FURTHER MATHEMATICS

Paper 9231/11

Paper 11

Key messages

- Candidates need to be careful to read the question in detail and answer as indicated.
- Attention needs to be paid to making sure workings are carried out at a sufficient level of accuracy to ensure the accuracy of the final answers.

General comments

The scripts for this paper were of a generally good quality. There were a fair number of high quality scripts and many showing evidence of sound learning. Work was well presented by the vast majority of candidates. Solutions were set out in a clear and logical order. The standard of numerical accuracy was good. Algebraic manipulation, where required, was of a sound standard. Some of the calculus work was extremely pleasing.

A very high proportion of scripts had substantial attempts at all eleven questions. Once again there were few misreads and few rubric infringements.

Candidates displayed a sound knowledge of most topics on the syllabus. As well as the calculus work, already mentioned, candidates tackled the questions on matrices, summation of series, roots of equations, and graphs of rational functions confidently. The question on proof by induction continued to cause difficulty for some candidates, as did the compulsory complex number question.

Comments on specific questions

Question 1

This question was generally well done and most candidates were able to get off to a good start. In part **(ii)**, those who wrote $\sum\alpha^3 = 7\sum\alpha^2 - 2\sum\alpha + 9$ were more successful than those who tried to remember a formula such as $\sum\alpha^3 = (\sum\alpha)^3 - 3\sum\alpha\sum\alpha\beta + 3\alpha\beta\gamma$.

Answers: **(i)** 45 **(ii)** 310

Question 2

Most candidates gained a mark for stating the inductive hypothesis and many for proving the base case. However, a significant number thought that the base case was for n = 1, rather than n = 2. A number of candidates attempted to prove that $\mathrm{H}_{k+1} \Rightarrow \mathrm{H}_k$, so only the best candidates produced a completely correct proof.

Question 3

Almost all candidates were able to show the initial result correctly and apply it to summing the series by the method of differences. A small number of candidates ignored the word ‘hence’ in the question, which meant the loss of one mark, if they correctly summed the series by finding three partial fractions. Common errors, by either method, were: incorrect cancellation of terms, or omission of a factor of ½. In the latter case a final mark for the sum to infinity could be earned on a follow through basis.

Answers: $\frac{1}{4} - \frac{1}{2}\left\{\frac{1}{(n+1)(n+2)}\right\}$ (OE) , $\sum_1^\infty \frac{1}{r(r+1)(r+2)} = \frac{1}{4}$

Question 4

Candidates generally made a good attempt to sketch the curve *C*. Those whose curve approached the pole from the wrong side of $\theta = \frac{3}{4}\pi$ lost a mark. If the complete cardioid was drawn, no penalty was incurred, provided it was clear that the curve reached the pole at $(0,\pi)$. The formula for calculating the area of a sector was well known and many correctly used the appropriate double angle formula to find the area of the region *R* correctly. In the final part marks were lost by those who did not read the question carefully, thus giving two identical answers to 3 significant figures.

Answers: 3π , 4.712 , 4.713

Question 5

This question was well done by the vast majority of candidates. Many candidates were able to form a 3×3 matrix, usually called **P**, from the eigenvectors, and a 3×3 diagonal matrix, usually called **D**, from the eigenvalues. Good attempts were then made to find $\mathbf{P}^{-1}$, since it was realised that $\mathbf{A} = \mathbf{PDP}^{-1}$. A significant number of candidates used nine letters for the elements of matrix **A** and then formed 3 sets of 3 linear equations, by using the result $\mathbf{Ae} = \lambda\mathbf{e}$. They then proceeded to solve all of these equations.

Answer: $\mathbf{A} = \begin{pmatrix} 3 & 4 & 2 \\ -11 & -27 & -13 \\ 21 & 54 & 26 \end{pmatrix}$

Question 6

Most candidates earned the first two marks for giving the argument of the five fifth roots of unity correctly.

The major obstacle, for many candidates, was to re-write the equation, which was given, in the form $z = \ldots$. Those who overcame this hurdle were often able to manipulate the right hand side to the required form given in the question, but only the best candidates were able to justify why the case $k = 0$ had to be rejected.

Answer: $\theta = \left(\frac{2k\pi}{5}\right)$ $k = 1, 2, 3, 4$

Question 7

There were many substantial attempts at this question and many candidates earned most of the marks. Many correctly reduced the matrices to echelon form and were able to deduce the dimension of each null space. Good attempts were then made at finding the basis for each null space. If everything was done correctly it was then easy to deduce that K_2 was a subspace of K_1.

Answer: $\left\{\begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ 3 \\ 0 \\ 1 \end{pmatrix}\right\}$ (OE)

Question 8

This question was among the best done questions on the paper. The method for solving the differential equation was well known to most candidates and showed evidence of good preparation for the paper. Good attempts were made to find the complementary function and particular integral, thus giving the general solution. The vast majority who got to this stage were able to find the values of the arbitrary constants from the conditions given in the question. There were a substantial number of candidates getting full marks for this question.

Answer: $y = \mathrm{e}^{-x}(3\cos 2x + 4\sin 2x) + 2\mathrm{e}^{-2x}$

Question 9

A good number of candidates realised that the first part of the question was best done by rearranging the equation to standard quadratic form and using the fact that, for real values of *x*, the discriminant would be non-negative. Those who first found the turning points, needed to say something about the curve being continuous for all values of *x*, in order to produce a sound argument. Most candidates made good attempts at finding the turning points on *C*. It should have been noted, from the wording of the question, that C only had one asymptote. Since two marks were available a suitable method, involving division, or division of numerator and denominator by x^2, was required. Those doing well on the first three parts were then able to get the shape of *C* correct in their sketch, but a significant number lost a mark by not giving the coordinates for the intersection of *C* with the asymptote.

Answers: $(-1, 1)$, $\left(2, 2\frac{1}{2}\right)$; $y = 2$

Question 10

The three formulae needed for this question were well known to most candidates. Careful candidates were able to score well on this question. Those candidates who required a substitution to evaluate the integral in the first part were, sometimes, less accurate than those who were able to write the result of integration by inspection. In calculating the coordinates of the centroid, the most common error was to get incorrect powers of 3 in either the numerator, or denominator, or both.

