CAMBRIDGE INTERNATIONAL EXAMINATIONS

Cambridge International Advanced Level

MARK SCHEME for the May/June 2015 series

9231 FURTHER MATHEMATICS

9231/11 Paper 1 (Paper 1), maximum raw mark 100

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge will not enter into discussions about these mark schemes.

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Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol √ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

- Note: B2 or A2 means that the candidate can earn 2 or 0.
  B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking g equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF Any Equivalent Form (of answer is equally acceptable)

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

BOD Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a 'fortuitous' answer

ISW Ignore Subsequent Working

MR Misread

PA Premature Approximation (resulting in basically correct work that is insufficiently accurate)

SOS See Other Solution (the candidate makes a better attempt at the same question)

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through √" marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

Qn & Part	Solution	Marks
1	$3 \times \frac{13\times14\times27}{6} - 5\times\frac{13\times14}{2} + 13 = 2015$	M1A1
	$\left[\frac{9\times10}{2}\right]^2 - 10 = 2015$ (Award M1 for subtracting 9 or 10 here.)	M1A1 (4) **Total: 4**
2	$\begin{bmatrix} 2 & -3 & 4 \\ 3 & -1 & 0 \\ 1 & 2 & k \end{bmatrix} = 0 \Rightarrow 7k = -28 \Rightarrow k = -4$	M1A1 (2)
	Add 1st and 3rd $\Rightarrow 3x - y = 2$ (Same as 2nd) (OE)	M1
	Set $x = t$ (for example) (OE)	M1
	$\Rightarrow y = 3t - 2,\ z = \frac{7}{4}t - \frac{5}{4}$ (OE) – many forms	A1A1 (4) **Total: 6**
3	$a_1 > 5$ (given) $\Rightarrow$ H_1 is true.	B1
	Assume H_k is true for some positive integer k, i.e. $a_k = 5 + \delta$, where $\delta > 0$.	B1
	$a_{k+1} - 5 = \frac{4a_k^2 + 25}{5a_k} - 5 = \frac{4a_k^2 + 25 - 25a_k}{5a_k} = \frac{(4a_k - 5)(a_k - 5)}{5a_k} > 0\ , \Rightarrow a_{k+1} > 5$	M1A1
	Or	
	$a_{k+1} = \frac{4}{5}(5+\delta) + \frac{5}{5+\delta},\ = 4 + \frac{4}{5}\delta + (1 - \frac{\delta}{5} + \frac{\delta^2}{25} - ...)$ for $0 < \delta < 5$	(M1) (A1)
	$= 5 + \frac{3}{5}\delta + 0(\delta^2) \geqslant a_{k+1} > 5$, ($\delta \geqslant 5$ is trivial).	
	$H_k \Rightarrow H_{k+1}$ and H_1 is true, hence by mathematical induction, the result is true for all $n \in \mathbf{Z}^+$ (N.B. The minimum requirement is 'true for all positive integers'.)	A1 (5)
	$a_{k+1} - a_k = \frac{5}{a_k} - \frac{1}{5}a_k$	M1
	$\frac{5}{a_k} < 1$ and $\frac{1}{5}a_k > 1 \Rightarrow a_{k+1} - a_k < 0 \Rightarrow a_{k+1} < a_k$	A1 (2) **Total: 7**
4	$\alpha + \beta + \gamma = 7\ ,\ \alpha\beta + \beta\gamma + \gamma\alpha = 2\ ,\ \alpha\beta\gamma = 3$ (Stated or implied by working.)	B1
(i)	$\frac{1}{(\alpha\beta)(\beta\gamma)(\gamma\alpha)} = \frac{1}{(\alpha\beta\gamma)^2} = \frac{1}{9}$	B1
(ii)	$\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} = \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma} = \frac{7}{3}$	M1A1
(iii)	$\frac{1}{\alpha^2\beta\gamma} + \frac{1}{\alpha\beta^2\gamma} + \frac{1}{\alpha\beta\gamma^2} = \frac{1}{\alpha\beta\gamma}\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right) = \frac{1}{\alpha\beta\gamma}\left(\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}\right) = \frac{2}{9}$	M1A1 (6)
	$\Rightarrow x^3 - \frac{7}{3}x^2 + \frac{2}{9}x - \frac{1}{9} = 0 \Rightarrow\ 9x^3 - 21x^2 + 2x - 1 = 0$	M1A1√ (2) **Total: 8**

