Cambridge International A Level

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Cambridge International Advanced Level

FURTHER MATHEMATICS **9231/12**

Paper 1 **October/November 2018**

MARK SCHEME

Maximum Mark: 100

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

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Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1:

Marks must be awarded in line with:

- the specific content of the mark scheme or the generic level descriptors for the question
- the specific skills defined in the mark scheme or in the generic level descriptors for the question
- the standard of response required by a candidate as exemplified by the standardisation scripts.

GENERIC MARKING PRINCIPLE 2:

Marks awarded are always **whole marks** (not half marks, or other fractions).

GENERIC MARKING PRINCIPLE 3:

Marks must be awarded **positively**:

- marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate
- marks are awarded when candidates clearly demonstrate what they know and can do
- marks are not deducted for errors
- marks are not deducted for omissions
- answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.

GENERIC MARKING PRINCIPLE 4:

Rules must be applied consistently e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.

GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Abbreviations

AEF/OE Any Equivalent Form (of answer is equally acceptable) / Or Equivalent

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a 'fortuitous' answer

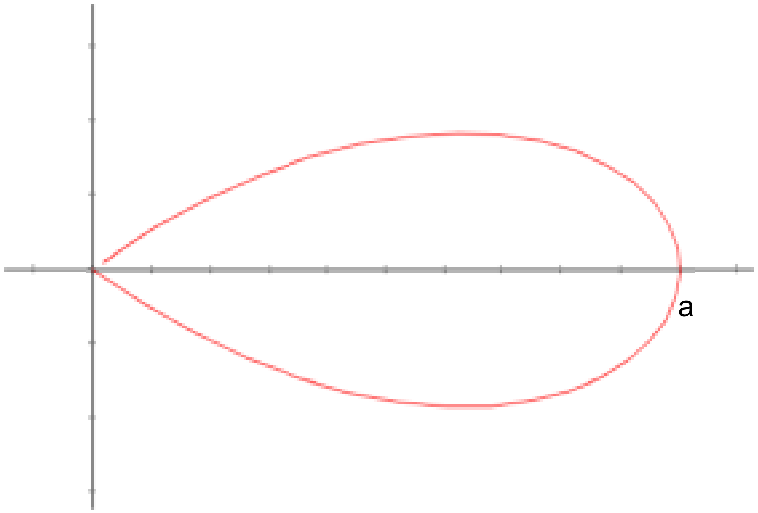
ISW Ignore Subsequent Working

SOI Seen or implied

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Question	Answer	Marks	Guidance
1(i)	$\alpha+\beta+\gamma=5$, $\alpha\beta+\alpha\gamma+\beta\gamma=13$	B1	Sum of roots and $\alpha\beta+\alpha\gamma+\beta\gamma$. SOI
	$\alpha^2+\beta^2+\gamma^2=5^2-2(13)$	M1	Uses $\sum\alpha^2=(\sum\alpha)^2-2(\sum\alpha\beta)$
	$=-1$	A1	www
		3	
1(ii)	$\alpha^3+\beta^3+\gamma^3=5(\alpha^2+\beta^2+\gamma^2)-13(\alpha+\beta+\gamma)+12$	M1	Uses $\alpha^3=5\alpha^2-13\alpha+4$.
	$=5(-1)-13(5)+12=-58$	A1	
	Alt method: Use formula e.g. $\sum\alpha^3=(\sum\alpha)(\sum\alpha^2-\sum\alpha\beta)+3\alpha\beta\gamma$ Or $(\sum\alpha)^3-3(\sum\alpha)(\sum\alpha\beta)+3\alpha\beta\gamma$		
		2	

