = 0 \quad \Rightarrow \quad \beta = \dfrac{3}{2}$	A1F	5	Finding α, β, γ
	Total		**8**	

Question 2

2 The unit vectors **i**, **j** and **k** are directed due east, due north and vertically upwards respectively.

Two helicopters, A and B, are flying with constant velocities of $(20\mathbf{i} - 10\mathbf{j} + 20\mathbf{k})\,\text{m s}^{-1}$ and $(30\mathbf{i} + 10\mathbf{j} + 10\mathbf{k})\,\text{m s}^{-1}$ respectively. At noon, the position vectors of A and B relative to a fixed origin, O, are $(8000\mathbf{i} + 1500\mathbf{j} + 3000\mathbf{k})\,\text{m}$ and $(2000\mathbf{i} + 500\mathbf{j} + 1000\mathbf{k})\,\text{m}$ respectively.

(a) Write down the velocity of A relative to B. *(2 marks)*

(b) Find the position vector of A relative to B at time t seconds after noon. *(3 marks)*

(c) Find the value of t when A and B are closest together. *(5 marks)*

Student response

2. $\underline{v}_A = (20\underline{i} - 10\underline{j} + 20\underline{k})\ \text{ms}^{-1}$

$\underline{v}_B = (30\underline{i} + 10\underline{j} + 10\underline{k})\ \text{ms}^{-1}$

a) ${}_B\underline{v}_A = \underline{v}_A - \underline{v}_B = (-10\underline{i} - 20\underline{j} + 10\underline{k})\ \text{ms}^{-1}$ ✓ 2

b) at noon $(t = 0)$ $\underline{r}_A = (8000\underline{i} + 1500\underline{j} + 3000\underline{k})\,\text{m}$

$\underline{r}_B = (2000\underline{i} + 500\underline{j} + 1000\underline{k})\,\text{m}$

${}_B\underline{r}_A = \underline{r}_A - \underline{r}_B = (6000\underline{i} + 1000\underline{j} + 2000\underline{k})\,\text{m}$ ✓

at time $t = t$,

${}_B\underline{r}_A = \underline{r}_0 + {}_B\underline{v}_A t$

$= (6000\underline{i} + 1000\underline{j} + 2000\underline{k}) + t(-10\underline{i} - 20\underline{j} + 10\underline{k})$

$= (6000 - 10t)\underline{i} + (1000 - 20t)\underline{j} + (2000 + 10t)\underline{k}$ ✓ 3

$= (6000 - 10t)^2 + (1000 - 20t)^2 + (2000 + 10t)^2$ ✓ M1

$\frac{d(a^2)}{dt} = -20(6000 - 10t) - 40(1000 - 20t) + 20(2000 + 10t)$ ✓ A1

$-[illegible]000 + 200t - 40{,}000 + 800t$ ✓ m1

$+ [illegible] + 200t = 0$ A1

$-12000 + 1200t = 0$ ✗ A0

$t = 10$

Commentary

The candidate finds the velocity of *A* relative to *B*, ${}_B v_A$, correctly by subtracting the velocity of *B* from the velocity of *A*. Although collecting like vector terms is not required for full marks in this part of the question, but, by doing so, the candidate reduces the risk of making errors in the subsequent parts of the question.

The candidate finds r_0, the initial position vector of *A* relative to *B*. The position vector of *A* relative to *B* at time *t*, ${}_B r_A$, is then found by using the relationship ${}_B r_A = r_0 + {}_B v_A\, t$. Evidently, the candidate's familiarity with this relationship helps the efficacy of the work done.

2c) The candidate recognises that the helicopters are closest together when $\left|{}_B r_A\right|^2$ is a minimum and that this implies $\frac{\mathrm{d}\left|{}_B r_A\right|^2}{\mathrm{d}t} = 0$. However, in implementing these requirement, the candidate commits an error and writes -12000+1200*t*=0 instead of –120000+1200*t*=0.

Mark Scheme

MM03 (cont)