Answers: $\bar{x} = \frac{15}{7}$ or 2,14 $\bar{y} = \frac{5}{8}$ or 0.625

Question 11 EITHER

Those choosing this option mostly knew the appropriate technique of calling the integral *I* and integrating by parts twice, in order to obtain the printed expression for *I* in the first part of the question. The middle part of the question produced a range of responses. There was some impressive work by those who did the middle part of the question completely. Other candidates only managed to get started with one application of integration by parts, while others, who were not able to establish the intermediate result, were able to use it to complete the middle section, thus gaining four or five marks out of the six available. Many of those doing this question were able to use the result from the first part and the reduction formula in order to find the mean value in the final part.

Answer: $\frac{3}{13\pi}\left(1 + e^{\pi}\right)$

Question 11 OR

Those choosing this alternative began well, usually finding the direction of the common perpendicular to the given lines by use of a vector product of their direction vectors, while a smaller number used a scalar product with the direction vector of each line separately. Most then successfully formed an equation for the unknown *m*, by use of the scalar product. They were usually able to manipulate this into the standard form of a quadratic equation and solve it. A considerable number of candidates did not explain that *m* had to be 2, because it was an integer, whereas the other solution was a rational number. This resulted in the loss of one mark. In the second part of the question, the majority of candidates found the magnitude of the vector product of either $\overrightarrow{AD}$ or $\overrightarrow{CD}$ with the unit vector in the direction $\overrightarrow{CA}$. A significant minority used the scalar product, followed by a Pythagoras calculation. The final part was well done by many candidates, who found the angle between the normal to each of the two planes.

Answers: $\sqrt{\frac{18}{17}}$ or 1.03

FURTHER MATHEMATICS

Paper 9231/12
Paper 12

Key messages

- Candidates need to be careful to read the question in detail and answer as indicated.
- Attention needs to be paid to making sure workings are carried out at a sufficient level of accuracy to ensure the accuracy of the final answers.

General comments

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Candidates displayed a sound knowledge of most topics on the syllabus. As well as the calculus work, already mentioned, candidates tackled the questions on matrices, summation of series, roots of equations, and graphs of rational functions confidently. The question on proof by induction continued to cause difficulty for some candidates, as did the compulsory complex number question.

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Almost all candidates were able to show the initial result correctly and apply it to summing the series by the method of differences. A small number of candidates ignored the word 'hence' in the question, which meant the loss of one mark, if they correctly summed the series by finding three partial fractions. Common errors, by either method, were: incorrect cancellation of terms, or omission of a factor of ½. In the latter case a final mark for the sum to infinity could be earned on a follow through basis.

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Answers: $(-1,1)$, $\left(2,2\frac{1}{2}\right)$; $y = 2$

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Those choosing this alternative began well, usually finding the direction of the common perpendicular to the given lines by use of a vector product of their direction vectors, while a smaller number used a scalar product with the direction vector of each line separately. Most then successfully formed an equation for the unknown m, by use of the scalar product. They were usually able to manipulate this into the standard form of a quadratic equation and solve it. A considerable number of candidates did not explain that m had to be 2, because it was an integer, whereas the other solution was a rational number. This resulted in the loss of one mark. In the second part of the question, the majority of candidates found the magnitude of the vector product of either $\overrightarrow{AD}$ or $\overrightarrow{CD}$ with the unit vector in the direction $\overrightarrow{CA}$. A significant minority used the scalar product, followed by a Pythagoras calculation. The final part was well done by many candidates, who found the angle between the normal to each of the two planes.

Answers: $\sqrt{\frac{18}{17}}$ or 1.03

FURTHER MATHEMATICS

Paper 9231/13
Paper 13

Key messages

- Candidates need to be careful to avoid sign errors.
- Candidates need to be careful to eliminate arithmetic and numerical errors in their answers.

General comments

The scripts for this paper were of a generally good quality. There were a considerable number of high quality scripts and many showing evidence of sound learning. Work was well presented by the vast majority of candidates. Solutions were set out in a clear and logical order. The standard of numerical accuracy was good. Algebraic manipulation, where required, was of a high standard. Vector work continued to be of an extremely high standard.

A very high proportion of scripts had substantial attempts at all eleven questions. Once again there were few misreads and few rubric infringements.

Candidates displayed a sound knowledge of most topics on the syllabus. As well as the vector work, already mentioned, candidates tackled the questions on summation of series, implicit differentiation, reduction formulae, matrices, rational functions and differential equations confidently. The question on proof by induction showed considerable improvement in understanding and presentation.

Comments on specific questions

Question 1

Most candidates were able to find the correct partial fractions and attempt to find the sum of the series by the method of differences. Those who wrote down too few terms, or who omitted the factor of ½, were still able to gain a mark for the sum to infinity on a follow through basis.

Answers: $\frac{1}{2}\left\{\frac{3}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right\}$ (acf), $S_\infty = \frac{3}{4}$

Question 2

The work on this fairly straightforward question on proof by mathematical induction showed considerable skill on the part of candidates. There were many correct proofs, with relatively few candidates losing the final mark for not stating a conclusion. A small number of candidates lost marks by showing that $\mathrm{H}_{k+1} \Rightarrow \mathrm{H}_k$, rather than its converse.

Question 3

There was a considerable amount of error free work on this question on implicit differentiation. It was surprising to find so many candidates expanding brackets before differentiating, which resulted in a fourteen term expression for the second derivative. A small number of candidates, having differentiated perfectly, lost the final mark, as they had the wrong sign for the answer.

Answer: $y''(1) = \frac{9}{32}$

Question 4

This question was done extremely well by the vast majority of candidates. They were able to use integration by parts, in the appropriate manner, to establish the reduction formula. Most found I_0, and rather fewer I_1, before using the reduction formula to evaluate I_3. Generally, candidates read the question carefully and gave its exact value.

