

General Certificate of Education
June 2005
Advanced Level Examination



MATHEMATICS
Unit Further Pure 4

MFP4

Wednesday 22 June 2005 Afternoon Session

In addition to this paper you will require:

- an 8-page answer book;
 - the **blue** AQA booklet of formulae and statistical tables.
- You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions

- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP4.
- Answer **all** questions.
- All necessary working should be shown; otherwise marks for method may be lost.

Information

- The maximum mark for this paper is 75.
- Mark allocations are shown in brackets.

Advice

- Unless stated otherwise, formulae may be quoted, without proof, from the booklet.

Answer **all** questions.

1 Solve the simultaneous equations

$$2x + 7y - 3z = 5$$

$$3x + y + 3z = 10$$

$$8x + 6y = 7$$

(4 marks)

2 (a) Find cartesian equations for the line with vector equation

$$\mathbf{r} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ 6 \\ 3 \end{bmatrix}$$

giving your answer in the form $\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$. (3 marks)

(b) Determine the direction cosines of this line. (2 marks)

3 (a) Evaluate the determinant of the matrix $\mathbf{M} = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 3 & 1 \\ 4 & 0 & -5 \end{bmatrix}$. (2 marks)

(b) A three-dimensional shape S , with volume 12 cm^3 , is transformed by a transformation having matrix \mathbf{X} . Find the volume of the image of S in the case when:

(i) $\mathbf{X} = \mathbf{M}$; (2 marks)

(ii) $\mathbf{X} = \mathbf{M}\mathbf{N}^2$, where \mathbf{N} is a 3×3 matrix and $\det \mathbf{N} = \frac{1}{3}$. (2 marks)

4 (a) Describe the transformation given by each of the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (4 \text{ marks})$$

(b) (i) Determine the matrix \mathbf{AB} . (2 marks)

(ii) Given that \mathbf{AB} represents a rotation, find the axis of rotation and the magnitude of the angle of rotation. (2 marks)

5 The points A , B and C have position vectors

$$\mathbf{a} = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k}, \quad \mathbf{b} = 3\mathbf{i} + \mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \mathbf{c} = 5\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}$$

respectively.

- (a) Write down the vectors \overrightarrow{AB} and \overrightarrow{AC} . (2 marks)
- (b) Find, in the form $\mathbf{r} \cdot \mathbf{n} = d$, a vector equation for the plane ABC . (4 marks)
- (c) Determine, to the nearest 0.1° , the acute angle between the plane ABC and the line with equation $\mathbf{r} = \mathbf{a} + t(5\mathbf{i} + \mathbf{j} + \mathbf{k})$. (4 marks)

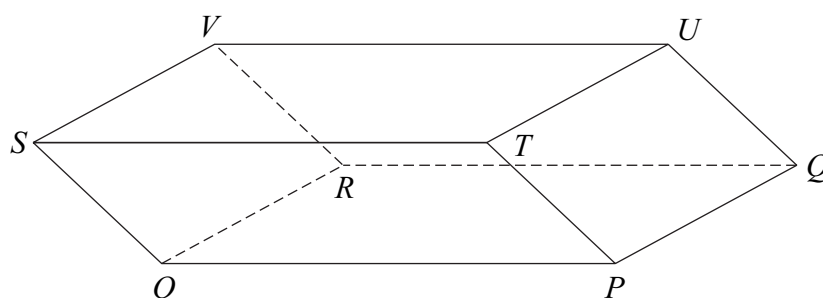
- 6 (a) (i) Explain why $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{a}) = 0$ for all vectors \mathbf{a} and \mathbf{b} . (2 marks)
- (ii) Hence show that

$$\mathbf{a} \cdot (\mathbf{b} \times (\mathbf{c} + \mathbf{a})) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

for all vectors \mathbf{a} , \mathbf{b} and \mathbf{c} . (2 marks)

- (b) The points P , R and S have position vectors $\mathbf{p} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$, $\mathbf{r} = 2\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$ and $\mathbf{s} = 7\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ respectively, relative to the origin O .

- (i) Evaluate $\mathbf{p} \cdot (\mathbf{r} \times \mathbf{s})$. (2 marks)
- (ii) Explain why \mathbf{p} , \mathbf{r} and \mathbf{s} are linearly independent. (1 mark)
- (iii) The parallelepiped $OPQRSTUV$ is shown in the diagram.



Write down the volume of the parallelepiped. (1 mark)

- (iv) Use the result of part (a)(ii) to show that

$$\mathbf{p} \cdot (\mathbf{r} \times \mathbf{t}) = \mathbf{p} \cdot (\mathbf{r} \times \mathbf{s})$$

where \mathbf{t} is the position vector of T . (2 marks)

Turn over ►

- 7 The transformation T is a stretch in a fixed direction, and maps the point (x, y) to the image point (x', y') , where

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 6.4 & -7.2 \\ -7.2 & 10.6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- (a) Determine the scale factor of the stretch. *(2 marks)*
- (b) By considering the invariant points of T, or otherwise, find an equation for the line of invariant points of T, giving your answer in the form $y = mx$. *(5 marks)*

- 8 (a) Show that $(a + b + c)$ is a factor of

$$\begin{vmatrix} a & b & c \\ b + c & c + a & a + b \\ b - c & c - a & a - b \end{vmatrix}$$

Express this determinant as the product of $(a + b + c)$ and a quadratic factor.

(5 marks)

- (b) Hence, or otherwise, show that there is a single real value of a for which the system of equations

$$ax + 3y + z = -5$$

$$4x + (1 + a)y + (a + 3)z = 9$$

$$2x + (1 - a)y + (a - 3)z = 15$$

does not have a unique solution, and find this value of a .

(5 marks)

9 The matrix \mathbf{M} is given by $\begin{bmatrix} 2 & 7 \\ 4 & k \end{bmatrix}$, where k is a constant. It is given that the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of \mathbf{M} .

(a) Show that $k = 5$ and find the eigenvalue corresponding to the eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
(3 marks)

(b) Determine the second eigenvalue of \mathbf{M} and find a corresponding eigenvector. (5 marks)

(c) Write down matrices \mathbf{U} and \mathbf{D} , having integer elements, such that \mathbf{M} can be expressed in the diagonalised form $\mathbf{M} = \mathbf{UDU}^{-1}$. (2 marks)

(d) Write down the matrix \mathbf{U}^{-1} . (1 mark)

(e) The matrix \mathbf{M}^{2n} , for positive integers n , is such that

$$\mathbf{M}^{2n} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

for integers a, b, c and d . Show that

$$a = p \times 4^n + q \times 81^n$$

where p and q are rational numbers to be determined. (4 marks)

END OF QUESTIONS

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