FURTHER MATHEMATICS

Paper 9231/01
Paper 1

General comments

Some scripts of outstanding quality and many of good quality were received in response to this examination. Once again there were very few poor scripts. Work was well presented by the vast majority of candidates. Solutions were set out in a clear and logical order. The standard of numerical accuracy was good, but there were lapses with algebraic manipulation by a small, but significant, minority of candidates.

There was no evidence to suggest that candidates had any difficulty completing the paper in the time allowed. A very high proportion of scripts had substantial attempts at all twelve questions. Once again there were few misreads and few rubric infringements. Only the occasional candidate, who could not fully answer one of the two alternatives for the final question, submitted two partial attempts.

The Examiners felt candidates had a sound knowledge of most topics on the syllabus. This year there was an improvement in the work on induction. Complex numbers and linear spaces seemed to be the areas of greatest uncertainty.

Comments on specific questions

Question 1

Many candidates were unsure what was required in this question. There were many attempts to find either or both coordinates of the triangular area that was being used to generate the cone.

Those who realised that

$$\bar{x} = \frac{\int_0^h xy^2 \mathrm{d}x}{\int_0^h y^2 \mathrm{d}x}$$

made good progress.

Question 2

The majority of candidates realised that u_n could be expressed as the difference of two logarithms. This enabled them to find the sum to N terms correctly, using the method of differences.

For part **(i)** it was necessary to see that x^{N+1} vanished as $N \to \infty$ when $-1 < x < 1$. This fact eluded a considerable number of candidates.

It was possible to get the correct result for part **(ii)** even if none of the earlier marks in the question had been gained.

Answers: $\ln\left(\frac{1+x^{N+1}}{1+x}\right)$; **(i)** $-\ln(1 + x)$; **(ii)** 0.

Question 3

A substantial number of candidates gained full marks on this question. For the initial proof, it was necessary to write down $\mathbf{Ae} = \lambda\mathbf{e}$ and $\mathbf{Be} = \mu\mathbf{e}$ and by adding corresponding left and right hand sides obtain $(\mathbf{A} + \mathbf{B})\mathbf{e} = (\lambda + \mu)\mathbf{e}$. A sizeable minority could not complete this proof, but were able to continue with the question.

In the second part it was only necessary to multiply the matrix **A** and the given eigenvector to find the corresponding eigenvalue. Some candidates wasted considerable time in finding and solving the characteristic equation.

The final stage was to obtain the eigenvalues 6, 4 and 3 for **A** + **B** by applying the result proved at the start of the question, and hence write down the required matrices **P** and **D**. Weaker candidates failed to match up the eigenvectors for **A** and **B** and so found incorrect eigenvalues for **A** + **B** and hence **D**.

Answers: 4; $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & -3 & -2 \end{pmatrix}$; $\begin{pmatrix} 1296 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 81 \end{pmatrix}$.

Question 4

Almost all candidates were able to find the correct value of θ for the intersection of C_1 and C_2. Some weaker candidates forgot to find the corresponding value of r.

Many candidates were able to sketch both graphs correctly.

Very few candidates were able to find the required area correctly. Many found the area between C_1, C_2 and the half-line $\theta = \pi$ rather than the half-line $\theta = 0$. A significant number of those who would, otherwise, have found the area correctly had a 2θ term, rather than a 4θ term, in the square of $(\theta + 2)$.

Answers: **(i)** (4, 2); **(iii)** $\frac{92}{15}$ or 6.13

Question 5

This question was well done by a large number of candidates. The first part was usually done by a substitution $y = x^3$. Only a small number of these candidates were unable to isolate the cube root term and then cube to find the required equation. Those who chose to find the coefficients of the required equation often ran into difficulty finding the coefficient of y in the required equation. Since the required equation was printed on the paper, full working was required. One way of doing this is to say

$$\sum \alpha^3\beta^3 = (1-\alpha)(1-\beta) + (1-\beta)(1-\gamma) + (1-\gamma)(1-\alpha)$$

$$= 3 - 2\sum\alpha + \sum\alpha\beta$$

$$= 3 - 0 + 1$$

$$= 4$$

In the second part most candidates employed $\sum \alpha^2 = \left(\sum\alpha\right)^2 - 2\sum\alpha\beta$.
A small number used a substitution $y = \sqrt{z}$, while a few used $S_{n+3} + S_{n+1} - S_n = 0$.

Answer: 1

Question 6

Most candidates were able to differentiate x and y correctly and thus obtain the first derivative. Many were then able to find the second derivative. The most common error was to find $\frac{\mathrm{d}^2y}{\mathrm{d}t^2}$ rather than $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

A good number of candidates were unable to get beyond writing down a definition of the mean value of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in the interval $0 \leqslant x \leqslant \frac{7}{4}$. Some who realised that the integral of the second derivative was the first derivative were unable to complete the question, because they failed to realise that $t = \frac{1}{2}$ corresponded to $x = \frac{7}{4}$.