Answer: $I_3 = \dfrac{4e^3 + 2}{27}$

Question 5

A number of candidates were able to establish the result $(\mathbf{A} + k\mathbf{I})\mathbf{e} = (\lambda + k)\mathbf{e}$, in order to gain the first two marks. A number omitted a step in the argument. The vast majority of candidates were able to gain the next four marks, for finding two eigenvectors and the third eigenvalue. A few made inevitable arithmetic errors. Those who used the initial result, since the question stated 'hence' mostly gained the final three marks. Those who found eigenvalues from the characteristic equation of the matrix **C,** and hence corresponding eigenvectors, could only earn two of the three marks.

Answers: $\begin{pmatrix}1\\-1\\1\end{pmatrix}$, $\begin{pmatrix}1\\1\\0\end{pmatrix}$; 6 ; – 6, 1, 3 ; $\begin{pmatrix}1\\-1\\1\end{pmatrix}$, $\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\begin{pmatrix}1\\-1\\-2\end{pmatrix}$ (OE)

Question 6

There were many completely correct solutions to this question on rational functions. A small, but significant, number of candidates did not continue their division sufficiently, in order to establish the correct equation of the oblique asymptote of the curve C. In which case they thought it to be $y = x$.

Answers: $x = 2$, $y = x + 2$, (0,0), (4,8)

Question 7

This question on complex numbers caused candidates more difficulty than any other question on the paper. Many thought that the given expression immediately gave the left hand side of the identity, failing to realise that a negative sign was involved. Most made reasonable attempts to expand the expression, not always by a very efficient method, which resulted in errors, or a loss of time. Most then managed to group terms in an appropriate manner, but because of the sign error, mentioned above, had the wrong signs for all of p, q, r and s. In the second part of the question, many forgot to change the limits of the integral, and a smaller number simply exchanged dx for dθ. Because of the sign error earlier, there was a good deal of wrong sign work in evaluating the integral, in an attempt to prove the required result.

Answers: $p = 4$, $q = 2$, $r = -4$, $s = -2$

Question 8

Few candidates had difficulty in obtaining the first two marks for the initial result. It was intended that this initial result should lead them to make the substitution $x = \left(\dfrac{1-u}{2}\right)$ in the original equation, in order to find the required cubic equation. A good number did this. However, almost as many found coefficients of the cubic equation by by establishing $\Sigma u = 1$, $\Sigma uv = -13$ and $uvw = -93$ directly from the properties of the roots of the original cubic, namely: $\Sigma\alpha = 1$, $\Sigma\alpha\beta = -3$ and $\alpha\beta\gamma = 10$. Using either method, many successfully found the required cubic equation in the second part of the question.

Answers: **(i)** $u^3 - u^2 - 13u + 93 = 0$ **(ii)** 10, $100v^3 - 10v^2 - 3v - 1 = 0$

Question 9

This question was generally well done. The vast majority, in the first part, found the vector product of the direction vectors of the given lines, in order to obtain the coefficients of x, y and z in the required cartesian equation of the plane. It was then a simple matter to find the constant term, using one of the known points in the plane. Rather fewer wrote equations for x, y and z, in terms of the parameters, which they then eliminated. The vast majority of candidates were able to use the scalar product correctly in order to find the acute angle between the planes, either in degrees or radians. Only a few candidates confused sine and cosine in the scalar product. In the final part, most candidates knew to use the vector product again, in order to find the direction of the line of intersection of the planes. Then, knowing a point on the line, gave its vector equation. A few candidates adopted an approach via solution of simultaneous equations, mostly successfully.

Answers: $6x + y + 4z = 13$, 83.3° or 1.45 rad., $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ -6 \\ 3 \end{pmatrix}$ (OE)

Question 10

Success on this question was somewhat mixed. Most candidates attempted the first part of the question using the augmented matrix, rather than attempting to solve simultaneous equations, where awkward numbers caused difficulty. Many realised that a could not take the values –3 or 5, but few could answer the question set. Many were able to show, satisfactorily, that a = –3 gave no solutions, whereas a = 5 gave infinite solutions. In the final part, some made arithmetical errors, but many found the correct solution.

Answers: All real values except –3 and 5 (formal set notation not required)

$x = -1,\quad y = \frac{7}{2},\quad z = -\frac{1}{2}$

Question 11 EITHER

This question was considerably less popular than the other alternative. Those attempting it were able to make good progress on the first three parts. The first part was done easily by expressing x and y in terms of r and θ. Most sketches were essentially correct, but some omitted a loop, while others added extra loops. The method for finding the area of a sector for a polar graph was well known and easily done in this example. The final part of the question was only done well by the best candidates. This was usually by differentiating the cartesian equation implicitly and setting dy/dx = 0 to obtain $r^2 = a^2/2$ and hence the polar coordinates of the points required, or by writing $y = (a\cos 2\theta)^{1/2}\sin\theta$, and setting d$y$/d$\theta$ = 0 in order to obtain $\cos 3\theta = 0$ or its equivalent.

Answers: $\frac{1}{2}a^2$, $\left(\frac{a}{\sqrt{2}}, \pm\frac{\pi}{6}\right)$ and $\left(\frac{a}{\sqrt{2}}, \pm\frac{5\pi}{6}\right)$

Question 11 OR

This question on a second order differential equation proved popular and was generally very well done. Most correctly applied the given substitution to derive the linear differential equation in x and z, although there was a lack of rigour from some weaker candidates. The complementary function was generally correctly obtained, as was the particular integral. Only weaker candidates failed to cope with repeated root from the auxiliary quadratic equation for the complementary function. Considerable numbers applied the boundary conditions correctly and went on to obtain a correct expression for the particular solution for y in terms of x, although some neglected the final step and so lost the final mark, with an otherwise correct solution.