Q	Solution	Marks	Total	Comments
2 (a)	${}_B\mathbf{v}_A = \mathbf{v}_A - \mathbf{v}_B$ $= (20\mathbf{i} - 10\mathbf{j} + 20\mathbf{k}) - (30\mathbf{i} + 10\mathbf{j} + 10\mathbf{k})$ $= -10\mathbf{i} - 20\mathbf{j} + 10\mathbf{k}$	M1A1	2	Simplification not necessary
(b)	${}_B\mathbf{r}_{0A} = (8000\mathbf{i} + 1500\mathbf{j} + 3000\mathbf{k})$ $-(2000\mathbf{i} + 500\mathbf{j} + 1000\mathbf{k})$ $= 6000\mathbf{i} + 1000\mathbf{j} + 2000\mathbf{k}$	M1		
	${}_\mathrm{B}\mathbf{r}_\mathrm{A} = (6000\mathbf{i} + 1000\mathbf{j} + 2000\mathbf{k})$ $+(-10\mathbf{i} - 20\mathbf{j} + 10\mathbf{k})t$ ${}_\mathrm{B}\mathbf{r}_\mathrm{A} = (6000 - 10t)\mathbf{i} + (1000 - 20t)\mathbf{j}$ $+(2000 + 10t)\mathbf{k}$	M1 A1F	3	Simplification not necessary
(c)	$\left|{}_\mathrm{B}\mathbf{r}_\mathrm{A}\right|^2 = (6000 - 10t)^2 + (1000 - 20t)^2$ $+(2000 + 10t)^2$	M1 A1F		
	The helicopters are closest when $\left|{}_B\mathbf{r}_A\right|^2$ is minimum. $y = (6000 - 10t)^2 + (1000 - 20t)^2$ $+(2000 + 10t)^2$ $\frac{\mathrm{d}y}{\mathrm{d}t} = 2(-10)(6000 - 10t)$ $+2(-20)(1000 - 20t)$ $+ 2(10)(2000 + 10t) = 0$	m1 A1F		
	$t = 100$	A1F	5	
	Alternative: $\begin{pmatrix} 6000 - 10t \\ 1000 - 20t \\ 2000 + 10t \end{pmatrix} \bullet \begin{pmatrix} -10 \\ -20 \\ 10 \end{pmatrix} = 0$	(M1) (A1F)		
	$-60000 + 100t - 20000 + 400t$ $+20000 + 100t = 0$	(m1) (A1F)		
	$600t = 60000$ $t = 100$	(A1F)	(5)	
	Total		**10**	

Question 3a&c

3 A particle P, of mass 2 kg, is initially at rest at a point O on a smooth horizontal surface. The particle moves along a straight line, OA, under the action of a horizontal force. When the force has been acting for t seconds, it has magnitude $(4t + 5)$ N.

(a) Find the magnitude of the impulse exerted by the force on P between the times $t = 0$ and $t = 3$. *(3 marks)*

(b) Find the speed of P when $t = 3$. *(2 marks)*

(c) The speed of P at A is $37.5\ \mathrm{m\,s^{-1}}$. Find the time taken for the particle to reach A. *(4 marks)*

Student Response

Leave blank

3, P 2kg

$F = (4t+5)N$

a, $I = Ft$

between $t = 0$ and $t = 3$

$\therefore I = 3(4(3)+5) - 0(4(0)+5)$ X

0

$I = 3(17) = 51Ns$

c, when $I = 2(37.5) - 2(0)$

$I = 75Ns.$

$\therefore 75 = t(4t+5)$ X

$4t^2 + 5t - 75 = 0$

$t = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$t = \dfrac{-5 \pm \sqrt{25 - 4 \times 4 \times -75}}{2 \times 4}$

$t = \dfrac{-5 \pm \sqrt{1225}}{8}$

$t = \dfrac{-5 \pm 35}{8}$

$t = -5$ or $t = 3.75$

t can't be negative

$\therefore t = 3.75s.$

0

2

Commentary

The candidate seems to know how to find the impulse of a constant force. But they do not know how to find the impulse of a variable force and treats the force as constant. The candidate simply multiplies the variable force by *t*.

For part (c), the candidate understood the Impulse-Momentum principle and gained one method mark. However, again, the candidate proceeded further by treating the force as constant and gained no more marks.

Mark Scheme

MM03 (cont)

Q	Solution	Marks	Total	Comments
3(a)	$I = \int_0^3 (4t+5)\mathrm{d}t$	M1		
	$= \left[2t^2 + 5t\right]_0^3$	m1		Or evaluation of constant
	$= 33$ Ns	A1	3	
	Alternative:			
	$I =$ Area under the Force–Time graph	(M1)		
	$= \frac{17+5}{2} \times 3$	(m1)		
	$= 33$ Ns	(A1)	(3)	
(b)	$I = mv - mu$			
	$33 = 2v - 2(0)$	M1		
	$v = 16.5 \text{ ms}^{-1}$	A1F	2	
(c)	$I = \int_0^t (4t+5)\mathrm{d}t = 2(37.5) - 2(0)$	M1		
	$2t^2 + 5t - 75 = 0$	A1		
	$t = \frac{-5 \pm \sqrt{25 + 8 \times 75}}{4}$	m1		
	$t = 5$	A1F	4	For one value of t identified only
	Total		**9**	

Question3b

3 A particle P, of mass 2 kg, is initially at rest at a point O on a smooth horizontal surface. The particle moves along a straight line, OA, under the action of a horizontal force. When the force has been acting for t seconds, it has magnitude $(4t + 5)$ N.