Question 7

Candidates were happy with a more straightforward case of proof by induction and many correct proofs were given for the first part of the question. There were, pleasingly, very few cases where 2 was taken as the base case, rather than 1. Only a minority of candidates gave an inadequate conclusion.

The vast majority of candidates knew what to do in the second part of the question but some candidates lost the accuracy mark with poor algebra.

Question 8

In part **(i)** most candidates realised they had to integrate by parts twice to establish the reduction formula. Some lost accuracy marks by not inserting limits at all, or inserting incorrect limits. There were, however, many completely correct derivations of the reduction formula.

In part **(ii)** most candidates gained the mark for using the length of arc formula and many simplified the integral correctly. Some of those who got this far did not realise that they had $2I_4$ and some who did, worked out I_4 but forgot to double it at the end.

Candidates who evaluated the integral directly on graphical calculators earned three of the five possible marks, in line with the instruction on the front of the paper. Nevertheless, there were a pleasing number of candidates who gained full marks on this question.

Answer: **(ii)** $\pi^3 - 24\pi + 48$ or 3.61.

Question 9

Almost all candidates were able to find the equations of the asymptotes correctly. If they did this they were not penalised if they did not explicitly state that they were independent of λ.

A surprising number of candidates realised that C would touch the x-axis at (1, 0), but did not state a value for λ. Both facts should have been obvious by observing that $(x - 1)^2 \equiv x^2 - 2x + 1$, the first from the repeated factor and the second by direct comparison. The graph was usually drawn essentially correctly, but forms at infinity were often wayward, which incurred a penalty. Some drew no graph in this part.

The graph in the case $\lambda = -4$ again was frequently drawn correctly, but, as above, forms at infinity were wayward. Sometimes decimal values were given instead of the exact values for the intersections with the x-axis, which also incurred a penalty.

Answers: 1; $(1 + \sqrt{5}, 0)$, $(1 - \sqrt{5}, 0)$.

Question 10

This was the least well done question on the paper. The first part of the question required candidates to realise that the sum of the series was the real part of the sum of the appropriate geometric progression. This was most elegantly done by saying

$$\text{Re}\left(\frac{z(1-z^{2n})}{1-z^2}\right) = \text{Re}\left(\frac{(1-z^{2n})}{(z^{-1}-z)}\right) = \text{Re}\left(\frac{1-\cos 2n\theta - \text{i}\sin 2n\theta}{-2\text{i}\sin\theta}\right) = \frac{\sin 2n\theta}{2\sin\theta}.$$

The second part of the question required candidates to see that the result could be obtained by differentiating both sides of the result in the first part, with respect to θ, and putting $\theta = \frac{\pi}{N}$. An extremely small number of candidates realised this and did it correctly.

Question 11

There were few complete answers to the first part of the question. Many were able to differentiate, correctly, twice, the given expression for y, with respect to x, and substitute the resulting expressions in the differential equation. Few saw that the suitable value for α was -2. Even fewer were able to do the necessary algebra to complete this part of the question.

The second part of the question was done well, except that the last mark was lost, because candidates gave the general solution for w and not y.

Answer: $y = x^{-2}(Ae^{-\frac{x}{2}} + Be^{-x} - \cos 2x)$.

Question 12 EITHER

The only practical way to start this question was to use the vector product of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ in order to find a vector perpendicular to both of them. It was then necessary to find the scalar product of either $\overrightarrow{AC}$ or $\overrightarrow{BC}$ with the unit normal vector and equate to 3 (the shortest distance between the two lines). On simplifying the resulting equation, this gave the required quadratic in λ. Those choosing this alternative were mostly familiar with this method and performed it accurately in many cases. Those who made an error were still able to solve the quadratic and continue with the rest of the question.

Having solved the quadratic, in part **(ii)**, it was necessary to find a vector perpendicular to the plane ABD for each of the two possible positions of D. The angle between the two planes was then the angle between these two normal vectors, which was found using the scalar product. Many candidates showed familiarity and accuracy in using this method. Those who made small errors with arithmetic, or algebra, still managed to score well with method marks.

Answer: **(ii)** 12.1°.