Answers: $y = 2x\mathrm{e}^{-3x} + x^2\mathrm{e}^{-3x} + 5x\sin 2x - 12x\cos 2x$

FURTHER MATHEMATICS

Paper 9231/21

Paper 21

Key messages

To score full marks in the paper candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

All steps of the argument or derivation should be written down in those questions which require a given result to be shown. Although less detail may suffice in questions where an unknown value or result is to be found, candidates are recommended to show sufficient working so that credit may be earned for their method of solution even if an error occurs.

General comments

Most candidates attempted all the questions, and while very good answers were frequently seen, the paper discriminated well between different levels of ability. In the only question which offered a choice, namely **Question 11**, there was a strong preference for the Statistics option, though the small minority of candidates who chose the Mechanics option frequently produced good attempts. Indeed all questions were answered well by some candidates, most frequently **Questions 1** and **6**. **Question 5** and the first part of **Question 8** were found to be challenging by some, but not all, candidates.

Recommendations to candidates to set out their work clearly, with any corrections legible, and the replacements to deleted attempts readily identifiable, were seemingly adopted by candidates. When relevant in Mechanics questions it is helpful to include diagrams which show, for example, what forces are acting and also their directions as in **Question 5**, and the directions of motion of particles as in **Question 2**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. Thus in **Questions 7**, **9** and **11** any symbols used when stating the hypotheses should either be in standard notation or should be correctly defined.

Many candidates appreciated the need to explain their working clearly in those questions which require certain given results to be shown to be true, rather than finding unknown results. It is particularly advisable to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, such as a numerical value in a Statistics question, candidates are well advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetic error.

The rubric for the paper specifies that non-exact numerical answers be given to 3 significant figures, and while most candidates abided by this requirement, some rounded intermediate results to this same accuracy with consequent risk that the final result is in error in the third significant figure. Such premature approximations should be avoided, particularly when the difference of two numbers of similar sizes is to be taken, as can happen for example when taking the difference of means or estimating variances.

Comments on specific questions

Question 1

The angle through which the flywheel rotates can be found from energy considerations, and hence the time taken to do so. However, most successful candidates preferred instead to first find the angular deceleration $0{\cdot}8$ rad s^{-2} and hence the time for the specified reduction in angular speed. Candidates who consider the angular acceleration should make clear it is negative, not positive. Those who are unfamiliar with rotational motion should not deduce the mass *m* of the flywheel from its radius and moment of inertia, and then apply

Newton's law $F = ma$ for the acceleration a of a particle of mass m in a straight line, followed by $v = u + at$. This incorrect method produces an incorrect time of 10 s.

Answer: 5 s

Question 2

The speeds $2u/3$ and $3u$ of the two particles after the collision may be found by solving any two of the three possible momentum equations. Two of these follow from equating the change of momentum of each particle to the impulse, with considerable care taken over the signs, while the third (dependent) equation comes from conservation of momentum for the system. It is not necessary or helpful to introduce the restitution equation. Almost all candidates knew in principle how to find the total loss of kinetic energy. The loss in kinetic energy is given in its simplified form in the questions. Therefore, candidates who arrive at a different result are well advised to check their working if time permits.

Question 3

Most candidates found the given magnitude of the reaction successfully by combining conservation of energy (giving an expression for the speed v) with a radial resolution of the forces on the particle P. These candidates also realised that this expression for the magnitude of the reaction should be equated to zero when P loses contact with the sphere in order to find the value $-\frac{1}{2}$ of $\cos\theta$. This value is then used to find the speed of P from the previously derived expression for v. Although replacement of θ by $\frac{1}{2}\pi$ in this expression for the speed on the surface of the sphere happens to give the correct speed when P passes through the horizontal plane containing O, candidates should realise that this is so because the same energy considerations apply whether or not P is still in contact with the sphere. Alternatively the horizontal and vertical components of the speed can be found and combined. A few candidates assumed wrongly that only the vertical component of the speed, or the elapsed time was required.

Answers: **(i)** $\sqrt{(\frac{1}{2}ga)}$ **(ii)** $\sqrt{(3ga/2)}$

Question 4

The moment of inertia of the lamina about A can be found by subtracting the moment of the inertia of the removed disc (of mass m) about O from that of the original disc D and then applying the parallel axis theorem, or equivalently by using the theorem to find the moments of inertia about A of each of the two discs before taking their difference. This rarely caused difficulty, unlike the second part of the question. In the second part the obvious method of solution is to equate the rotational energy $\frac{1}{2}I\omega^2$ of the augmented lamina when B is at the lowest point to the loss of potential energy, giving $\omega^2 = 42g/55a$, and then note that the speed of B at this point is $6a\omega$ from which k may be found. Candidates need to be alert to several possible pitfalls in this process: the addition of the particle at B changes the total moment of inertia to $220\ ma^2$; the centre of mass of the lamina falls a distance $6a$ while the particle falls $12a$ (or the centre of mass of the combined body of mass $11m$ falls $84a/11$); and the speed of B is $6a\omega$ rather than $a\omega$ or $3a\omega$.

Answer: $k = 5{\cdot}24$

Question 5

In order to minimize the time and effort in answering questions such as this, it is advisable to choose those moment and resolution equations which will lead most quickly to the desired result, and also to avoid the introduction of unknown forces which are not needed and must be eliminated, such as here the forces at the hinge B. Thus taking moments about B for the rod AB will not only yield immediately the friction $F_A = 3mg/2$ at A for later use, but also enable the required reaction R_C at C to be found from a vertical resolution of the forces on the system. The friction $F_C = 3mg$ at C follows from taking moments about B for the rod BC, and a horizontal resolution of forces shows that this is equal to the normal reaction R_A at A. These four friction and reaction forces can be found in other ways, but extra effort and a consequent added possibility of error may well be involved. Finally the limiting values $\frac{1}{2}$ and $6/13$ of F_A/R_A and F_C/R_C are calculated, and the larger of the two selected as the least possible value of μ for which the rods do not slip at either point.