(a) Find the magnitude of the impulse exerted by the force on P between the times $t = 0$ and $t = 3$. *(3 marks)*

(b) Find the speed of P when $t = 3$. *(2 marks)*

(c) The speed of P at A is $37.5\ \mathrm{m\,s^{-1}}$. Find the time taken for the particle to reach A. *(4 marks)*

Student Response

b) at noon $(t = 0)$ $\quad \underline{r}_A = (8000\underline{i} + 1500\underline{j} + 3000\underline{k})$ m

$\underline{r}_B = (2000\underline{i} + 500\underline{j} + 1000\underline{k})$ m

${}_B\underline{r}_A = \underline{r}_A - \underline{r}_B = (6000\underline{i} + 1000\underline{j} + 2000\underline{k})$ m ✓

at time $t = t$,

${}_B\underline{r}_A = \underline{r}_0 + {}_B\underline{v}_A t$

$= (6000\underline{i} + 1000\underline{j} + 2000\underline{k}) + t(-10\underline{i} - 20\underline{j} + 10\underline{k})$

$= (6000 - 10t)\underline{i} + (1000 - 20t)\underline{j} + (2000 + 10t)\underline{k}$ ✓

3

Commentary

The candidate knows the Impulse-Momentum principle and uses it correctly to arrive at the required result of $16.5\ ms^{-1}$.

Mark Scheme

(b)	$I = mv - mu$ $33 = 2v - 2(0)$ $v = 16.5\ \mathrm{ms^{-1}}$	M1 A1F	2	

Question 4

4 Two small smooth spheres, A and B, of equal radii have masses 0.3 kg and 0.2 kg respectively. They are moving on a smooth horizontal surface directly towards each other with speeds $3\,\mathrm{m\,s^{-1}}$ and $2\,\mathrm{m\,s^{-1}}$ respectively when they collide. The coefficient of restitution between A and B is 0.8.

(a) Find the speeds of A and B immediately after the collision. *(6 marks)*

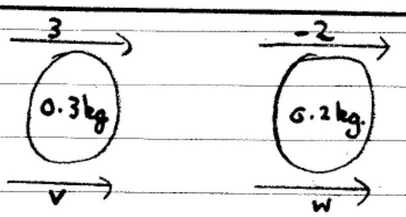
(b) Subsequently, B collides with a fixed smooth vertical wall which is at right angles to the path of the sphere. The coefficient of restitution between B and the wall is 0.7.

Show that B will collide again with A. *(3 marks)*

Student Response

(cont on next page)

4a.



$e = 0.8$

blank

Conservation of momentum:

$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$

$0.3\times 3 + (0.2\times -2) = 0.3V + 0.2W$

$0.9 - 0.4 = 0.3V + 0.2W$

$0.5 = 0.3V + 0.2W$ ①

Restitution:

$e = \dfrac{\text{Speed of separation}}{\text{Speed of approach}}$

$0.8 = \dfrac{W-V}{3-(-2)}$

$0.8 = \dfrac{W-V}{5}$

$W - V = 4$ ②

② × 0.2 $\quad 0.2W - 0.2V = 0.8$ ③

③ − ① $\quad -0.5V = 0.3$

$V = -0.6\ ms^{-1}$

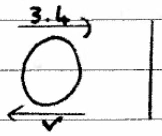
Sub in ① $\quad 0.5 = (0.3\times -0.6) + 0.2W$

$0.5 = -0.18 + 0.2W$

$0.2W = 0.68$

$W = 3.4\ ms^{-1}$

∴ immediately after the collision A is travelling at $0.6\ ms^{-1}$ in the opposite direction to initial and B is travelling at $3.4\ ms^{-1}$ in the opposite direction to initial.



$e = 0.7$

Restitution $\quad e = \dfrac{\text{speed of separation}}{\text{speed of approach}}$ ~~of spee~~

$0.7 = \dfrac{v}{3.4}$

$v = 3.4\times 0.7 = 2.38\ ms^{-1}$

6

Leave blank

b. $I = m(v-u)$

$33 = 2(v-0)$

$33 = 2v$

$v = 16.5\ ms^{-1}$

when $t = 3$ the speed of P is $16.5\ ms^{-1}$

2

Commentary

The work is clearly presented and every step in the solution shown. The candidate uses the principle of conservation of linear momentum and Newton's experimental law correctly (without any sign errors). The resulting equations are then solved simultaneously to give the speeds of *A* and *B* as well as their directions.

For part (b), the candidate uses the experimental law for the sphere *B* to find its speed immediately after collision with the wall. The given result of 2.38 ms^{-1} is accurate. The candidate then comments clearly on both speeds and directions of *A* and *B* to explain why *B* would collide again with *A*.