Question 12 OR

Candidates choosing this alternative generally performed less well than those candidates who chose the other alternative. In many cases little was done beyond the first two parts. In part **(i)** candidates were well used to reducing the matrix to echelon form, but some made arithmetical errors or stopped well short and lost a mark in doing so. The mark for the dimension of *V* could be obtained nevertheless. In part **(ii)**, the most popular approach was to show that if a linear combination of the vectors gave the zero vector then the only solution was the trivial one. A good number, however, used row reduction methods with matrices. A number of candidates did not appreciate the demands of the phrase 'show that' and thought that a statement of what linear dependence meant was sufficient. Part **(iii)** following on from part **(ii)** should have been obvious, but some ignored part **(ii)**. In part **(iv)**, *W* was not a vector space and the absence of the zero vector or non-closure under addition were sufficient reasons. In part **(v)**, again, 'show that' was misinterpreted by some of those candidates who made an attempt at this part. The best approach was probably to use row reduction on an augmented matrix to show that $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in V \Leftrightarrow y - z - t = 0$ hence $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in W \Rightarrow y - z - t \neq 0$. Other approaches, via a system of equations, were equally acceptable.

Answers: **(i)** 3; **(iii)** $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 3 \\ -4 \end{pmatrix} \right\}$; **(iv)** No, no zero vector or other valid reason.

FURTHER MATHEMATICS

Paper 9231/02
Paper 2

General comments

Good answers were seen to both the Mechanics and Statistics questions, though it was noticeable that very few candidates chose the Mechanics alternative in **Question 11**. There were also frequently unsuccessful attempts at **Question 6**. As has usually been the case in previous examinations, the paper worked well in discriminating between candidates, producing a range of performance varying from low to very high marks, and there was little sign of undue time pressure on candidates.

Comments on specific questions

Question 1

The great majority of candidates successfully found the magnitude of the impulse I from the change in momentum of the bullet, and hence the resisting force by dividing I by the time. The infrequent errors were most often caused by confusion over the signs of the two momentum terms, resulting in their effective addition rather than subtraction.

Answers: **(i)** 4.6 N s; **(ii)** 1150 N.

Question 2

The time for the particle to move directly from B to B' is clearly double that from O to B' say, and the latter is readily found from the usual SHM formula $x = a \sin \omega t$ with $x = \frac{1}{2}a$ and $\omega = \frac{2\pi}{T}$. Some of those candidates who used the cosine form inadvertently found instead the time from A' to B', essentially by not considering which position the particle is at when $t = 0$. Those few candidates who implicitly assumed that the particle moves with constant speed, leading to an answer such as $\frac{1}{4}T$, gained no credit. The second part was often answered correctly by using the SHM formula $v^2 = \omega^2 (a^2 - x^2)$, and taking the ratio of the two values of v corresponding to $x = \frac{1}{2}a$ and 0. Occasionally the ratio for v^2 or the inverse ratio for v was mistakenly given as the answer.

Answers: **(i)** $\frac{T}{6}$; **(ii)** $\frac{\sqrt{3}}{2}$.

Question 3

The force exerted by the inclined plane on the rod at B was usually found correctly by taking moments for the rod about A, though very few candidates wrongly assumed the force acts in a vertical direction rather than normal to the plane. In some cases it would have assisted candidates to show all the forces on a diagram. The friction F and reaction R_A at A were then usually found correctly, from vertical and horizontal resolutions of the forces acting on the rod or taking moments about some point other than A. The question was answered less well thereafter, though, with a large number of candidates taking the coefficient of friction μ to equal $\frac{F}{R_A}$, with no mention of this being either the limiting case or a lower bound on μ. The underlying problem in many such cases may have been a lack of facility with inequalities. The final step is to note how this bound on μ varies with x and hence deduce that the least possible value of μ permitting equilibrium at any point on the half of the rod nearer to B follows from taking $x = 0.4$.

Answers: $25x$; $\frac{1}{\sqrt{3}}$.

Question 4

The value of k may be found in a variety of ways, though all effectively reduce to equating the initial vertical speeds of the two balls. The most common fault was probably to equate instead their horizontal speeds. While many candidates then attempted to obtain the horizontal speed of each ball after the collision, using conservation of momentum and the restitution equation, this is unnecessary. Instead only the difference in horizontal speeds is required, which follows from the latter equation, since combining this relative speed with the time to the ground of $\frac{4u}{5g}$ yields the required distance between the balls when they first hit the ground.

Answers: $\frac{4}{3}$; $\frac{4eu^2}{3g}$.

Question 5

The moments of inertia about O of both the original disc and the removed rectangle are first found from the standard formulae. Taking their difference yields the moment of inertia about O of the lamina, and application of the parallel axes theorem produces the required moment of inertia about A. Alternatively the theorem may be applied separately to the disc and the rectangle before the difference is taken. The fact that the final result is given in the question may have assisted some candidates to refine their working. While many candidates appreciated that they should then apply conservation of energy to the rotation of the lamina, very few realised that the critical position to consider is when the centre of gravity O is at the highest point, implying that O is at a vertical distance 0.4 above its initial position. Instead a variety of other vertical distances were wrongly inserted into the energy equation.

Answer: 6.23 .