Answer: $\mu_{min} = \frac{1}{2}$

Question 6

The first probability was usually found correctly from $q^n p$, where $p = 0{\cdot}01$ and $q = 1 - p = 0{\cdot}99$, and candidates should give this and other numerical results to 3 significant figures as required by the instructions for the paper. Candidates are recommended to be sufficiently familiar with this topic to be able to write down the value of E(N) from $1/p$, and not produce instead some expression involving N. The final part of this question is a little more challenging, requiring the least value of the integer n which satisfies $1 - q^n > 0{\cdot}9$.

Answers: 0·00914, 100, 230

Question 7

When formulating and testing hypotheses concerning the difference of population means in questions such as this, it is important that candidates choose the most appropriate test, which in this case is a paired-sample t-test. To carry out the test, the difference is taken between each of the paired times and then the mean $\overline{d}$ of this sample of 8 differences calculated. An unbiased estimate s^2 of the variance of the corresponding population is also found and then a t-value of 1·09 calculated using $\overline{d}/(s/\sqrt{8})$ and compared with the tabular value 2·365. This leads to acceptance of the null hypothesis of there being no non-zero difference between the two mean times.

Question 8

In the challenging first part of this question, candidates should state explicitly that the probability of there being no flaws in a length x of ribbon is equal to the probability of the distance between two successive flaws being greater than x. The working of some candidates who utilised this equality was confused by using the symbol X to denote both the distance between two successive flaws, as this symbol is defined in the question, and the number of flaws in a particular length of ribbon. Other candidates appeared to believe that the given distribution function of X is so obvious as to require no argument, or that it is sufficient to simply note that the derivative of the given F(x) is in the appropriate form for a negative exponential distribution. However, neither of these attempts is adequate. By contrast, most candidates stated the mean and found the median distance correctly. The final result follows immediately from $P(X \geq 50) = 1 - F(50)$, though F(50) was frequently seen instead.

Answers: 62·5 (**i**) 43·3 (**ii**) 0·449

Question 9

As in all such tests, the hypotheses should be stated in terms of the population mean and not the sample mean. The latter is here 5·3125, and the temptation to approximate to 3 significant figures such intermediate results in this and other questions should be resisted since this may produce a lower accuracy in the final result. The unbiased estimate 2·217 of the population variance may be used to calculate a t-value of 1·54. Since it is a one-tail test, comparison with the tabulated value of 1·895 leads to acceptance of the null hypothesis, namely that the population mean is not greater than 4·5. The required 95% confidence interval is centred on the sample mean and not 4·5 as some candidates mistakenly thought, and finding it uses the same estimated variance but a different tabular t-value of 2·365.

Answer: (4·07, 6·56)

Question 10

Candidates generally produced good answers to the first part of this question, apart from confusion between the null and alternative hypotheses and in the final conclusion. The null hypothesis is one of independence, and this is accepted since the calculated χ^2-value 2·44 is less than the tabular value 5·991. The key to the second part is to appreciate that the calculated value of χ^2 increases by a factor N, so that dependence at the 1% significance level requires that $2{\cdot}44N > 9{\cdot}21$.

Answer: Preferences are independent, 4

Question 11 EITHER

Although much less popular than the Statistics alternative discussed below, those candidates who chose this optional question often made very good attempts. The extension is readily seen to be $\frac{1}{4}l$ at the equilibrium point, and use of Newton's law $F = ma$ at a distance x from this point, with F comprised of mg and the tension, yields the usual form of the SHM equation with $\omega^2 = 4g/l$. This confirms the given expression for the period, since it equals $2\pi/\omega$. The speed $\frac{1}{4}\sqrt{(gl)}$ at E is most easily found from the standard SHM formula $\omega\sqrt{(x_0^2 - x^2)}$, with the amplitude $x_0 = l/8$ and $x = 0$. This leads to a speed of $\frac{1}{4}\sqrt{(15gl)}$ immediately before striking the plane, after which the component parallel to the plane is unchanged and the normal component is changed by a factor $\frac{1}{3}$. Combining these components confirms the given speed immediately after impact.

Question 11 OR

Calculation of the product moment correlation coefficient r for the sample is straightforward, with the relevant expression given in the *List of Formulae*. Apart from an occasional arithmetical error most candidates experienced little difficulty with this question. However, candidates should retain additional significant figures in their intermediate results in order to ensure 3 significant figure accuracy in r. The test requires an explicit statement of the null and alternative hypotheses, $\rho = 0$ and $\rho < 0$, and here candidates should be aware that r and ρ are not the same entity. Comparison of the magnitude of the previously calculated value of r with the tabular value 0·707 leads to a conclusion of there being no evidence of negative correlation, suggesting that the opening of the new restaurant has not affected the takings of the existing one. In part **(iii)**, many candidates probably consulted the table of critical values for the product moment correlation coefficient for values close to 0·431 when deducing a range of possible values of N. As well as stating this range, however, they are well advised to quote the particular values which lead to their conclusion, since otherwise an incorrect range may gain no credit. The most relevant values are the one-tail 5% ones for n = 15 and 16, namely 0·441 and 0·426, between which 0·431 lies.

Answers: **(i)** -0·358, **(iii)** $N \geq 1{\cdot}6$

FURTHER MATHEMATICS

Paper 9231/22

Paper 22

Key messages

To score full marks in the paper candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

All steps of the argument or derivation should be written down in those questions which require a given result to be shown. Although less detail may suffice in questions where an unknown value or result is to be found, candidates are recommended to show sufficient working so that credit may be earned for their method of solution even if an error occurs.

General comments

Most candidates attempted all the questions, and while very good answers were frequently seen, the paper discriminated well between different levels of ability. In the only question which offered a choice, namely **Question 11**, there was a strong preference for the Statistics option, though the small minority of candidates who chose the Mechanics option frequently produced good attempts. Indeed all questions were answered well by some candidates, most frequently **Questions 1** and **6**. **Question 5** and the first part of **Question 8** were found to be challenging by some, but not all, candidates.

Recommendations to candidates to set out their work clearly, with any corrections legible, and the replacements to deleted attempts readily identifiable, were seemingly adopted by candidates. When relevant in Mechanics questions it is helpful to include diagrams which show, for example, what forces are acting and also their directions as in **Question 5**, and the directions of motion of particles as in **Question 2**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. Thus in **Questions 7**, **9** and **11** any symbols used when stating the hypotheses should either be in standard notation or should be correctly defined.