Mark Scheme

4(a)	Conservation of momentum :			
	$0.3(3) - 0.2(2) = 0.3v_A + 0.2v_B$	M1A1		
	$3v_A + 2v_B = 5$ ---------------(1)			
	Newton's experimental law :			
	$0.8 = \frac{v_B - v_A}{5}$	M1		
	$v_B - v_A = 4$ ----------------(2)	A1		For both (1) and (2)
	Solving (1) and (2)	m1		Dependent on both M1s
	$v_B = 3.4$ $v_A = -0.6$	A1F	6	For both solutions
(b)	$0.7 = \frac{v}{3.4}$	M1		
	$v = 2.38$	A1F		
	Speed of B (2.38) $\succ$ Speed of A (0.6)			
	$\therefore$ B collides again with A	E1	3	Cannot be gained without A1F
	Total		**9**	

Question 5a

5 A ball is projected with speed $u\,\mathrm{m\,s^{-1}}$ at an angle of elevation α above the horizontal so as to hit a point P on a wall. The ball travels in a vertical plane through the point of projection. During the motion, the horizontal and upward vertical displacements of the ball from the point of projection are x metres and y metres respectively.

(a) Show that, during the flight, the equation of the trajectory of the ball is given by

$$y = x\tan\alpha - \frac{gx^2}{2u^2}(1+\tan^2\alpha)$$

(6 marks)

(b) The ball is projected from a point 1 metre vertically below and R metres horizontally from the point P.

(i) By taking $g = 10\,\mathrm{m\,s^{-2}}$, show that R satisfies the equation

$$5R^2\tan^2\alpha - u^2R\tan\alpha + 5R^2 + u^2 = 0$$

(2 marks)

(ii) Hence, given that u and R are constants, show that, for $\tan\alpha$ to have real values, R must satisfy the inequality

$$R^2 \leqslant \frac{u^2(u^2-20)}{100}$$

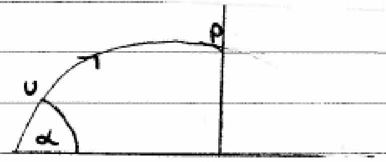
(2 marks)

(iii) Given that $R = 5$, determine the minimum possible speed of projection.

(3 marks)

Student Response

5a.



(9)

Vertically:

$s = y \quad u = u\sin\alpha \quad v = \times \quad a = -g \quad t = ?$

Horizontally

$s = x \quad u = u\cos\alpha \quad v = \times \quad a = \times \quad t = ?$

Horizontally: $s = ut$ so $t = \dfrac{s}{u}$

$\therefore\ t = \cancel{u\cos\alpha}\ \dfrac{x}{u\cos\alpha}$ ✓

Vertically: $s = ut + \frac{1}{2}at^2$

$y = u\sin\alpha\left(\dfrac{x}{u\cos\alpha}\right) - \dfrac{g}{2}\left(\dfrac{x}{u\cos\alpha}\right)^2$ ✓

$y = \dfrac{xu\sin\alpha}{u\cos\alpha} - \dfrac{g}{2}\left(\dfrac{x^2}{u^2\cos^2\alpha}\right)$

$y = x\dfrac{\sin\alpha}{\cos\alpha} - \dfrac{gx^2}{2u^2\cos^2\alpha}$ ✓

$y = x\tan\alpha - \dfrac{gx^2\sec^2\alpha}{\cancel{2u^2}\ 2u^2}$ ✓

$y = x\tan\alpha - \dfrac{gx^2}{2u^2}(1+\tan^2\alpha)$ ✓ $\boxed{1+\tan^2\alpha = \sec^2\alpha}$ 6

Commentary

The candidate considers the motion of the ball horizontally to find an expression for the time in terms of *x*, *u* and $\cos\alpha$. The expression is then substituted into the equation for the vertical motion. Some algebraic manipulation and the identities $\frac{\sin\alpha}{\cos\alpha} = \tan\alpha$ and $\frac{\text{i}}{\cos^2\alpha} = 1 + \tan^2\alpha$ are used to give the equation of the trajectory.

Mark Scheme

MM03 (cont)

Q	Solution	Marks	Total	Comments
5(a)	$y = ut\sin\alpha - \frac{1}{2}gt^2$	M1 A1		
	$x = ut\cos\alpha$	M1		
	$t = \frac{x}{u\cos\alpha}$	A1		
	$y = u\left(\frac{x}{u\cos\alpha}\right)\sin\alpha - \frac{1}{2}g\left(\frac{x}{u\cos\alpha}\right)^2$	M1		
	$y = x\tan\alpha - \frac{gx^2}{u^2\cos^2\alpha}$			
	$y = x\tan\alpha - \frac{gx^2}{2u^2}(1+\tan^2\alpha)$	A1	6	Answer given

Question 5b

$2u^2$

(b) The ball is projected from a point 1 metre vertically below and R metres horizontally from the point P.