Question	Answer	Marks	Guidance
2(i)	2	**B1**	Stated
		1	
2(ii)	Negative eigenvalue $=-2$	**B1**	Stated
	$\mathbf{A}+2\mathbf{I}=\begin{pmatrix}4&3&1\\0&0&1\\0&0&3\end{pmatrix}$ $\begin{vmatrix}i&j&k\\4&3&1\\0&0&1\end{vmatrix}$	**M1**	**Uses** vector product (or equations) to find corresponding eigenvector.
	$\begin{pmatrix}3\\-4\\0\end{pmatrix}$	**A1**	Accept any non-zero scalar multiple of $\begin{pmatrix}3\\-4\\0\end{pmatrix}$.
		3	
2(iii)	An eigenvalue of $A+A^6$ is $2+2^6=66$, 62 or 2	**B1**	
	Corresponding eigenvector is $\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\begin{pmatrix}3\\-4\\0\end{pmatrix}$ or $\begin{pmatrix}-6\\1\\3\end{pmatrix}$ oe	**B1**	
		2	

Question	Answer	Marks	Guidance
3(i)		**B1**	Just one loop, correct shape at extremities
		B1	Correct position including (a, 0) labelled or in table.
		2	

Question	Answer	Marks	Guidance
3(ii)	$\frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} a^2 cos^2 3\theta \, d\theta$	**M1**	For using correct formula
	$\frac{a^2}{4}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (\cos 6\theta + 1) \, d\theta$	**M1**	Using double angle formula correctly
	$= \frac{a^2}{4}\left[\frac{1}{6}\sin 6\theta + \theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = \frac{\pi a^2}{12}$	**A1**	
		3	
3(iii)	$r = a\cos\theta\left(4\cos^2\theta - 3\right) \Rightarrow r = a\left(\frac{x}{r}\right)\left(4\left(\frac{x}{r}\right)^2 - 3\right)$	**B1**	Uses $x = r\cos\theta$ **and** $x^2 + y^2 = r^2$.
	$\Rightarrow r^4 = ax\left(4x^2 - 3r^2\right) \Rightarrow \left(x^2 + y^2\right)^2 = ax\left(4x^2 - 3\left(x^2 + y^2\right)\right)$	**M1**	For eliminating θ
	$\Rightarrow \left(x^2 + y^2\right)^2 = ax\left(x^2 - 3y^2\right)$	**A1**	Any equivalent cartesian form without fractions.
		3	

Question	Answer	Marks	Guidance
4(i)	$m^2+2m+1=0 \Rightarrow (m+1)^2=0 \Rightarrow m=-1$	M1	Forms and solves auxiliary equation.
	CF: $(A+Bt)e^{-t}$	A1	States CF.
	PI: $x = p\sin t + q\cos t$	M1	Uses correct form of PI and differentiates twice.
	$\Rightarrow \dot{x} = p\cos t - q\sin t \Rightarrow \ddot{x} = -p\sin t - q\cos t$	A1	
	$-p\sin t - q\cos t + 2(p\cos t - q\sin t) + p\sin t + q\cos t = 4sint$	M1	Compares coefficients and attempts to solve
	$2p=0 \Rightarrow p=0$. $-2q=4 \Rightarrow q=-2$.	A1	
	GS: $\boldsymbol{x}=(A+Bt)e^{-t}-2\cos t$	A1FT	States general solution. FT on correct form only
		7	
4(ii)	$x \approx -2\cos t$	B1FT	
		1	

Question	Answer	Marks	Guidance
5(i)	$\begin{pmatrix} 3 & 2 & 0 & 1 \\ 6 & 5 & -1 & 3 \\ 9 & 8 & -2 & 5 \\ -3 & -2 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 3 & 2 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	**M1**	Attempt to row reduction.
	$\to \begin{pmatrix} 3 & 2 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	**A1**	Two correct rows only
	$r(\mathbf{M}) = 4 - 2 = 2$	**A1**	Obtains rank.
		3	