Many candidates appreciated the need to explain their working clearly in those questions which require certain given results to be shown to be true, rather than finding unknown results. It is particularly advisable to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, such as a numerical value in a Statistics question, candidates are well advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetic error.

The rubric for the paper specifies that non-exact numerical answers be given to 3 significant figures, and while most candidates abided by this requirement, some rounded intermediate results to this same accuracy with consequent risk that the final result is in error in the third significant figure. Such premature approximations should be avoided, particularly when the difference of two numbers of similar sizes is to be taken, as can happen for example when taking the difference of means or estimating variances.

Comments on specific questions

Question 1

The angle through which the flywheel rotates can be found from energy considerations, and hence the time taken to do so. However, most successful candidates preferred instead to first find the angular deceleration 0·8 rad s^{-2} and hence the time for the specified reduction in angular speed. Candidates who consider the angular acceleration should make clear it is negative, not positive. Those who are unfamiliar with rotational motion should not deduce the mass *m* of the flywheel from its radius and moment of inertia, and then apply

Newton's law $F = ma$ for the acceleration a of a particle of mass m in a straight line, followed by $v = u + at$. This incorrect method produces an incorrect time of 10 s.

Answer: 5 s

Question 2

The speeds $2u/3$ and $3u$ of the two particles after the collision may be found by solving any two of the three possible momentum equations. Two of these follow from equating the change of momentum of each particle to the impulse, with considerable care taken over the signs, while the third (dependent) equation comes from conservation of momentum for the system. It is not necessary or helpful to introduce the restitution equation. Almost all candidates knew in principle how to find the total loss of kinetic energy. The loss in kinetic energy is given in its simplified form in the questions. Therefore, candidates who arrive at a different result are well advised to check their working if time permits.

Question 3

Most candidates found the given magnitude of the reaction successfully by combining conservation of energy (giving an expression for the speed v) with a radial resolution of the forces on the particle P. These candidates also realised that this expression for the magnitude of the reaction should be equated to zero when P loses contact with the sphere in order to find the value $-\frac{1}{2}$ of $\cos\theta$. This value is then used to find the speed of P from the previously derived expression for v. Although replacement of θ by $\frac{1}{2}\pi$ in this expression for the speed on the surface of the sphere happens to give the correct speed when P passes through the horizontal plane containing O, candidates should realise that this is so because the same energy considerations apply whether or not P is still in contact with the sphere. Alternatively the horizontal and vertical components of the speed can be found and combined. A few candidates assumed wrongly that only the vertical component of the speed, or the elapsed time was required.

Answers: **(i)** $\sqrt{(\frac{1}{2}ga)}$ **(ii)** $\sqrt{(3ga/2)}$

Question 4

The moment of inertia of the lamina about A can be found by subtracting the moment of the inertia of the removed disc (of mass m) about O from that of the original disc D and then applying the parallel axis theorem, or equivalently by using the theorem to find the moments of inertia about A of each of the two discs before taking their difference. This rarely caused difficulty, unlike the second part of the question. In the second part the obvious method of solution is to equate the rotational energy $\frac{1}{2}I\omega^2$ of the augmented lamina when B is at the lowest point to the loss of potential energy, giving $\omega^2 = 42g/55a$, and then note that the speed of B at this point is $6a\omega$ from which k may be found. Candidates need to be alert to several possible pitfalls in this process: the addition of the particle at B changes the total moment of inertia to $220\ ma^2$; the centre of mass of the lamina falls a distance $6a$ while the particle falls $12a$ (or the centre of mass of the combined body of mass $11m$ falls $84a/11$); and the speed of B is $6a\omega$ rather than $a\omega$ or $3a\omega$.

Answer: $k = 5{\cdot}24$

Question 5

In order to minimize the time and effort in answering questions such as this, it is advisable to choose those moment and resolution equations which will lead most quickly to the desired result, and also to avoid the introduction of unknown forces which are not needed and must be eliminated, such as here the forces at the hinge B. Thus taking moments about B for the rod AB will not only yield immediately the friction $F_A = 3mg/2$ at A for later use, but also enable the required reaction R_C at C to be found from a vertical resolution of the forces on the system. The friction $F_C = 3mg$ at C follows from taking moments about B for the rod BC, and a horizontal resolution of forces shows that this is equal to the normal reaction R_A at A. These four friction and reaction forces can be found in other ways, but extra effort and a consequent added possibility of error may well be involved. Finally the limiting values $\frac{1}{2}$ and $6/13$ of F_A/R_A and F_C/R_C are calculated, and the larger of the two selected as the least possible value of μ for which the rods do not slip at either point.

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Answers: 0·00914, 100, 230

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In the challenging first part of this question, candidates should state explicitly that the probability of there being no flaws in a length x of ribbon is equal to the probability of the distance between two successive flaws being greater than x. The working of some candidates who utilised this equality was confused by using the symbol X to denote both the distance between two successive flaws, as this symbol is defined in the question, and the number of flaws in a particular length of ribbon. Other candidates appeared to believe that the given distribution function of X is so obvious as to require no argument, or that it is sufficient to simply note that the derivative of the given F(x) is in the appropriate form for a negative exponential distribution. However, neither of these attempts is adequate. By contrast, most candidates stated the mean and found the median distance correctly. The final result follows immediately from $P(X \geq 50) = 1 - F(50)$, though F(50) was frequently seen instead.

Answers: 62·5 (**i**) 43·3 (**ii**) 0·449

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Answer: (4·07, 6·56)

Question 10

Candidates generally produced good answers to the first part of this question, apart from confusion between the null and alternative hypotheses and in the final conclusion. The null hypothesis is one of independence, and this is accepted since the calculated χ^2-value 2·44 is less than the tabular value 5·991. The key to the second part is to appreciate that the calculated value of χ^2 increases by a factor N, so that dependence at the 1% significance level requires that $2{\cdot}44N > 9{\cdot}21$.