(i) By taking $g = 10\,\mathrm{m\,s^{-2}}$, show that R satisfies the equation

$$5R^2\tan^2\alpha - u^2R\tan\alpha + 5R^2 + u^2 = 0 \qquad \textit{(2 marks)}$$

Student Response

Leave blank

b) $P(R, 1)$

i) $g = 10\,\mathrm{ms}^{-1}$, sub P into y equation

$$1 = R\tan\alpha - \frac{\cancel{10}^{5}R^2}{\cancel{2}u^2}(1+\tan^2\alpha) \checkmark$$

$$u^2 = u^2R\tan\alpha - 5R^2 - 5R^2\tan^2\alpha$$

$$\therefore 5R^2\tan^2\alpha - u^2R\tan\alpha + 5R^2 + u^2 = 0 \checkmark$$

2

ii) $b^2 - 4ac \geq 0$

$$(-u^2R)^2 - 4[(5R^2)\times(5R^2+u^2)] \geq 0 \checkmark$$

$$u^4R^2 - 4(25R^4 + 5R^2U^2) \geq 0$$

$$u^4R^2 - 100R^4 - 20R^2u^2 \geq 0$$

$u, R.$ constants, $R^2 > 0$

$$u^4 - 100R^2 - 20u^2 \geq 0 \checkmark$$

$$u^4 - 20u^2 \geq 100R^2$$

$$R^2 \leq \frac{u^2(u^2-20)}{100} \checkmark$$

2

Commentary

Clearly, by writing P(R,1), the candidate understands the axes of reference correctly and recognises that when y=1 (as opposed to –1), x=R. These coordinates are substituted into the equation of the trajectory and then the equation is manipulated to give the required result.

For the part (b)(ii), the candidate recognises that the quadratic equation in $\tan\alpha$ has real roots if its discriminant is non-negative. The candidate forms this inequality and carries out the necessary algebraic manipulation to give the required result.

Mark Scheme

(b)(i)	$1 = R\tan\alpha - \frac{10R^2}{2u^2}(1+\tan^2\alpha)$	M1		
	$5R^2\tan^2\alpha - u^2R\tan\alpha + 5R^2 + u^2 = 0$	A1	2	Answer given
(ii)	For real solutions of the quadratic :			
	$u^4R^2 - 20R^2(5R^2+u^2) \geq 0$	M1		
	$R^2 \leq \frac{u^4 - 20u^2}{100}$			
	$R^2 \leq \frac{u^2(u^2-20)}{100}$	A1	2	Answer given
(iii)	$5^2 \leq \frac{u^2(u^2-20)}{100}$			
	$u^4 - 20u^2 - 2500 \geq 0$	M1		Condone equation
	${u_{\min}}^2 = 61.0$ (or $10+\sqrt{2600}$)	A1		
	$u_{\min} = 7.81$	A1F	3	3 sf required
	Total		**13**	

Question 5b iii

(iii) Given that $R = 5$, determine the minimum possible speed of projection.
(3 marks)

Student Response

100

iii. $5^2 \leq \frac{u^2(u^2-20)}{100}$

$u^2(u^2-20) \geq 2500$

$u^2 \geq 2500$ or $u^2 \geq 2500+20$ X

$u \geq \mp 50$ $u \geq \mp\sqrt{2520}$

since $u>0$, $u \geq 50$ or $\sqrt{2520}$

$\therefore$ min. speed = $50\ ms^{-1}$

0

Commentary

The candidate substitutes 5 for *R* in the given inequality and then multiplies both sides of the inequality by 100 to write $u^2(u^2-20) \geq 2500$. Then, in attempting to solve the inequality, the candidate falsely asserts that either $u^2 \geq 2500$ or $u^2 - 20 \geq 2500$ and gains no marks for this part of the question.

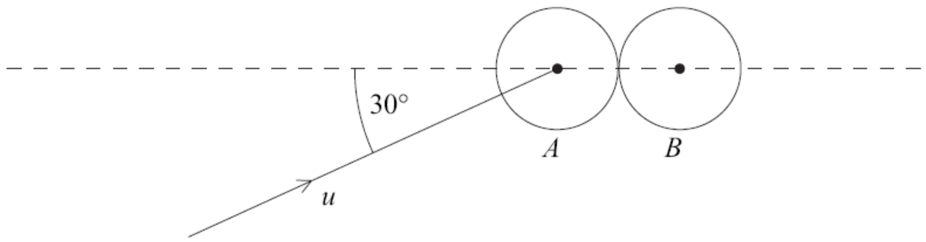
Mark Scheme

(iii)	$5^2 \leq \frac{u^2(u^2-20)}{100}$			
	$u^4 - 20u^2 - 2500 \geq 0$	M1		Condone equation
	$u_{\min}{}^2 = 61.0$ (or $10+\sqrt{2600}$)	A1		
	$u_{\min} = 7.81$	A1F	3	3 sf required

Question 6

6 A smooth spherical ball, A, is moving with speed u in a straight line on a smooth horizontal table when it hits an identical ball, B, which is at rest on the table.