Question	Answer	Marks	Guidance
5(ii)	$3x + 2y \quad + t = 0$ $y - z + t = 0$	M1	Solves homogeneous system of equations.
	$\Rightarrow t = \mu,\ z = \lambda,\ y = \lambda - \mu,\ x = -\frac{2}{3}\lambda + \frac{1}{3}\mu$	M1	Using 2 parameters
	A basis is $\left\{\begin{pmatrix}-2\\3\\3\\0\end{pmatrix}, \begin{pmatrix}1\\-3\\0\\3\end{pmatrix}\right\}$ or $\left\{\begin{pmatrix}-1\\0\\3\\3\end{pmatrix}, \begin{pmatrix}0\\-1\\1\\2\end{pmatrix}\right\}$ or equivalent	A1	AEF
		3	
5(iii)	$\mathbf{M}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}2\\5\\8\\-2\end{pmatrix}$ so a particular solution is $\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$	B1	Finds a particular solution.
	General solution: $(\mathbf{x} =)\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}-2\\3\\3\\0\end{pmatrix} + \mu\begin{pmatrix}1\\-3\\0\\3\end{pmatrix}$	M1	Using correct format
		A1FT	
		3	

Question	Answer	Marks	Guidance
6	$y^{(1)} = e^x u^{(1)} + ue^x = e^x\left(\binom{1}{0}u + \binom{1}{1}u^{(1)}\right) \Rightarrow H_1$ is true	**M1A1**	Shows base case using product rule
	Assume that $H_k : y^{(k)} = e^x\left(\binom{k}{0}u + \binom{k}{1}u^{(1)} + \cdots + \binom{k}{r}u^{(r)} + \cdots + \binom{k}{k}u^{(k)}\right)$	**B1**	States inductive hypothesis.
	Then $y^{(k+1)} = e^x\left(\binom{k}{0}u + \binom{k}{1}u^{(1)} + \cdots + \binom{k}{r}u^{(r)} + \cdots + \binom{k}{k}u^{(k)}\right) +$	**M1**	Differentiates using product rule
	$e^x\left(\binom{k}{0}u^{(1)} + \binom{k}{1}u^{(2)} + \cdots + \binom{k}{r}u^{(r+1)} + \cdots + \binom{k}{k}u^{(k+1)}\right)$ $= e^x\left(\binom{k}{0}u + \ldots\left(\binom{k}{r} + \binom{k}{r-1}\right)u^r + \ldots\binom{k}{k}u^{(k+1)}\right)$	**M1A1**	Shows application of $\binom{k}{r} + \binom{k}{r-1} = \binom{k+1}{r}$.
	$= e^x\left(\binom{k+1}{0}u + \ldots\binom{k+1}{r}u^r + \ldots\binom{k+1}{k+1}u^{(k+1)}\right)$	**B1**	Shows reasoning for first and last term correctly
	So H_k *implies* H_{k+1} so, by induction, H_n is true for all $n \geqslant 1$.	**A1**	States conclusion.
		8	

Question	Answer	Marks	Guidance
7(i)	$\sum_{r=1}^{N}(3r+1)(3r+4)=9\sum_{r=1}^{N}r^2+15\sum_{r=1}^{N}r+4N$	**M1**	Expands
	$9\left(\frac{1}{6}N(N+1)(2N+1)\right)+15\left(\frac{1}{2}N(N+1)\right)+4N$	**M1**	Substitutes formulae for $\sum r$ and $\sum r^2$.
	$=N\left(\frac{9}{6}(2N^2+3N+1)+\frac{15}{2}N+\frac{15}{2}+4\right)$ $=N(3N^2+12N+13)$	**A1**	Shows simplification to the given answer (AG).
		3	
7(ii)	$\frac{1}{(3r+1)(3r+4)}=\frac{1}{3}\left(\frac{1}{3r+1}-\frac{1}{3r+4}\right)$	**B1**	Finds partial fractions.
	$T_N=\frac{1}{3}\left(\frac{1}{4}-\frac{1}{7}+\frac{1}{7}-\frac{1}{10}+\cdots+\frac{1}{3(N-1)+1}-\frac{1}{3N+4}\right)$	**M1**	Expresses terms as differences.
	$\frac{1}{3}\left(\frac{1}{4}-\frac{1}{3N+4}\right)=\frac{1}{12}-\frac{1}{3(3N+4)}$	**A1**	Cancels to given answer (AG).
		3	
7(iii)	$T_N=\frac{N}{4(3N+4)}\Rightarrow\ \frac{S_N}{T_N}=4(3N+4)(3N^2+12N+13)$	**M1**	Writes $\frac{S_N}{T_N}$ as a polynomial
	So $\frac{S_N}{T_N}$ is an integer because all terms are integers	**A1**	Justifies expression being integer
		2	