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Although much less popular than the Statistics alternative discussed below, those candidates who chose this optional question often made very good attempts. The extension is readily seen to be $\frac{1}{4}l$ at the equilibrium point, and use of Newton's law $F = ma$ at a distance x from this point, with F comprised of mg and the tension, yields the usual form of the SHM equation with $\omega^2 = 4g/l$. This confirms the given expression for the period, since it equals $2\pi/\omega$. The speed $\frac{1}{4}\sqrt{(gl)}$ at E is most easily found from the standard SHM formula $\omega\sqrt{(x_0^2 - x^2)}$, with the amplitude $x_0 = l/8$ and $x = 0$. This leads to a speed of $\frac{1}{4}\sqrt{(15gl)}$ immediately before striking the plane, after which the component parallel to the plane is unchanged and the normal component is changed by a factor $\frac{1}{3}$. Combining these components confirms the given speed immediately after impact.

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Answers: **(i)** -0·358, **(iii)** $N \geq 1{\cdot}6$

FURTHER MATHEMATICS

Paper 9231/23

Paper 23

Key messages

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Recommendations to candidates in previous reports to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable were seemingly adopted by candidates. When relevant in Mechanics questions it is helpful to include diagrams which show, for example, what forces are acting and also their directions as in **Question 11**, and the directions of motion of particles as in **Question 1**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. Thus in **Questions 6**, **10** and **11** any symbols used when stating the hypotheses should either be in standard notation or should be correctly defined.

Many candidates appreciated the need to explain their working clearly in those questions which require certain given results to be shown to be true, rather than finding unknown results. It is particularly advisable to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, such as a numerical value in a Statistics question, candidates are well advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetic error.

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Comments on specific questions

Question 1

Almost all candidates rightly appreciated that the question requires the application of conservation of momentum and Newton's restitution equation, followed by solution of the two resulting simultaneous equations for the speeds $v_A = \frac{1}{3}(1\text{-}2e)u$ and $v_B = \frac{1}{3}(1+e)u$ of the two spheres after the collision. These speeds may then be substituted into $\frac{1}{2}m(3u^2 - 3v_A^2 - 6v_B^2)$ to find the given expression for the loss in kinetic energy. Since the question requires that the validity of the given expression be shown, candidates are recommended to show at least some of the intermediate steps.

Question 2

The given differential equation results from Newton's Law $F = ma$, with the force F being the component towards O of the two equal tensions T in the strings, and thus equal to $2Tx/AP$ with the appropriate sign. This tension T follows from Hooke's Law with extension ($AP - a$), and $AP = \sqrt{(4a^2 + x^2)}$. These various components are then combined and re-arranged. It is sensible to read questions carefully, which might here prevent candidates wrongly treating P as being vertically above O rather than on the same horizontal surface. Having frequently found the first part of the question challenging, many candidates were able to successfully neglect the second and higher powers of x/a and hence obtain the usual form of the SHM equation with $\omega = 2g/a$ and find the period from $2\pi/\omega$. In this context neglecting means taking equal to zero, and not setting equal to unity as some candidates mistakenly thought.

Answer: $\pi\sqrt{(2a/g)}$.

Question 3

Almost all candidates applied the standard SHM formula $v^2 = \omega^2(x_0^2 - x^2)$ using the pairs of values of v and x given in the question to produce two simultaneous equations which they solved for the amplitude x_0 and for $\omega = ½$, leading to the period $2\pi/\omega$. Although the second part of the question may be solved with two applications using $x = 6$ and 8 of either $x = x_0 \sin \omega t$ or $x = x_0 \cos \omega t$, candidates must take care when combining the resulting times. Thus $2 \cos^{-1} 0{\cdot}6$, for example, corresponds to the time taken by P to travel to the point A from $x = 10$ rather than from O. The inverse trigonometric functions should be evaluated in radians and not degrees.

Answers: 4π s $\quad \pi$ s

Question 4

Derivation of the given expression for the magnitude of the reaction R requires that the net radial force acting on the particle as a result of R and $mg \cos \theta$ be equated to mv^2/a, followed by substitution for the speed v using conservation of energy relative to the initial position. Finding $\cos \alpha$ when the particle leaves the surface of the sphere simply requires, as almost all candidates appreciated, that R be equated to zero. The last part, concerning the subsequent motion, requires more thoughtful analysis. Firstly the value of $\cos \alpha$ is used to find the speed $v_1 = \sqrt{(¾\, ga)}$ when the particle loses contact. The vertical component of the velocity of P as it strikes the plane then follows from the square root of $v_1^2 \sin^2 \alpha + 2ga(1 + \cos \alpha)$. Instead of the first term in this square root, either $u^2 \sin^2 \alpha$ or just v_1^2 were quite often seen. This final part can alternatively be solved using energy, but such attempts were often incorrect, potentially as it is the vertical component of the velocity which is required and not the overall speed.

Answers: **(i)** ¾ **(ii)** $\sqrt{(245ga/64)}$

Question 5

Since candidates are required to show that the moment of inertia is as given in the question, they are recommended to include at least some brief explanation in their answer instead of simply writing down the sum of several terms such as $12ma^2 + 2ma^2 + 64ma^2$. Most candidates made a good attempt at this first part of the question, but were often much less successful in the remaining two parts. The required expression for the second derivative is best found by dividing $23mga \sin \theta$ by the total moment of inertia, which is no longer just that of the lamina since the attached particle increases it to $150ma^2$. Care is needed over the sign due to the definition of θ. The final part follows from conservation of energy, and here it is necessary to remember that the system's initial rotational energy is non-zero and again that a particle has been attached to the original lamina and rod.