Just before the collision, the direction of motion of A makes an angle of $30°$ with the line of the centres of the two balls, as shown in the diagram.



The coefficient of restitution between A and B is e.

(a) Given that $\cos 30° = \frac{\sqrt{3}}{2}$, show that the speed of B immediately after the collision is

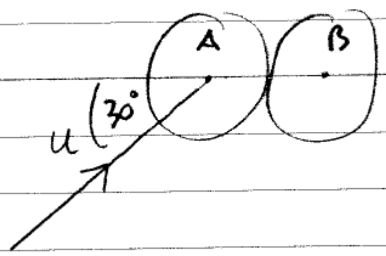
$$\frac{\sqrt{3}}{4}u(1+e)$$ *(5 marks)*

(b) Find, in terms of u and e, the components of the velocity of A, parallel and perpendicular to the line of centres, immediately after the collision. *(3 marks)*

(c) Given that $e = \frac{2}{3}$, find the angle that the velocity of A makes with the line of centres immediately after the collision. Give your answer to the nearest degree. *(3 marks)*

Student Response

6.



$e = e$

$m = m$

$\perp$ to line of centres, ~~motion~~ velocity is unchanged:

$U_{VA} = V_{VA} = u \sin 30°$

Leave blank

$=$ to line of centres:

$$\textcircled{A} \xrightarrow{u} \qquad \textcircled{B}$$

$p: \sum mu = \sum mv$

$$(m)(u\cos 30) + (m)(0) = m(V_A) + m(V_B)$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2}u = V_A + V_B \qquad (1)$$

$e: \quad v_2 - v_1 = -e(u_2 - u_1)$

$$V_B - V_A = -e(0 - u\cos 30)$$

$$\Rightarrow \quad V_B - V_A = \frac{\sqrt{3}}{2}eu \qquad (2)$$

$(1) + (2)$:

$$2V_B = \frac{\sqrt{3}}{2}u + \frac{\sqrt{3}}{2}eu$$

$$\Rightarrow \quad \underline{\underline{V_B = \frac{\sqrt{3}}{4}u(1+e)}}$$

5

(b) From earlier, $\perp$ to line of centres is unchanged:

$$\underline{\underline{V_{A\,PERP}}} = u\sin 30 \quad \underline{\underline{= \frac{u}{2}}}$$

$=$: $V_{A\,PAR} = V_B - \frac{\sqrt{3}}{2}eu$

$$= \frac{\sqrt{3}}{4}u(1+e) - \frac{2\sqrt{3}}{4}eu$$

$$= \frac{\sqrt{3}}{4}u\left[1 + e - 2e\right]$$

$$\underline{\underline{= \frac{\sqrt{3}}{4}u(1-e)}}$$

3

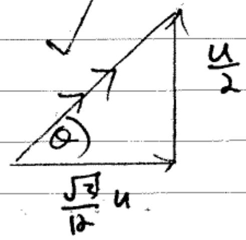
Leave blank

(c) $V_{A\perp} = \frac{u}{2}$

$V_{A=} = \frac{\sqrt{3}}{4}u(1-e)$ $\quad [e = \frac{2}{3}]$

$= \frac{\sqrt{3}}{4}u(1-\frac{2}{3})$ ✓

$= \frac{\sqrt{3}}{12}u$ ✓



$\tan\theta = \frac{u/2}{\frac{\sqrt{3}}{12}u}$ ✓ $\quad \frac{1/2}{\frac{\sqrt{3}}{12}} = \frac{6}{\sqrt{3}}$ ✓

$\Rightarrow \theta = \tan^{-1}\left(\frac{6}{\sqrt{3}}\right)$ ✓

$= 73.8978862...^\circ$ ✓

$= 74^\circ$ (2 s.f.) ✓

3

11

Commentary

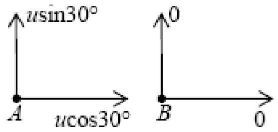
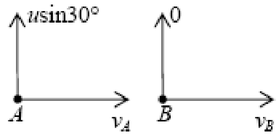
The candidate applies the principle of conservation of linear momentum and the law of restitution along the line of centers of the two balls. The applications are free from sign errors. The candidate solves these equations simultaneously to show the requested result.

For the part (b), the candidate completes the solution of the simultaneous equations to give the speed of *A* parallel to the line of centers in terms of *u* and *e*. The candidate recognises that the speed of *A* perpendicular to the line of centers is unchanged and states it as $u \sin 30$.

The candidate forms the tangent ratio involving the velocity components of *A* perpendicular and parallel to the line of centers to find the angle requested. The ratio is simplified correctly and calculator's inverse tangent function used to find the angle.