Qn & Part	Solution	Marks
5	$\sin\frac{1}{2}\theta = \frac{1}{2} \Rightarrow \theta = \frac{1}{3}\pi$; Intersection at $\left(\frac{1}{\sqrt{2}}, \frac{1}{3}\pi\right)$ (Accept 1.05 for $\frac{1}{3}\pi$.)	M1A1 (2)
	C_1: Circle centre at pole and radius $1/\sqrt{2}$ C_2: Curve, approx. correct orientation, from (0,0) to $(1,\pi)$. Completely correct correct shape.	B1 B1 B1 (3)
	$\frac{1}{6}\times\pi\times\frac{1}{2} - \frac{1}{2}\int_0^{\frac{1}{3}\pi}\sin\frac{1}{2}\theta\,\mathrm{d}\theta$	M1A1
	$= \frac{1}{12}\pi\times\frac{1}{2}\left[-2\cos\frac{1}{2}\theta\right]_0^{\frac{1}{3}\pi} = \frac{1}{12}\pi + \frac{\sqrt{3}}{2} - 1$	M1A1 (4) **Total: 9**
6	$2x - 6\left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right) + 50y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ At (3,1) $6 - 18y' - 6 + 50y' = 0 \Rightarrow y' = 0$ (AG)	M1A1 A1 A1 (4)
	$2 - 6(xy'' + y' + y') + 50[(y')^2 + yy''] = 0$	B1B1
	At (3,1) $2 - 18y'' + 50y'' = 0 \Rightarrow y'' = -\frac{1}{16}$	M1A1
	$\Rightarrow$ maximum. (All previous marks required for final mark, i.e. CSO)	A1 (5) **Total: 9**
7	$I_n = \int_0^{\frac{\pi}{2}} x^n \sin x\,\mathrm{d}x = [-x^n\cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} nx^{n-1}\cos x\,\mathrm{d}x$ (LNR)	M1A1
	$= 0 + \left[nx^{n-1}\sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} n(n-1)x^{n-2}\sin x\,\mathrm{d}x$ (LR)	M1A1
	$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$ (AG)	A1 (5)
	$I_0 = \int_0^{\frac{\pi}{2}} x^n \sin x\,\mathrm{d}x = \left[-\cos x\right]_0^{\frac{\pi}{2}} = 1$	B1
	$I_2 = \pi - 2$, using reduction formula.	M1A1
	$I_4 = 4\times\left(\frac{\pi}{2}\right)^3 - 12(\pi - 2) = \frac{1}{2}\pi^3 - 12\pi + 24$ (If I_2 found by integration M1A1 (if correct) then M1A1 for I_4 from reduction formula.)	A1 (4) **Total: 9**