Answers: **(i)** $-(23g/150a) \sin \theta$ **(ii)** $\sqrt{(2g/3a)}$

Question 6

The null and alternative hypotheses were usually stated correctly, though all candidates should express these in terms of the population and not the sample mean, for example by writing $\mu = 8{\cdot}05$ and $\mu > 8{\cdot}05$. Most candidates went on to estimate the variance of the population correctly, and hence calculate a *t*-value of 2·03, apart from an occasional confusion between biased and unbiased estimates of the population variance. Note that it is quite unnecessary to find Σx^2 in order to use the first form of the variance expression quoted in the *List of Formulae* since in this question the second form given there can be applied directly to the data. Comparison with the critical *t*-value of 1·833 then leads to the conclusion that the mean of X is greater than 8·05.

Question 7

Most candidates were aware that the standard deviation for this exponential distribution is $1/\lambda$ and so did not need to relate it to λ by integration, though apparent confusion over the standard deviation and variance caused some to use $1/\lambda^2$ or $1/\sqrt{\lambda}$ instead, thereby affecting the following parts as well. The probability in **(ii)** is readily found from F(10) – F(5), while the median value m of T follows from equating F(m) to ½.

Answers: **(i)** 0·125 **(ii)** 0·249 **(iii)** 5·55

Question 8

Many candidates made a good attempt at the first part of this question, finding the expected values correctly to at least 3 and preferably 4 significant figures and then comparing the calculated value 12·4 of χ^2 with the tabular value 11·14 in order to conclude that the alternative hypothesis should be accepted. As in all similar tests it is necessary to formulate appropriate hypotheses, and in this case the alternative hypothesis is that the grade of reliability is not independent of the town. In order to identify which town makes the greatest contribution to the test statistic candidates need to examine which set of three terms corresponding to each of the towns A, B and C contributes most to the calculated value of χ^2. Since the sums of the relevant terms are respectively 0·83, 3·42 and 8·18 it is clearly C, though B was a much more popular choice, potentially as it has the most respondents. Relating this to the context of the question might well focus on the reliability of the digital television signal being poorer than expected in town C, but suitable comments were rarely seen.

Question 9

When integrating f(x) to find the distribution function F(x) in the range ($-a$, a), candidates should be aware that the answer is not simply an indefinite integral. Instead they should either integrate over the interval ($-a$, x) or equivalently introduce a constant of integration which may be found from either F($-a$) = 0 or F(a) = 1. Candidates should also realise that having f(x) = 0 other than in $-a \le x \le a$ does not imply that F(x) = 0 also; in fact F(x) = 1 when $x > a$. Since the question asks that candidates find the distribution function G(y) of Y rather than show that it takes a given form, they need only equate G(y) to F(ln y) without justifying this, though care should be taken over relevant intervals for y. Having found G(y), the value of k in the final part follows from solving 1 – G(k) = ¼.

Answers: $F(x) = 0\ (x < -a),\ (x + a)/2a\ (-a \le x \le a),\ 1\ (a < x)$;
$G(y) = 0\ (y < e^{-a}),\ (\ln y + a)/2a\ (e^{-a} \le y \le e^{a}),\ 1\ (e^{a} < y)$; $k = 7{\cdot}39$.

Question 10

Since the question does not invite candidates to make any statistical assumptions, the most appropriate approach is to find unbiased estimates $s_P^2 = 1{\cdot}469$ and $s_Q^2 = 1{\cdot}241$ of the population variances from the data given for the two towns P and Q, and hence an estimate s^2 for the combined sample using $s_P^2/50 + s_Q^2/40$. This enables the required confidence interval to be found from $0{\cdot}6 \pm 1{\cdot}96s$, and then a z-value of 2·44 from $0{\cdot}6/s$ in the second part. Comparison with the tabular value 2·326 causes the null hypothesis $\mu_P = \mu_Q$ to be rejected in favour of $\mu_P > \mu_Q$, and as always these hypotheses should be clearly stated in terms of the population means. A large number of candidates proceeded instead on the assumption that the populations of speeds for P and Q have equal variance, in which case this assumption should be stated explicitly. The numerical results are not very different, with a calculated z-value of 2·42 for example.

Answer: (0·118, 1·082)

Question 11 EITHER

Although much less popular than the Statistics alternative discussed below, those candidates who chose this optional question often made very good attempts. As is usual in such questions, there are several possible approaches and candidates are well advised to select one which will lead most quickly and easily to the required result. Since the friction F_A and reaction R_A at the point A are clearly required, one obvious approach is to find two equations involving only these unknowns, and this can be done by taking moments about C for the rod and resolving the forces on it in a direction parallel to it. Even though it introduces a force which is not directly required, namely the reaction R_C at C, an equally effective approach is to first find R_C by taking moments for the rod about A, since F_A and R_A can then be found immediately from it by resolving forces horizontally and vertically. In either event, the given lower bound on μ follows from $F_A \le \mu R_A$. The required set of values of k may then be found by using the given value 0·9 of μ in this inequality.

Answer: $k \le 91/43$

Question 11 OR

Most candidates were aware that the sample product moment correlation coefficient r is related to the gradients of the two regression lines by $r^2 = (-0{\cdot}5)(-1{\cdot}2)$, though they must remember to take the square root and to select the appropriate sign when doing so. The test requires an explicit statement of the null and alternative hypotheses, $\rho = 0$ and $\rho \neq 0$, and here candidates should be aware that r and ρ are not the same entity. Comparison of the magnitude of the previously calculated value of r with the tabular value 0·878 leads to acceptance of the null hypothesis, namely that the population product moment correlation coefficient does not differ from zero. In the final part two linear simultaneous equations for p and q may be obtained by recalling that the mean values of the sample data satisfy both regression line equations, or alternatively (and equivalently) the regression line equations can be solved for the mean values $\overline{x}$ and $\overline{y}$ and then p and q found from the data. An approach favoured by some candidates of substituting the sample data into the expression for the gradients given in the *List of Formulae* does also produce simultaneous equations for p and q, but unfortunately they are nonlinear and can lead to much fruitless effort in seeking a solution. Candidates should also be aware that a data pair (x, y) will only satisfy the regression equations if it is the mean point, so that inserting (p, q) in the equations produces an incorrect answer.

Answer: $r = -0{\cdot}775$, $p = 7$, $q = 1$