Question 6

While a variety of approaches were seen to this question, most of them were invalid. It is necessary to consider the normal distribution of the differences in the diameters of the tubes and rods following the system failure, and in particular the probability that the expected value of the difference will be less than (or exceed, as appropriate) zero. Even among those candidates who formulated the ratio of the expected value and the standard deviation, few used the correct standard deviation 0.03289 corresponding to the distribution of the differences of the increased diameters.

Answer: 0.431 .

Question 7

The appropriate expression for the required confidence interval was widely known and correctly applied. The only common fault was to use an incorrect tabular t-value instead of 2.262, though some candidates used a biased estimate of the population variance instead of an unbiased one. No change is needed in the latter estimate in the second part of the question since the sample consists of the same ten specimens, though the value of n in the denominator of the confidence interval expression is then 20 rather than 10, while the tabular value of t changes to 2.093. It is not of course necessary to give the actual confidence interval, since only its width is requested.

Answers: [23.5, 33.6]; 6.61 .

Question 8

In order to give a clear explanation in part **(i)**, it is necessary to relate $P(X > x)$ to the distribution of the number of serious faults N on a randomly chosen x km stretch of motorway, and more specifically to note that it equals the probability of zero faults, $P(N = 0)$. Since this probability is the first term in a Poisson distribution with mean $2.1x$, the given result follows. Most candidates failed to distinguish between the distributions of N and X, though, and their explanations were at best unclear and at worst invalid. However many went on to make a reasonable attempt at the distribution function $F(x)$, which is simply $1 - P(X > x)$, and to differentiate $F(x)$ in order to obtain the probability density function $f(x)$. The mean distance was usually found correctly from $\frac{1}{2.1}$, as was the median by equating $F(x)$ to $\frac{1}{2}$, or equivalently by using $f(x)$.

Answers: **(ii)** $1 - e^{-2.1x}$, $2.1\,e^{-2.1x}$; **(iii)** 0.476 km; **(iv)** 0.330 km .

Question 9

The key to answering this question correctly is to base the t-test on the changes in the cholesterol levels for the single sample of 8 people, rather than to conduct a two-sample test. The relevant assumption is that the population of differences is normal, and the test itself produces a calculated t-value of 2.61. This is greater than the appropriate tabular t-value 1.415, leading to the conclusion that the population mean cholesterol level is reduced. A very similar t-test, using the same unbiased estimate 238.21 of the population variance, leads to the calculated value 1.695, and comparison with the same tabular value as previously suggests that the reduction is more than 5 units.

Question 10

Most candidates knew how to find the estimated regression line of y on x, the general form of which is given in the List of Formulae, and then use it to evaluate y for $x = 0.40$. The calculation of the coefficient b is, however, very susceptible to rounding error, and candidates who made premature approximations risked producing answers which were not correct to the specified 3 significant figures. The product moment correlation coefficient r was often found correctly, apart from arithmetic errors, though those candidates who chose to find it from the square root of the product of the slopes in the regression lines of y on x and x on y often gave a positive rather than a negative value of r. The hypotheses are best expressed using a different symbol than r, $\rho = 0$ and $\rho \neq 0$ say, since r is essentially an estimate of ρ derived from the sample. Since the magnitude of r is less than the relevant tabular value 0.576, the test leads to the conclusion that the two variables are not correlated.

Answers: 7.31; –0.194 .

Question 11 EITHER

This optional question was attempted by a very small proportion of the candidates, and few produced complete answers to it. The first part is straightforward, requiring that $\sin^2 kt$ be differentiated twice and the resulting expression rearranged into the form $b + c\theta$. SHM can then be shown by, for example, changing the dependent variable to $\varphi = \theta - \frac{1}{2}$ so as to produce the standard form of the SHM equation in φ, and the centre and amplitude of the motion are then obvious. The expressions previously found for the first and second derivatives yield the radial and tangential components respectively of the resulting force acting on P, namely $mak^2\sin^2 2kt$ and $2mak^2\cos 2kt$, which may be combined to give the required result.

Answers: $b = 2k^2$, $c = -4k^2$; $\frac{1}{2}$; $\frac{1}{2}$.

Question 11 OR

Although some candidates specified a binomial or Poisson distribution, most correctly stated a geometric one, and then estimated p by taking the reciprocal of the sample mean 1.84. The expected values were usually found correctly by those who chose a geometric distribution, except for the case $x > 6$, which should be calculated so as to ensure the total of the expected frequencies is 100. The last three cells should be combined before the value 2.27 of χ^2 is calculated. Due to the combining of cells, there are now 2 degrees of freedom, and hence the appropriate tabular value is 7.378, leading to the conclusion that the geometric distribution with the estimated value of p fits the data. In answering the final part of the question, many candidates wrongly proposed alternative methods, instead of changes to the present method as requested. These changes are the use of $\frac{1}{2}$ for p, and a consequent increase of 1 in the number of degrees of freedom.