8	$\sum_{r=1}^{n} z^{2r-1} = \dfrac{z(1-[z^2]^n)}{1-z^2} = \dfrac{z-z^{2n+1}}{1-z^2}$	M1A1
	$= \dfrac{1-z^{2n}}{z^{-1}-z} = \dfrac{1-(\cos 2n\theta + i\sin 2n\theta)}{\cos\theta - i\sin\theta - (\cos\theta + i\sin\theta)} = \dfrac{1-\cos 2n\theta - i\sin 2n\theta}{-2i\sin\theta}$	M1A1 A1
	Equating imaginary parts: $\sum_{r=1}^{n} \sin(2r-1)\theta = \dfrac{2\sin^2 n\theta}{2\sin\theta} = \dfrac{\sin^2 n\theta}{\sin\theta}$ (AG)	M1A1 (7)
	Differentiating: $\sum_{r=1}^{n} \sin(2r-1)\cos(2r-1)\theta = 2n\sin n\theta\cos n\theta\,\mathrm{cosec}\,\theta - \sin^2 n\theta\,\mathrm{cosec}\,\theta\cot\theta$	M1A1
	Putting $\theta = \dfrac{\pi}{2n}$: $\sum_{r=1}^{n} (2r-1)\cos(2r-1)\left(\dfrac{\pi}{2n}\right) = n\sin\dfrac{\pi}{2}\cos\dfrac{\pi}{2}\mathrm{cosec}\left(\dfrac{\pi}{2n}\right) - \sin^2\dfrac{\pi}{2}\mathrm{cosec}\left(\dfrac{\pi}{2n}\right)\cot\left(\dfrac{\pi}{2n}\right)$	dM1
	$\Rightarrow \sum_{r=1}^{n} (2r-1)\cos\left[\dfrac{(2r-1)\pi}{2n}\right] = -\,\mathrm{cosec}\left(\dfrac{\pi}{2n}\right)\cot\left(\dfrac{\pi}{2n}\right)$. (AG)	A1 (4) **Total: 11**
9	$\dot{x} = 4 + 3t^{\frac{1}{2}}$ $\quad \dot{y} = 4 - 3t^{\frac{1}{2}}$	B1
	$\dot{x}^2 + \dot{y}^2 = 32 + 18t$	B1
	$s = \int_0^4 (32+18t)^{\frac{1}{2}}\,dt = \left[\dfrac{1}{27}(32+18t)^{\frac{3}{2}}\right]_0^4$	M1A1
	$= \dfrac{1}{27}\left(104^{\frac{3}{2}} - 32^{\frac{3}{2}}\right) = 32.6$ (CAO)	M1A1 (6)
	$\int_0^{32} y\,dx = \int_0^4 y\dfrac{dx}{dt}\,dt = \int_0^4 t\left(4-2\sqrt{t}\right)\left(4+3\sqrt{t}\right)dt = \int_0^4 \left(16t + 4t^{\frac{3}{2}} - 6t^2\right)dt$	M1
	$= \left[8t^2 + \dfrac{8}{5}t^{\frac{5}{2}} - 2t^3\right]_0^4$ $\quad$ (= 51.2)	M1A1
	MV $= \dfrac{\int_b^a y\,dx}{b-a} = \; = \dfrac{51.2}{32} = 1.6$ (CAO)	M1A1 (5) **Total: 11**

10	$\begin{pmatrix} 2 & 2 & -3 \\ 2 & 2 & 3 \\ -3 & 3 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ -3 \end{pmatrix} \Rightarrow$ eigenvalue is -3.	M1A1 (2)
	$\lambda = 4: \quad \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & -3 \\ -3 & 3 & -1 \end{vmatrix} = \begin{pmatrix} 7 \\ 7 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ (Or by equations)	M1A1
	$\lambda = 6: \quad \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 2 & -3 \\ -3 & 3 & -3 \end{vmatrix} = \begin{pmatrix} 3 \\ -3 \\ -6 \end{pmatrix} \sim \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$	A1 (3)
	$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & -2 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$	B1√B1√ (2) M1;A1
	$\mathbf{B = QAQ^{-1} = QPAP^{-1}Q^{-1}; = QPA\ (QP)^{-1}}$	B1
	$\mathbf{QP} = \begin{pmatrix} -2 & 15 & -17 \\ -1 & 5 & -7 \\ 0 & 3 & -3 \end{pmatrix}$ **(CAO)**	
	Eigenvalues are -3, 4 and 6 (same as those for **A**). (CAO)	B1
	Eigenvectors are: $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 15 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 17 \\ 7 \\ 3 \end{pmatrix}$.	B1 (5) **Total: 12**
11 (e)	$\dfrac{dv}{dx} = -\dfrac{1}{y^2}\dfrac{dy}{dx}$	B1
	$\dfrac{d^2v}{dx^2} = -\dfrac{1}{y^2}\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 \dfrac{d^2v}{dy^2} = -\dfrac{1}{y^2}\dfrac{d^2y}{dx^2} + \dfrac{2}{y^3}\left(\dfrac{dy}{dx}\right)^2$	M1A1
	$\dfrac{2}{y^3}\left(\dfrac{dy}{dx}\right)^2 - \dfrac{1}{y^2}\dfrac{d^2y}{dx^2} - \dfrac{2}{y^2}\dfrac{dy}{dx} + \dfrac{5}{y} = 17 + 6x - 5x^2 \sim \dfrac{d^2v}{dx^2} + 2\dfrac{dv}{dx} + 5v = 17 + 6x - 5x^2$ (AG)	A1 (4)
	$m^2 + 2m + 5 = 0 \Rightarrow m = -1 \pm 2i \Rightarrow CF: e^{-x}(A\cos 2x + B\sin 2x)$	M1A1
	PI : $v = px^2 + qx + r \Rightarrow v' = 2px + q \Rightarrow v'' = 2p$	M1
	Equate coefficients: $5p = -5 \quad 4p + 5q = 6 \quad 2p + 2q + 5r = 17$	M1
	$p = -1,\ q = 2,\ r = 3$	A1
	GS : $v = e^{-x}(A\cos 2x + B\sin 2x) + 3 + 2x - x^2$	A1
	When $x = 0 \quad y = \dfrac{1}{2} \Rightarrow v = 2$ and $\dfrac{dy}{dx} = -1 \Rightarrow \dfrac{dv}{dx} = 4$	