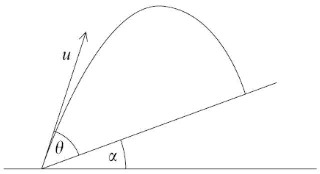
Mark Scheme

MM03 (cont)

Q	Solution	Marks	Total	Comments
6(a)	Before: 			
	After: 			
	Con. of Mom. along the line of centres: $mu\cos 30° = mv_A + mv_B$	M1		
	$v_A + v_B = \frac{\sqrt{3}}{2}u$ ----------(1)	A1		
	Newton's experimental law : $e = \frac{v_B - v_A}{u\cos 30° - 0}$	M1		
	$v_B - v_A = \frac{\sqrt{3}}{2}ue$ ----------(2)	A1		
	Solving (1) and (2) : $v_B = \frac{\sqrt{3}}{4}u(1+e)$	A1	5	Answer given
(b)	$\perp$ $u\sin 30° = \frac{1}{2}u$	B1		$u$sin30 accepted
	$\parallel$ $v_A = \frac{\sqrt{3}}{2}u - \frac{\sqrt{3}}{4}u(1+e)$ $v_A = \frac{\sqrt{3}}{4}u(1-e)$	M1 A1F	3	Simplification not needed
(c)	$\alpha = \tan^{-1}\dfrac{\frac{1}{2}u}{\frac{\sqrt{3}}{4}u\left(1-\frac{2}{3}\right)}$	M1 A1F		
	$\alpha = \tan^{-1}\frac{6}{\sqrt{3}}$ $\alpha = 74°$	A1F	3	To the nearest degree required
	Total		**11**	

Question 7

7 A particle is projected from a point on a plane which is inclined at an angle α to the horizontal. The particle is projected up the plane with velocity u at an angle θ above the plane. The motion of the particle is in a vertical plane containing a line of greatest slope of the inclined plane.



(a) Using the identity $\cos(A+B) = \cos A\cos B - \sin A\sin B$, show that the range up the plane is

$$\frac{2u^2\sin\theta\cos(\theta+\alpha)}{g\cos^2\alpha}$$ *(8 marks)*

(b) Hence, using the identity $2\sin A\cos B = \sin(A+B) + \sin(A-B)$, show that, as θ varies, the range up the plane is a maximum when $\theta = \frac{\pi}{4} - \frac{\alpha}{2}$. *(3 marks)*

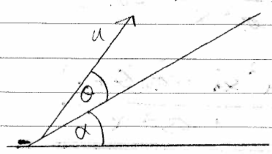
(c) Given that the particle strikes the plane at right angles, show that

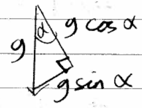
$$2\tan\theta = \cot\alpha$$ *(4 marks)*

Student Response

Question number

7. (a) ~~⊥ to slope~~





⊥ to slope:

$u = u\sin\theta \quad a = -g\cos\alpha \quad s = 0 \quad t = t$

$0 = ut\sin\theta - \frac{g}{2}t^2\cos\alpha$

$\Rightarrow t = 0$ or $u\sin\theta = \frac{g}{2}t\cos\alpha$

$\Rightarrow t = \frac{2u\sin\theta}{g\cos\alpha}$

= to slope:

$u = u\cos\theta \quad s = R \quad t = \quad a = -g\sin\alpha$

$R = u\cos\theta\left(\frac{2u\sin\theta}{g\cos\alpha}\right) - \frac{g}{2}\sin\alpha\left(\frac{2u\sin\theta}{g\cos\alpha}\right)^2$

$\Rightarrow R = \frac{2u^2\sin\theta\cos\theta}{g\cos\alpha} - \frac{2u^2\sin\alpha\sin^2\theta}{g\cos^2\alpha}$

$\Rightarrow R = \frac{2u^2\sin\theta\cos\theta\cos\alpha - 2u^2\sin\alpha\sin^2\theta}{g\cos^2\alpha}$

umber

Leave blank

$$\Rightarrow \quad R = \frac{2u^2 \sin\theta}{g\cos^2\alpha}\left(\cos\theta\cos\alpha - \sin\theta\sin\alpha\right)$$

$$\Rightarrow \quad R = \frac{2u^2 \sin\theta}{g\cos^2\alpha}\left(\cos(\theta+\alpha)\right)$$

$$\Rightarrow \quad R = \frac{2u^2 \sin\theta\cos(\theta+\alpha)}{g\cos^2\alpha}$$

8

(b) $\Rightarrow \quad R = \dfrac{u^2}{g\cos^2\alpha}\left(2\sin\theta\cos(\theta+\alpha)\right)$

$$= \frac{u^2}{g\cos^2\alpha}\left(\sin(2\theta+\alpha) + \sin(-\alpha)\right)$$

$$= \frac{u^2}{g\cos^2\alpha}\left(\sin(2\theta+\alpha) - \sin\alpha\right)$$

As α is fixed, maximum range occurs when

$\sin(2\theta+\alpha) = 1$ (max value of sine is 1)

$\Rightarrow \quad 2\theta + \alpha = \dfrac{\pi}{2}$ [ignore higher solutions ($> 2\pi$)]

$\Rightarrow \quad 2\theta = \dfrac{\pi}{2} - \alpha$

$\Rightarrow \quad \theta = \dfrac{\pi}{4} - \dfrac{\alpha}{2}$

3

Leave blank

[7] (c) This implies that when the particle strikes the plane, its component of velocity parallel to the plane is zero.