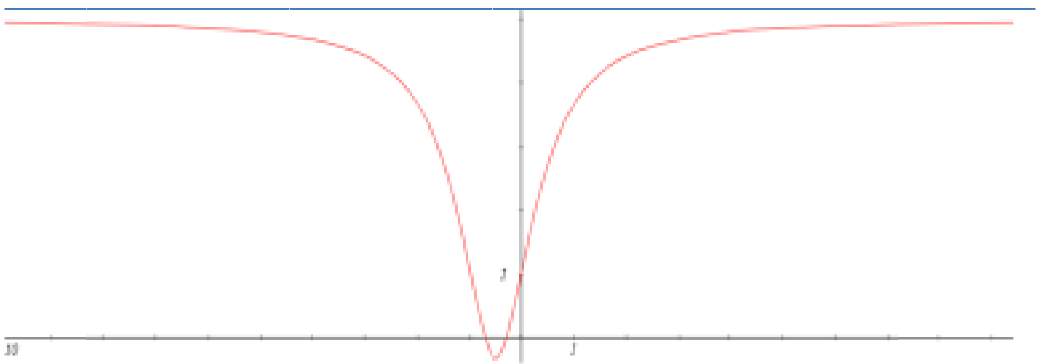
Question	Answer	Marks	Guidance
7(iv)	$\frac{S_N}{N^3 T_N} = \frac{4(3N+4)(3N^2+16N+9)}{N^3}$	M1	Divides expression in **(iii)** by N^3 and takes limit
	$\to 4(3)(3) = 36$	A1	
		2	

Question	Answer	Marks	Guidance
8(i)	$z + z^{-1} = 2\cos\theta$	B1	Use of $z + z^{-1} = 2\cos\theta$.
	$(z+z^{-1})^6 = (z^6+z^{-6}) + 6(z^4+z^{-4}) + 15(z^2+z^{-2}) + 20$	M1A1	Expands and groups.
	$64\cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$	M1A1	Substitutes $z^n + z^{-n} = 2\cos n\theta$.
	$\Rightarrow \cos^6\theta = \frac{1}{32}(10 + 15\cos 2\theta + 6\cos 4\theta + \cos 6\theta)$	A1	(Allow $p = 10$, $q = 15, r = 6, s = 1$.)
		6	

Question	Answer	Marks	Guidance
8(ii)	$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^6 \frac{x}{2} dx = \frac{1}{32}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} 10 + 15\cos x + 6\cos 2x + \cos 3x dx$	**M1**	Applies part **(i)**
		M1	Integrates correctly (3/4 terms correct).
	$\frac{1}{32}\left[10x + 15\sin x + 3\sin 2x + \frac{1}{3}\sin 3x\right]_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}$	**M1**	Inserts limits and evaluates.
	$= \frac{1}{32}\left\{\left(5\pi + 15 + 0 - \frac{1}{3}\right) - \left(-5\pi - 15 + 0 + \frac{1}{3}\right)\right\} = \frac{1}{16}\left(5\pi + \frac{44}{3}\right)$	**A1**	
		4	

Question	Answer	Marks	Guidance
9(i)	$y = 5 - \frac{4}{x^2 + x + 1}$	**M1**	Alt method: Finding limit
	As $x \to \pm\infty$, $y \to 5 \therefore y = 5$ CAO	**A1**	
		2	