	$\Rightarrow 2 = A + 3 \Rightarrow A = -1$	B1
	$v' = e^{-x}\ (-2A \sin 2x + 2B \cos 2x) - e^{-x}\,(A \cos 2x + B \sin 2x)$	M1
	$\Rightarrow 4 = 2B + 1 + 2 \Rightarrow 2B = 1 \Rightarrow B = \dfrac{1}{2}$	A1
	$\Rightarrow y = \left[e^{-x}\left(\dfrac{1}{2}\sin 2x - \cos 2x\right) + 3 + 2x - x^2 \right]^{-1}$	A1 (10) **Total: 14**
110 (i)	$\mathbf{p} = \begin{pmatrix} 8+\lambda \\ 2-2\lambda \\ 3 \end{pmatrix},\ \mathbf{q} = \begin{pmatrix} 5 \\ 3+2\mu \\ -14-3\mu \end{pmatrix} \Rightarrow \overrightarrow{QP} = \begin{pmatrix} 3+\lambda \\ -1-2\lambda-2\mu \\ 17+3\mu \end{pmatrix}$	M1A1
	$\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} . \begin{pmatrix} 3+\lambda \\ -1-2\lambda-2\mu \\ 17+3\mu \end{pmatrix} = 0 \Rightarrow 5\lambda + 4\mu = -5$	M1A1
	$\begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} . \begin{pmatrix} 3+\lambda \\ -1-2\lambda-2\mu \\ 17+3\mu \end{pmatrix} = 0 \Rightarrow -4\lambda - 13\mu = 53$	A1
	Solving : $\lambda = 3, \mu = -5$	M1A1
	Whence: $\mathbf{p} = \begin{pmatrix} 11 \\ -4 \\ 3 \end{pmatrix}, \mathbf{q} = \begin{pmatrix} 5 \\ -7 \\ 1 \end{pmatrix}$.	A1 (8)
(ii) (a)	$\overrightarrow{AP} = \begin{pmatrix} 3 \\ -6 \\ 0 \end{pmatrix}, \overrightarrow{AQ} = \begin{pmatrix} -3 \\ -9 \\ -2 \end{pmatrix}$	B1
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & 0 \\ -3 & -9 & -2 \end{vmatrix} = \begin{pmatrix} 12 \\ 6 \\ -45 \end{pmatrix}$ (CAO) ; Area $= \dfrac{1}{2}\sqrt{2205}$ (=23.5)	M1A1 A1
(b)	$\overrightarrow{\mathrm{AB}} = \begin{pmatrix} -3 \\ 1 \\ -17 \end{pmatrix} \Rightarrow$ Volume $= \dfrac{1}{3} \times \dfrac{1}{2}\sqrt{2205} \left\lvert \dfrac{\begin{pmatrix} -3 \\ 1 \\ -17 \end{pmatrix} . \begin{pmatrix} 12 \\ 6 \\ -45 \end{pmatrix}}{\sqrt{2205}} \right\rvert = \dfrac{735}{6}$ (= 122.5)	M1A1 (6) **Total: 14**

	Alternative: for marks 3,4,5,6 and 7 in part **(i)**	
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 0 \\ 0 & 2 & -3 \end{vmatrix} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$	(M1A1)
	$6k - \lambda = 3$ $3k + 2\lambda + 2 = -1$ $2k - 3\mu = 17$	(A1)
	Solving; $k = 1$, $\lambda = 3$ and $\mu = -5$ (If k missing, or assumed to be 1, deduct 1 mark.)	(M1A1)