$=$ to plane:

$u = u\cos\theta \quad a = -g\sin\alpha \quad v = 0 \quad t = \dfrac{2u\sin\theta}{g\cos\alpha}$ ✓

from earlier

$v = u + at$

$0 = u\cos\theta - g\sin\alpha\left(\dfrac{2u\sin\theta}{g\cos\alpha}\right)$ ✓

$0 = u\cos\theta - 2u\sin\theta\left(\dfrac{g\sin\alpha}{g\cos\alpha}\right)$

$0 = u\cos\theta - 2u\sin\theta\tan\alpha$ ✓

$\Rightarrow \quad 2u\sin\theta\tan\alpha = u\cos\theta$

$\Rightarrow \quad 2\tan\alpha = \dfrac{\cancel{u}\cos\theta}{\cancel{u}\sin\theta}$ ✓

$\Rightarrow \quad 2\tan\alpha = \cot\theta$ ✓ [7]

$\Rightarrow \quad \frac{1}{2}\cot\alpha = \tan\theta$ ✓

$\Rightarrow \quad \underline{\underline{\cot\alpha = 2\tan\theta}}$ ✓

4.

15

Commentary

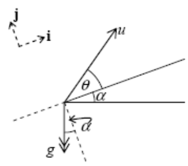
The candidate considers the motion of the particle perpendicular to the slope. The components of the initial velocity and acceleration perpendicular to the slope are stated. It is recognised and stated that for range the displacement *s*=0. The constant acceleration formula involving *s*, *u*, *a* and *t* is used to give and solve the equation for *t*. The candidate then considers motion parallel to the slope, stating and using the relevant expressions for *u*, *s*, *t* and *a* in the constant acceleration formula. Correct algebraic manipulation of the resulting equation and the use of the given identity yield the required expression for the range up the plane.

For the part (b), the candidate uses the given trigonometric identity correctly to separate ϑ from α. The candidate asserts that, as α is fixed, for maximum range, $\sin(2\vartheta+\alpha)=1$. The candidate solves this simple trigonometric equation to show the requested result.

In attempting the last part of this question, the candidate recognises and uses the two essential implications arising from the particle striking the plane at right angles, viz., the component of the velocity parallel to the plane being zero and the displacement perpendicular to the plane being zero. The accurate algebraic manipulation leads to the correct result.

Mark Scheme

MM03 (cont)

Q	Solution	Marks	Total	Comments
7(a)				
	$y=ut\sin\theta-\frac{1}{2}gt^2\cos\theta$	M1A1		
	$y=0 \quad\Rightarrow\quad t=\frac{2u\sin\theta}{g\cos\alpha}$	A1F		
	$x=ut\cos\theta-\frac{1}{2}gt^2\sin\alpha$	M1A1		
	$R=u\frac{2u\sin\theta}{g\cos\alpha}\cos\theta-\frac{1}{2}g\left(\frac{2u\sin\theta}{g\cos\alpha}\right)^2\sin\alpha$	M1		
	$R=\frac{2u^2\sin\theta\cos(\theta+\alpha)}{g\cos^2\alpha}$	m1 A1	8	Dependent on M1s Answer given
(b)	$R=\frac{2u^2\times\frac{1}{2}[\sin(2\theta+\alpha)+\sin(-\alpha)]}{g\cos^2\alpha}$	B1		
	R is maximum when $\sin(2\theta+\alpha)=1$	M1		
	or $2\theta+\alpha=\frac{\pi}{2}$			
	$\therefore\quad\theta=\frac{\pi}{4}-\frac{\alpha}{2}$	A1	3	Answer given
(c)	$y=0 \quad\Rightarrow\quad t=\frac{2u\sin\theta}{g\cos\alpha}$	M1 A2,1		For using y=0 and $\dot{x}=0$ A2 for both correct
	$\dot{x}=0 \quad\Rightarrow\quad t=\frac{u\cos\theta}{g\sin\alpha}$			
	$\frac{2u\sin\theta}{g\cos\alpha}=\frac{u\cos\theta}{g\sin\alpha}$			
	$2\tan\theta=\cot\alpha$	A1	4	Answer given
				N.B. A problem arose which ultimately affected the marking of part 7(c). Please see the Report on the Examination for details.
	Total		15	
	TOTAL		75	