Question	Answer	Marks	Guidance
9(ii)	$yx^2 + yx + y = 5x^2 + 5x + 1$ $\Rightarrow (y-5)x^2 + (y-5)x + (y-1) = 0$	**B1**	Forms quadratic equation in x.
	For real x, $(y-5)^2 - 4(y-5)(y-1) \geqslant 0$ (condone $>$)	**M1**	Uses discriminant
	$\Rightarrow (y-5)(3y+1) \leqslant 0$	**M1**	Factorising
	$\Rightarrow -\frac{1}{3} \leqslant y < 5$, because $y = 5$ is an asymptote (www)	**A1**	Explaining strict upper inequality (AG)
		4	
9(iii)	$y' = 0 \Rightarrow (x^2 + x + 1)(10x + 5) - (5x^2 + 5x + 1)(2x + 1) = 0$	**M1**	Differentiates and equates to 0.
	$\Rightarrow 4(2x+1) = 0 \Rightarrow x = -\frac{1}{2}, y = -\frac{1}{3}$	**A1**	
		2	

Question	Answer	Marks	Guidance
9(iv)		B1FT	Positive y-intercept at (0,1), FT dep on minimum point from **(iii)**.
		B1	Correct asymptote and completely correct graph.
		2	

Question	Answer	Marks	Guidance
10(i)	$\overrightarrow{AB}=\begin{pmatrix}2\\3\\2\end{pmatrix}, \overrightarrow{CD}=\begin{pmatrix}1\\1.5\\1\end{pmatrix}=\frac{1}{2}\overrightarrow{AB}$	B1	Or shows if parallel, then m=3/2
		1	

Question	Answer	Marks	Guidance
10(ii)	$\overrightarrow{AB}=\begin{pmatrix}2\\3\\2\end{pmatrix}$, $\overrightarrow{CD}=\begin{pmatrix}1\\m\\1\end{pmatrix}$ **and** $\overrightarrow{AC}=\begin{pmatrix}-2\\-1\\0\end{pmatrix}$ **or** AD $=\begin{pmatrix}-1\\m-1\\1\end{pmatrix}$ **or** BC$=\begin{pmatrix}-4\\-4\\-2\end{pmatrix}$	**B1**	
	$\mathbf{n}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&3&2\\1&m&1\end{vmatrix}=\begin{pmatrix}3-2m\\0\\2m-3\end{pmatrix}$ (so parallel to $\begin{pmatrix}1\\0\\-1\end{pmatrix}$)	**M1A1**	Finds common perpendicular using cross product.
	$\frac{\lvert AC.\boldsymbol{n}\rvert}{\lvert\boldsymbol{n}\rvert}=\frac{\lvert-2(3-2m)+0+0\rvert}{\sqrt{(3-2m)^2+(2m-3)^2}}$ o.e.	**M1**	Uses formula for shortest distance.
	$=\frac{2}{\sqrt{2}}=\sqrt{2}$	**A1**	
		5	
10(iii)	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&3&2\\-2&-1&0\end{vmatrix}=\begin{pmatrix}2\\-4\\4\end{pmatrix}\sim\begin{pmatrix}1\\-2\\2\end{pmatrix}$ o.e.	**M1A1**	Finds normal to plane ABC (AEF).
	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&3&2\\-1&1&1\end{vmatrix}=\begin{pmatrix}1\\-4\\5\end{pmatrix}$ o.e.	**A1**	Finds normal to ABD (AEF).
	$\cos\theta=\frac{1+8+10}{\sqrt{1^2+2^2+2^2}\sqrt{1^2+4^2+5^2}}\left(=\frac{19}{\sqrt{378}}\right)$	**M1A1FT**	Uses formula for angle between two lines.
	$\Rightarrow\theta=12.2°$	**A1**	CAO.
		6	

Question	Answer	Marks	Guidance
11E(i)	$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{12t^{\frac{1}{2}}}{18-2t}=\frac{6t^{\frac{1}{2}}}{9-t}$	**B1**	AEF.
	$\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{dy}{dx}\right)\times\frac{dt}{dx}$	**M1**	Uses chain rule again to find second derivative.
	$=\frac{3t^{-\frac{1}{2}}(9-t)+6t^{\frac{1}{2}}}{(9-t)^2(18-2t)}$	**dM1**	Uses quotient (or product rule)
	$=\frac{3t^{-\frac{1}{2}}(9-t+2t)}{2(9-t)^3}=\frac{3(9+t)}{2t^{\frac{1}{2}}(9-t)^3}$	**A1**	AG.
		4	
11E(ii)	$\frac{1}{56}\int_0^{56}\frac{d^2y}{dx^2}dx$	**B1**	Uses correct formula for mean value
	$=\frac{1}{56}\left[\frac{6t^{\frac{1}{2}}}{9-t}\right]_{t=0}^{t=4}$	**M1**	Finding limits correctly
		M1	Using expression
	$=\frac{1}{56}\left(\frac{6\sqrt{4}}{9-4}\right)=\frac{3}{70}.$	**A1**	Inserts correct values of t and obtains answer (AG.)
		4	

Question	Answer	Marks	Guidance
11E(iii)	$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=(18-2t)^2+144t=4(t+9)^2$	**M1A1**	Simplifies $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2$.
	$2\pi\int_0^4\left(8t^{\frac{3}{2}}\right)(2(t+9))dt=32\pi\int_0^4 t^{\frac{5}{2}}+9t^{\frac{3}{2}}dt$	**M1A1**	Uses $2\pi\int y\frac{ds}{dt}dt$.
	$=32\pi\left[\frac{2}{7}t^{\frac{7}{2}}+\frac{18}{5}t^{\frac{5}{2}}\right]_0^4$	**M1**	Integrates term by term.
	$=\frac{2^{11}\times 83}{35}\pi=\frac{169984}{35}\pi=15300$	**A1**	Accept exact answer or decimal rounding to 15300.
		6	
11O(i)	$I_n=\left[x\left(x^2-1\right)^n\right]_1^{\sqrt{2}}-2n\int_1^{\sqrt{2}}x^2\left(x^2-1\right)^{n-1}dx$	**M1A1**	Integrates by parts.
	$=\sqrt{2}-2n\int_1^{\sqrt{2}}\left(x^2-1+1\right)\left(x^2-1\right)^{n-1}dx$	**M1**	Uses $x^2=x^2-1+1$.
	$=\sqrt{2}-2nI_n-2nI_{n-1}$	**A1**	
	$\Rightarrow(2n+1)I_n=\sqrt{2}-2nI_{n-1}$	**A1**	AG.
		5	

Question	Answer	Marks	Guidance
11O(ii)	$\frac{dx}{d\theta} = \tan\theta\sec\theta$	**M1A1**	Differentiates $\sec\theta$.
	$\sec\theta = \sqrt{2} \Rightarrow \theta = \frac{\pi}{4}$ $\sec\theta = 1 \Rightarrow \theta = 0$	**B1**	Changes limits.
	$I_n = \int_0^{\frac{\pi}{4}} \left(\sec^2\theta - 1\right)^n \tan\theta\sec\theta\,\mathrm{d}\theta = \int_0^{\frac{\pi}{4}} \tan^{2n+1}\theta\sec\theta\,\mathrm{d}\theta$	**B1**	Uses $\sec^2\theta - 1 = \tan^2\theta$. (AG.)
		4	
11O(iii)	$\int_0^{\frac{\pi}{4}} \frac{\sin^7\theta}{\cos^8\theta}\,\mathrm{d}\theta = I_3$	**B1**	Deduces that integral is I_3.
	$I_0 = \sqrt{2} - 1$ (or $I_1 = \frac{2-\sqrt{2}}{3}$)	**B1**	Calculates I_0 or I_1
	$3I_1 = \sqrt{2} - 2I_0 \Rightarrow I_1 = \frac{2-\sqrt{2}}{3}$.	**M1A1**	Uses reduction formula to find I_1 *or* I_2 ………
	$I_2 = \frac{\sqrt{2}}{5} - \frac{4}{5}I_1 = \frac{7\sqrt{2}-8}{15}$		Finds I_2.
	$I_3 = \frac{\sqrt{2}}{7} - \frac{6}{7}I_2 = \frac{16-9\sqrt{2}}{35}$	**A1**	Finds I_3.
		5	